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ON JAMES AND JORDAN–VON NEUMANN TYPE CONSTANTS AND NORMAL STRUCTURE IN BANACH SPACES

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Abstract. The weakly convergent sequence coefficient $\mathrm{WCS}(X)$ is estimated by the James type constant $J_{X,t}(\tau)$, Jordan–von Neumann type constant $C_t(X)$ and the Domínguez Benavides coefficient $R(1,X)$, which enable us to obtain some sufficient conditions for normal structure. The results obtained in this paper are more general than other previously known sufficient conditions for normal structure.

1. Introduction

Let X be a nontrivial Banach space, we will use B_X and S_X to denote the unit ball and unit sphere of X, respectively. Recall that a Banach space X is

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called uniformly nonsquare if there exists $\delta > 0$ such that for all $x, y \in S_X$

$$\min\{\|x+y\|, \|x-y\|\} \le 2(1-\delta).$$

It is well known that if X is uniformly nonsquare then X is reflexive. A bounded convex subset K of a Banach space X is said to have normal structure if for every convex subset H of K that contains more than one point there exists a point $x_0 \in H$ such that

$$\sup\{\|x_0 - y\| : y \in H\} < \sup\{\|x-y\| : x, y \in H\}.$$

A Banach space X is said to have weak normal structure if every weakly compact convex subset of X that contains more than one point has normal structure. In reflexive spaces, both notions coincide. Weak normal structure and normal structure play an important role in metric fixed point theory for nonexpansive mappings, since it was proved by Kirk that every reflexive Banach space with normal structure has the fixed point property (see [10]). Many geometrical properties of Banach spaces implying weak normal structure or normal structure have been studied (see [2]–[4], [6]–[9], [11]–[16], [18]–[20]).

Throughout this paper, we assume that X does not have the Schur property. The weakly convergent sequence coefficient $\mathrm{WCS}(X)$, introduced by Bynum, was reformulated by Sims and Smyth in [13] in the following equivalent form:

$$\mathrm{WCS}(X) = \inf\Big\{\lim_{n,m;n\neq m} \|x_n - x_m\|,\ x_n \xrightarrow{\mathrm{w}} 0,\ \|x_n\| = 1\Big\},$$

where the infimum is taken over all weakly null sequences $\{x_n\}$ in X such that $\lim\limits_{n,m;n\neq m} \|x_n - x_m\|$ exists. It is clear that $1 \le \mathrm{WCS}(X) \le 2$ and it is known that $\mathrm{WCS}(X) > 1$ implies that X has weak normal structure.

The coefficient $R(1,X)$ was defined by Domínguez Benavides in [5] as

$$R(1,X) = \sup\Big\{\liminf_{n\to\infty}\{\|x_n + x\|\Big\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \le 1$ and all weakly null sequences $\{x_n\}$ in B_X such that

$$D[(x_n)] := \limsup_{n\to\infty}\limsup_{m\to\infty} \|x_n - x_m\| \le 1.$$

Obviously, $1 \le R(1,X) \le 2$.

The aim of this paper is to estimate the lower bounds for $\mathrm{WCS}(X)$ in terms of James type constant $J_{X,t}(\tau)$, Jordan–von Neumann type constant $C_t(X)$ and Domínguez Benavides coefficient $R(1,X)$. By means of these bounds we identify several geometrical properties implying normal structure. We show that properties obtained in this paper are more general than other previously known sufficient conditions for normal structure.

