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GLOBAL WELL-POSEDNESS AND ATTRACTOR FOR DAMPED WAVE EQUATION WITH SUP-CUBIC NONLINEARITY AND LOWER REGULAR FORCING ON $\mathbb{R}^3$

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Abstract. The dissipative wave equation with sup-cubic nonlinearity and lower regular forcing term which belongs to $H^{-1}(\mathbb{R}^3)$ in the whole space $\mathbb{R}^3$ is considered. Well-posedness of a translational regular solution is achieved by establishing extra space-time translational regularity of an energy solution. Furthermore, a global attractor in the naturally defined energy space $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ is built.

1. Introduction

In this paper we study the following weakly damped wave equation defined on $\mathbb{R}^3$:

$$\begin{cases} u_{tt} + \gamma u_t - \Delta u + mu + f(x,u) = g(x) & \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ u(x,0) = u_0, \quad u_t(x,0) = u_1 & \text{for } x \in \mathbb{R}^3. \end{cases} \tag{1.1}$$

Here $\gamma, m > 0$ and the initial data (u_0, u_1) belongs to the energy space $\mathcal{H} = H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, $g \in H^{-1}(\mathbb{R}^3)$ is independent of time, $f \in C^1(\mathbb{R}^4)$, $f(0) = 0$

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and satisfies the following conditions:

$$|\partial_s f(x,s)| \leq C(1+|s|^{p-1}), \tag{1.2}$$

$$\liminf_{|x|+|s|\to\infty} \frac{f(x,s)}{s} \geq 0, \tag{1.3}$$

where $3 < p < 4$. Here (and throughout) the symbol C stands for a generic constant indexed occasionally for the sake of clarity, the different positive constants C_i, $i \in \mathbb{N}$, are also used for special differentiation, and for the convenience, we denote $u(t) = u(x,t)$ and $u_t(t) = u_t(t,x)$ throughout the paper.

When the spacial variable x belongs to a bounded domain, the existence of global attractors for weakly damped wave equations (1.1) has been studied extensively, see [1], [2], [4], [7]–[9], [17]–[19], [23], [26] and references therein. In the case that the spatial domain is unbounded, the typical Sobolev embedding is continuous but no longer compact, and the spaces $L^p(\mathbb{R}^{\mathbb{N}})$ are not nested, so there is a substantial difficulty in dealing with the compactness of the corresponding operator semigroup, which is a key point to obtain the existence of global attractor. To overcome this difficulty, several classical methods have been established. For example, in [14], Karachalios and Stavrakakis have employed weighted Sobolev spaces to deal with (1.1). However, when working in weighted spaces we have to impose an additional condition that the initial data and forcing term also belong to the corresponding weighted spaces. Combining with the idea of "tail estimate" in [27], Djiby and You proved that solutions of (1.1) are uniformly small for large spacial and time variables, and then obtained the asymptotic compactness in [10].

On the other hand, mathematical properties of (1.1) including well-posedness and asymptotic behavior also depend strongly on the growth rate of the nonlinearity f. For a long time, exactly the cubic growth rate of the nonlinearity f has been considered as critical in bounded domains, see for instance [2], [3], [23] and the literature cited therein. Based on the recent progress in Strichartz estimates for bounded domains (for the dimension $N = 3$), Burq et al. [6] proved the global well-posedness of weak solutions with quintic nonlinearity growth of hyperbolic equation, Kalantarov et al. [16] established the corresponding attractor theory for the dissipative wave equation in bounded domains. For the case of unbounded domains, since the Strichartz estimates have been known a little earlier, based on that, Kapitanski and Feireisl constructed global attractors in [15] and [11], [12], respectively.

Concerning regularity of the external force term g, all papers mentioned above require $g \in L^2$ or specially $g = 0$. Still, a larger range of g, i.e. $g \in H^{-1}$, can also be considered. We refer the reader to [22], [25], [28], in which the existence and asymptotic regularity of global attractor have been discussed for the strong damping wave equations. As mentioned in [28], the strong damping