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NODAL SOLUTIONS FOR A CLASS OF DEGENERATE ONE DIMENSIONAL BVP'S

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Abstract. In [7], a family of degenerate one dimensional boundary value problems was studied and the existence of positive (and negative) solutions and solutions that possess one interior node was shown for a range of values of a parameter, λ. It was conjectured that there is a natural extension of these results giving solutions with any prescribed number of interior nodes. This conjecture will be established here.

1. Introduction

In the recent papers [7], [8] and [9], the existence of nodal solutions for the degenerate boundary value problem

$$\begin{cases} -du'' = \lambda u - a(x)f(u)u & \text{for } x \in [0,L], \\ u(0) = u(L) = 0, \end{cases} \tag{1.1}$$

was studied. The functions a and f satisfy

$$0 \leq a \in \mathcal{C}[0,1], \quad a^{-1}(0) = [\alpha,\beta], \quad 0 < \alpha < \beta < L, \tag{1.2}$$

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and

$$f \in \mathcal{C}^1(\mathbb{R}), \quad f(0) = 0, \quad \xi f'(\xi) > 0 \quad \text{for } \xi \neq 0, \quad \text{and} \quad \lim_{|\xi| \to \infty} f(\xi) = \infty. \tag{1.3}$$

By (1.3), $f(\xi) > 0$ for $\xi \neq 0$.

In (1.1), d is a positive constant, which without lost of generality can be set equal to 1. Thus instead of (1.1), the problem

$$\begin{cases} -u'' = \lambda u - a(x) f(u) u & \text{for } x \in [0, L], \\ u(0) = u(L) = 0, \end{cases} \tag{1.4}$$

will be considered. Problem (1.4) is degenerate in the sense that the function a vanishes on a subinterval of $[0, L]$. The analysis of the classical case when $a(x) > 0$ for all $x \in [0, L]$ can be found in [12], where it was established that for every integer $n \geq 1$, (1.4) admits a solution with $n-1$ (interior) zeroes, or nodes, in $(0, L)$ if and only if $\lambda > (n\pi/L)^2$, the n-th eigenvalue of $-D^2$, $D = d/dx$, in $(0, L)$ under Dirichlet boundary conditions. Later it was shown in [2] that the degenerate problem (1.4) admits a positive solution if and only if

$$\left(\frac{\pi}{L}\right)^2 < \lambda < \left(\frac{\pi}{\beta - \alpha}\right)^2.$$

Note that $(\pi/(\beta - \alpha))^2$ is the first eigenvalue of $-D^2$ in (α, β) under Dirichlet boundary conditions. More recent results, Theorems 4.1 and 4.2 of [7], tell us that (1.4) has a solution with one node in $(0, L)$ if and only if

$$\left(\frac{2\pi}{L}\right)^2 < \lambda < \left(\frac{2\pi}{\beta - \alpha}\right)^2.$$

It was further conjectured in [7] that in the general case when $n \geq 1$, (1.4) possesses a solution with $n - 1$ (interior) nodes in $(0, L)$ if and only if

$$\left(\frac{n\pi}{L}\right)^2 < \lambda < \left(\frac{n\pi}{\beta - \alpha}\right)^2. \tag{1.5}$$

Note that for every $n \geq 1$, $(n\pi/L)^2$ and $(n\pi/(\beta - \alpha))^2$ are the n-th eigenvalues of $-D^2$ in $(0, L)$ and (α, β), respectively, under Dirichlet boundary conditions. Our main goal here is to establish this conjecture as well as to further study the structure of the set of solutions of (1.4). To do so, first some preliminaries will be carried out in Section 2. Then, to obtain the conjecture, some nonstandard a priori bounds for solutions will be obtained in Section 3 and the existence argument, which employs Leray–Schauder degree theory, will be given in Section 4. In fact a stronger result will be obtained: for each λ satisfying (1.5), (1.4) has at least two solutions with $n - 1$ interior nodes, u^+ and u^-, such that $(u^+)'(0) > 0$ and $(u^-)'(0) < 0$.

To describe our further results and state our main result precisely, some definitions and notation are required. Throughout this paper, the solutions