2. Preliminaries

Before going to the results, let us recall some concepts and results which will be used in the following sections. The James type modulus $J_{X,t}(\tau)$ was introduced by Takahashi in [14] as a generalization of the James constant $J(X)$. For $\tau \geq 0$ and $-\infty \leq t < \infty$, the modulus $J_{X,t}(\tau)$ is defined as

$$J_{X,t}(\tau) = \sup\big\{\mathcal{M}_t(\|x+\tau y\|, \|x-\tau y\|) : x, y \in S_X\big\},$$

where $\mathcal{M}_t(a,b)$ is the generalized mean defined by

$$\begin{aligned} \mathcal{M}_t(a,b) &:= \left(\frac{a^t+b^t}{2}\right)^{1/t} \quad (-\infty < t < \infty \text{ and } t \neq 0),\\ \mathcal{M}_{-\infty}(a,b) &:= \lim_{t\to-\infty} \mathcal{M}_t(a,b) = \min\{a,b\},\\ M_0(a,b) &:= \lim_{t\to 0} \mathcal{M}_t(a,b) = \sqrt{ab}, \end{aligned}$$

where a and b are two positive real numbers. Particular values of $J_{X,t}(\tau)$ include some known constants (see [1], [6]–[7], [11], [15]) such as $J(X) = J_{X,-\infty}(1)$, $T(X) = J_{X,0}(1)$, $A_2(X) = J_{X,1}(1)$, $E(X) = 2J_{X,2}(1)^2$, $C_{\mathrm{G}}(\tau, X) = J_{X,0}(\tau)$, $\rho_X(\tau) = J_{X,1}(\tau) - 1$ and $\gamma(\tau) = J_{X,2}(\tau)^2$. Some geometric properties of Banach spaces X in terms of the modulus $J_{X,t}(\tau)$ were investigated in [14], [18].

(i) X is uniformly nonsquare $\Leftrightarrow$ $J_{X,t}(1) < 2$ $\Leftrightarrow$ $J_{X,t}(\tau) < 1+\tau$ for some $0 < \tau < +\infty$.

(ii) X is uniformly smooth $\Leftrightarrow$ $\lim_{\tau\to 0^+} \frac{\rho_X(\tau)}{\tau} = 0$ $\Leftrightarrow$ $\lim_{\tau\to 0^+} \frac{J_{X,t}(\tau)-1}{\tau} = 0$ for $1 \leq t \leq 2$.

Let $-\infty \leq t < \infty$, the Jordan–von Neumann type modulus $C_t(X)$ is defined as follows (see [14]):

$$C_t(X) = \sup\left\{\frac{J^2_{X,t}(\tau)}{1+\tau^2} : 0 \leq \tau \leq 1\right\}.$$

The choice $t = -\infty$ gives the constant $C_{-\infty}(X)$ which has been discussed in [14], [16], [20]. It is clear that the Jordan–von Neumann type modulus is a generalization of the Jordan–von Neumann constant $C_{\mathrm{NJ}}(X) = C_2(X)$ and the Zbăganu constant $C_{\mathrm{Z}}(X) = C_0(X)$. Some properties and inequalities among them have been indicated in [14], [16].

(i) Let $-\infty < t < \infty$ and $t \neq 0$, then for any Banach space X,

$$C_t(X) = \sup\left\{\frac{[(\|x+y\|^t + \|x-y\|^t)/2]^{2/t}}{\|x\|^2 + \|y\|^2} : x \in S_X,\, y \in B_X\right\}.$$

Let $t = -\infty$, then

$$C_{-\infty}(X) = \sup\left\{\frac{\min\{\|x+y\|^2, \|x-y\|^2\}}{\|x\|^2+\|y\|^2} : (x,y) \neq (0,0)\right\}.$$

(ii) Let $-\infty \leq t < \infty$, X is uniformly nonsquare if and only if $C_t(X) < 2$.

(iii) $J^2(X)/2 \le C_{-\infty}(X) \le C_{\mathrm{Z}}(X) \le C_{\mathrm{NJ}}(X) \le J(X)$. Moreover, the above inequalities are strict in some Banach spaces.

