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THREE ZUTOT

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Abstract. Three topics in dynamical systems are discussed. First we deal with cascades and solve two open problems concerning, respectively, product recurrence, and uniformly rigid actions. Next we provide a new example that displays some unexpected properties of strictly ergodic actions of non-amenable groups.

Introduction

We collect in this paper three short notes ([1]). They are independent of each other and are collected here just because they occurred to us in recent discussions. The first two actually solve some open problems concerning, respectively, product recurrence, and uniformly rigid actions admitting a weakly mixing fully supported invariant probability measure. The third provides a new interesting example that displays some unexpected properties of strictly ergodic actions of non-amenable groups.

1. On product recurrence

A *dynamical system* here is a pair (X, T) where X is a compact metric space and T a self-homeomorphism. The reader is referred to [4] for most of the notions used below and for the necessary background.

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([1]) Zuta is minutia (or miniature) in Hebrew; zutot is the plural, minutiae.

In [4, Theorem 9.11, p. 181], Furstenberg has shown that a point x of a dynamical system (X,T) is *product-recurrent* (i.e. has the property that for every dynamical system (Y,S) and a recurrent point $y \in Y$, the pair (x,y) is a recurrent point of the product system $X \times Y$) if and only if it is a distal point (i.e. a point which is proximal only to itself). In [2], Auslander and Furstenberg posed the following question: if (x,y) is recurrent for all *minimal* points y, is x necessarily a distal point? Such a point x is called a *weakly product recurrent point*. This question is answered in the negative in [11].

It turns out (see also [3, Theorem 4.3]) that a negative answer was already at hand for Harry Furstenberg when he and Joe Auslander posed this question. In fact, many years earlier, he proved a theorem according to which an F-flow ([2]) is disjoint from every minimal system [4]. As a direct consequence of this theorem, if X is an F-flow, x a transitive point in X, Y any minimal system and y any point in Y, then the pair (x,y) has a dense orbit in $X \times Y$. In particular, (x,y) is a recurrent point of the product system $X \times Y$. Thus a transitive point x in an F-flow is weakly product recurrent. Since such a point is never distal, one concludes that x is indeed weakly product recurrent but not distal.

In [11, Question 5.3] the authors pose the following natural question:

Problem 1.1. Is every *minimal* weakly product recurrent point a distal point?

(This was also repeated in [3, Question 9.2].)

In this note we show that, here again, the answer is negative. The counter example is based on a result of [5] concerning POD systems and on the existence of doubly minimal systems (see [13] and [14]). A minimal dynamical system (X,T) is called *proximal orbit dense* (POD) if it is totally minimal and for any distinct points u and v in X, there exists $0 \neq n \in \mathbb{Z}$ such that $\Gamma_n = \{(T^n x, x) : x \in X\}$ is contained in $\overline{\mathcal{O}_{T\times T}(u,v)}$, the orbit closure of (u,v) in the product system $X \times X$.

A minimal (X,T) is called *doubly minimal* [14] (or a system having *topologically minimal self-joinings in the sense of del Junco* [13]) if the only orbit closures of $T \times T$ in $X \times X$ are the graphs $\Gamma_m = \{(T^m x, x) : x \in X\}$, $m \in \mathbb{Z}$, and all of $X \times X$. Clearly a doubly minimal system is POD. In [5], the authors prove the following striking property of POD systems:

Theorem 1.2. *If (Y,S) is POD then any minimal (X,T) that is not an extension of (Y,S) is disjoint from it.*

For the reader's convenience we reproduce the short proof:

([2]) Recall that a dynamical system (X,T) is an *F-flow* if it is (i) totally transitive (i.e. every power T^n, $n \neq 0$, is transitive) and (ii) the periodic points are dense in X. E.g. every weakly mixing finite type subshift is an F-flow.