3. Main results

THEOREM 3.1. *Let $\tau \ge 0$ and $t \in [-\infty, +\infty)$. Then for any Banach space X,*

$$\mathrm{WCS}(X) \ge \frac{1+\tau/R(1,X)}{J_{X,t}(\tau)}.$$

PROOF. If $J_{X,t}(\tau) = 1+\tau$, the estimate is trivial since $\mathrm{WCS}(X) \ge 1$ and $1 \le R(1,X) \le 2$. Suppose that $J_{X,t}(\tau) < 1+\tau$, then X is uniformly nonsquare and therefore reflexive. Let $\{x_n\}$ be a weakly null sequence in S_X such that $d = \lim_{n,m,n\neq m} \|x_n - x_m\|$ exists. We consider a normalized functional sequence $\{x_n^*\}$ such that $x_n^*(x_n) = 1$. Note that the reflexivity of X guarantees, by passing to a subsequence if necessary, that $x_n^* \xrightarrow{\mathrm{w}^*} x^*$. Let $0 < \varepsilon < 1$ and choose N large enough so that $|x^*(x_N)| < \varepsilon/2$ and $d - \varepsilon < \|x_m - x_N\| < d + \varepsilon$ for all $m > N$. Note that

$$\lim_{n\neq m} \left\| \frac{x_m - x_n}{d+\varepsilon} \right\| \le 1, \qquad \left\| \frac{x_N}{d+\varepsilon} \right\| \le 1.$$

By the definition of $R(1,X)$, we can choose $M > N$ large enough such that

(i) $x_N^*(x_M) < \varepsilon$;
(ii) $|(x_M^* - x^*)(x_N)| < \varepsilon/2$;
(iii) $||(x_M + x_N)/(d+\varepsilon)|| \le R(1,X) + \varepsilon$.

Hence

$$|x_M^*(x_N)| \le |(x_M^* - x^*)(x_N)| + |x^*(x_N)| < \varepsilon.$$

Denote $R := R(1,X)$. Let us put

$$x = \frac{x_N - x_M}{d+\varepsilon}, \qquad y = \frac{x_N + x_M}{(d+\varepsilon)(R+\varepsilon)}.$$

It is easy to check that $x, y \in B_X$ and for $\tau \ge 0$

$$\begin{aligned}
(d+\varepsilon)\|x+\tau y\| &= \left\| \left(1+\frac{\tau}{R+\varepsilon}\right)x_N - \left(1-\frac{\tau}{R+\varepsilon}\right)x_M \right\| \\
&\ge \left(1+\frac{\tau}{R+\varepsilon}\right)x_N^*(x_N) - \left(1-\frac{\tau}{R+\varepsilon}\right)x_N^*(x_M) \\
&\ge 1+\frac{\tau}{R+\varepsilon} - \varepsilon, \\
(d+\varepsilon)\|x-\tau y\| &= \left\| \left(1+\frac{\tau}{R+\varepsilon}\right)x_M - \left(1-\frac{\tau}{R+\varepsilon}\right)x_N \right\| \\
&\ge \left(1+\frac{\tau}{R+\varepsilon}\right)x_M^*(x_M) - \left(1-\frac{\tau}{R+\varepsilon}\right)x_M^*(x_N) \\
&\ge 1+\frac{\tau}{R+\varepsilon} - \varepsilon.
\end{aligned}$$

By the definition of $J_{X,t}(\tau)$, for $t \in [-\infty, +\infty)$,

$$(d+\varepsilon)J_{X,t}(\tau) \geq 1 + \frac{\tau}{R+\varepsilon} - \varepsilon.$$

Since the sequence $\{x_n\}$ and ε are arbitrary,

$$\mathrm{WCS}(X) \geq \frac{1+\tau/R(1,X)}{J_{X,t}(\tau)}. \qquad \square$$

Corollary 3.2. *Let X be a Banach space with*

$$J_{X,t}(\tau) < 1 + \frac{\tau}{R(1,X)}$$

for some $\tau \geq 0$ and $t \in [-\infty, +\infty)$, then X has normal structure. In particular, if $J(X) < 1 + 1/R(1,X)$ then X has normal structure.

Proof. Firstly, observe that $J_{X,t}(\tau) < 1 + \tau/R(1,X) \leq 1 + \tau$ and therefore X is uniformly nonsquare, then X is reflexive so weak normal structure coincides with normal structure and it is sufficient to prove that $\mathrm{WCS}(X) > 1$. By the assumption that $J_{X,t}(\tau) < 1 + \tau/R(1,X)$, it follows that

$$\mathrm{WCS}(X) \geq \frac{1+\tau/R(1,X)}{J_{X,t}(\tau)} > 1. \qquad \square$$

Remark 3.3. (1) It was proved by Mazcuñán-Navarro in [12] that $J(X) \geq R(1,X)$. Take $t = -\infty$ and $\tau = 1$ in Theorem 3.1, then we obtain [4, Theorem 3.2]. More precisely, it is clear that

$$\mathrm{WCS}(X) \geq \frac{1+1/R(1,X)}{J(X)} = \frac{J(X)+J(X)/R(1,X)}{[J(X)]^2} \geq \frac{J(X)+1}{[J(X)]^2}.$$

Meanwhile, Corollary 3.2 also improves Proposition 26 in [12], Corollary 11 in [3], and Corollary 4 in [8].

(2) Let $t = -\infty$ and $\tau = 1$ in Theorem 3.1. In the particular case that X is a Hilbert space, it follows that

$$\mathrm{WCS}(X) \geq \frac{1+1/R(1,X)}{J(X)} = \frac{1+1/\sqrt{3/2}}{\sqrt{2}} \approx 1.28.$$

In [3], Casini et al. obtained the following bound for $\mathrm{WCS}(X)$:

$$\mathrm{WCS}(X) \geq \frac{2}{2J(X)+1-\sqrt{5}}.$$

In the particular case that X is a Hilbert space, it follows that

$$\mathrm{WCS}(X) \geq \frac{2}{2J(X)+1-\sqrt{5}} = \frac{2}{2\sqrt{2}+1-\sqrt{5}} \approx 1.25.$$

However, the real value of $\mathrm{WCS}(X)$ is $\sqrt{2}$ in this case, so the result in Theorem 3.1 is better than Casini's result in some space.

Theorem 3.4. *Let X be a Banach space, for $t \in [-\infty, +\infty)$,*

$$\mathrm{WCS}(X)^2 \geq \frac{1 + 1/R(1,X)^2}{C_t(X)}.$$

Proof. If $C_t(X) = 2$, the estimate is trivial since $\mathrm{WCS}(X) \geq 1$ and $1 \leq R(1,X) \leq 2$. Suppose that $C_t(X) < 2$, then X is uniformly nonsquare and therefore reflexive. Let $\{x_n\}$ be a weakly null sequence in S_X such that $d = \lim_{n \neq m} \|x_n - x_m\|$. Take x_N, x_M, x_N^*, x_M^* as in Theorem 3.1, we can choose $M > N$ large enough such that

(i) $x_N^*(x_M) < \varepsilon$, $x_M^*(x_N) < \varepsilon$;
(ii) $|(x_M^* - x^*)(x_N)| < \varepsilon/2$;
(iii) $||(x_M + x_N)/(d+\varepsilon)|| \leq R(1,X) + \varepsilon$.

Denote $R := R(1,X)$, let us put

$$x = \frac{x_N - x_M}{d+\varepsilon}, \qquad y = \frac{x_N + x_M}{(d+\varepsilon)(R+\varepsilon)^2},$$

it is easy to check that $x \in B_X$, $\|y\| \leq 1/(R+\varepsilon) \leq 1$, so

$$\begin{aligned}
(d+\varepsilon)\|x+y\| &= \left\| \left(1 + \frac{1}{(R+\varepsilon)^2}\right) x_N - \left(1 - \frac{1}{(R+\varepsilon)^2}\right) x_M \right\| \\
&\geq \left(1 + \frac{1}{(R+\varepsilon)^2}\right) x_N^*(x_N) - \left(1 - \frac{1}{(R+\varepsilon)^2}\right) x_N^*(x_M) \\
&\geq \left(1 + \frac{1}{(R+\varepsilon)^2}\right)(1-\varepsilon), \\
(d+\varepsilon)\|x-y\| &= \left\| \left(1 + \frac{1}{(R+\varepsilon)^2}\right) x_M - \left(1 - \frac{1}{(R+\varepsilon)^2}\right) x_N \right\| \\
&\geq \left(1 + \frac{1}{(R+\varepsilon)^2}\right) x_M^*(x_M) - \left(1 - \frac{1}{(R+\varepsilon)^2}\right) x_M^*(x_N) \\
&\geq \left(1 + \frac{1}{(R+\varepsilon)^2}\right)(1-\varepsilon).
\end{aligned}$$

For $-\infty < t < \infty$ and $t \neq 0$, by the equivalent definition of $C_t(X)$,

$$C_t(X) \geq \frac{\left(\dfrac{\|x+y\|^t + \|x-y\|^t}{2}\right)^{2/t}}{\|x\|^2 + \|y\|^2} \geq \frac{\left(1 + \dfrac{1}{(R+\varepsilon)^2}\right)(1-\varepsilon)^2}{(d+\varepsilon)^2}.$$

For $t = -\infty$ and $t = 0$, we similarly get the following inequalities:

$$C_{-\infty}(X) \geq \frac{\min\{\|x+y\|^2, \|x-y\|^2\}}{\|x\|^2 + \|y\|^2} \geq \frac{\left(1 + \dfrac{1}{(R+\varepsilon)^2}\right)(1-\varepsilon)^2}{(d+\varepsilon)^2}.$$

$$C_{\mathrm{Z}}(X) \geq \frac{\|x+y\|\|x-y\|}{\|x\|^2 + \|y\|^2} \geq \frac{\left(1 + \dfrac{1}{(R+\varepsilon)^2}\right)(1-\varepsilon)^2}{(d+\varepsilon)^2}.$$

Since the sequence $\{x_n\}$ and ε are arbitrary, for $-\infty \le t < \infty$, we have

$$\mathrm{WCS}(X)^2 \ge \frac{1+1/R^2}{C_t(X)}. \qquad \square$$

COROLLARY 3.5. *Let X be a Banach space with*

$$C_t(X) < 1 + \frac{1}{R(1,X)^2}, \quad \textit{for some } t \in [-\infty, +\infty),$$

then X has normal structure. In particular, if $C_{-\infty}(X) < 1 + 1/R(1,X)^2$ then X has normal structure.

PROOF. Firstly, observe that $C_t(X) < 1 + 1/R(1,X)^2 \le 2$ for which $1 \le R(1,X) \le 2$, therefore X is uniformly nonsquare, so X is reflexive. It is sufficient to prove that $\mathrm{WCS}(X) > 1$. Since $C_t(X) < 1 + 1/R(1,X)^2$,

$$\mathrm{WCS}(X)^2 \ge \frac{1+1/R(1,X)^2}{C_t(X)} > 1. \qquad \square$$

THEOREM 3.6. *Let X be a Banach space, the conditions*

(a) $C_{\mathrm{Z}}(X) < (1+\sqrt{3})/2$;
(b) $C_{-\infty}(X) < (1+\sqrt{3})/2$;
(c) $C_{-\infty}(X) < 1 + 1/J(X)^2$;
(d) $C_{-\infty}(X) < 1 + 1/R(1,X)^2$

satisfy the chain of implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

PROOF. (a) $\Rightarrow$ (b) It is trivial due to the inequality $C_{-\infty}(X) \le C_{\mathrm{Z}}(X)$.

(b) $\Rightarrow$ (c) Since $2x(x-1) < 1$ if and only if $x \in ((1-\sqrt{3})/2, (1+\sqrt{3})/2)$, we have $2C_{-\infty}(X)(C_{-\infty}(X)-1) < 1$.

On the other hand, $C_{-\infty}(X) \ge J(X)^2/2$, so

$$J(X)^2(C_{-\infty}(X)-1) \le 2C_{-\infty}(X)(C_{-\infty}(X)-1) < 1$$

and then $C_{-\infty}(X) < 1 + 1/J(X)^2$.

(c) $\Rightarrow$ (d) It is trivial due to the well-known inequality $J(X) \ge R(1,X)$. $\square$

COROLLARY 3.7. *Let X be a Banach space with*

$$C_{-\infty}(X) < 1 + \frac{1}{J(X)^2},$$

then X has normal structure.

REMARK 3.8. (a) In [11], Llorens-Fuster et al. obtained the following result: if $C_{\mathrm{Z}}(X) < (1+\sqrt{3})/2$ then X has normal structure. Gao et al. proved in [8]: if $C_{\mathrm{Z}}(X) < 1 + 1/J(X)^2$ then X has normal structure. In view of Theorem 3.6 the hypotheses $C_{\mathrm{Z}}(X) < (1+\sqrt{3})/2$, $C_{\mathrm{Z}}(X) < 1+1/J(X)^2$ are more restrictive than the hypothesis of Corollary 3.5.

(b) We give an example showing that there are spaces satisfying the condition $C_{-\infty}(X) < 1 + 1/R(1,X)^2$ which do not satisfy $C_{-\infty}(X) < 1 + 1/J(X)^2$.

Consider for $\lambda \geq 1$ the space $X_\lambda := (l_2, |\cdot|_\lambda)$ with $\|x\|_\lambda = \max\{\|x\|_2, \lambda\|x\|_\infty\}$. Then

$$C_{-\infty}(X_\lambda) = \min\{\lambda^2, 2\}, \qquad J(X_\lambda) = \min\{\sqrt{2}\lambda, 2\},$$
$$R(1, X_\lambda) = \max\{\lambda/\sqrt{2}, \sqrt{3}/\sqrt{2}\}.$$

Then, for any $\lambda > \sqrt{3}/2$, $J(X) > R(1,X)$, and for any $\sqrt{(1+\sqrt{3})/2} < \lambda < \sqrt{5/3}$,

$$C_{-\infty}(X) > 1 + \frac{1}{J(X)^2} \quad \text{and} \quad C_{-\infty}(X_\lambda) < 1 + \frac{1}{R(1, X_\lambda)^2}.$$

(c) We use the space $l_{2,\infty}$ as a limiting case for Corollaries 3.2 and 3.5. The Bynum space $l_{2,\infty}$, which is the space l_2 renormed according to $\|x\|_{2,\infty} = \max\{\|x^+\|_2, \|x^-\|_2\}$, where x^+ and x^- are the positive and the negative part of x, respectively, defined as $x^+(i) = \max\{x(i), 0\}$ and $x^- = x^+ - x$. It is well known that $J(l_{2,\infty}) = 1 + 1/\sqrt{2}$, $C_{\mathrm{NJ}}(l_{2,\infty}) = 3/2$ (see [9]). From the inequality $C_{-\infty}(X) \leq C_{\mathrm{NJ}}(X)$, we get that $C_{-\infty}(l_{2,\infty}) \leq 3/2$, take $x = (-1, 1, 0, \ldots)$, $y = (1/2, 1/2, 0, \ldots) \in l_{2,\infty}$ and $\|x+y\| = \|x-y\| = 3/2$, and so $C_{-\infty}(X) \geq 3/2$, then $C_{-\infty}(l_{2,\infty}) = 3/2$. It was proved in [5] that $R(1, l_{2,\infty}) = \sqrt{2}$, so

$$J(X) = 1 + \frac{1}{R(1, l_{2,\infty})} \quad \text{and} \quad C_{-\infty}(l_{2,\infty}) = 1 + \frac{1}{R(1, l_{2,\infty})^2}.$$

However, $l_{2,\infty}$ lacks normal structure, so we conclude that the results obtained in the Corollaries 3.2 and 3.5 are sharp.

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