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BOUNDEDNESS IN A TWO-SPECIES QUASI-LINEAR CHEMOTAXIS SYSTEM WITH TWO CHEMICALS

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Abstract. We consider the two-species quasi-linear chemotaxis system generalizing the prototype

$$\begin{cases} u_t = \nabla\cdot(D_1(u)\nabla u) - \chi_1\nabla\cdot(S_1(u)\nabla v), & x\in\Omega,\ t>0,\\ 0=\Delta v - v + w, & x\in\Omega,\ t>0,\\ w_t = \nabla\cdot(D_2(w)\nabla w) - \chi_2\nabla\cdot(S_2(w)\nabla z), & x\in\Omega,\ t>0,\\ 0=\Delta z - z + u, & x\in\Omega,\ t>0, \end{cases} \tag{0.1}$$

under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega\subseteq\mathbb{R}^N$ $(N\geq 1)$. Here $D_i(u)=(u+1)^{m_i-1}$, $S_i(u)=u(u+1)^{q_i-1}$ $(i=1,2)$, with parameters $m_i\geq 1$, $q_i>0$ and $\chi_1,\chi_2\in\mathbb{R}$. Hence, (0.1) allows the interaction of attraction-repulsion, with attraction-attraction and repulsion-repulsion type. It is proved that

(i) in the attraction-repulsion case $\chi_1<0$: if $q_1<m_1+2/N$ and $q_2<m_2+2/N-(N-2)^+/N$, then for any nonnegative smooth initial data, there exists a unique global classical solution which is bounded;

(ii) in the doubly repulsive case $\chi_1=\chi_2<0$: if $q_1<m_1+2/N-(N-2)^+/N$ and $q_2<m_2+2/N-(N-2)^+/N$, then for any nonnegative smooth initial data, there exists a unique global classical solution which is bounded;

(iii) in the attraction-attraction case $\chi_1=\chi_2>0$: if $q_1<2/N+m_1-1$ and $q_2<2/N+m_2-1$, then for any nonnegative smooth initial data, there exists a unique global classical solution which is bounded.

In particular, these results demonstrate that the circular chemotaxis mechanism underlying (0.1) goes along with essentially the same destabilizing features as known for the quasi-linear chemotaxis system in the doubly attractive case. These results generalize the results of Tao and Winkler (Discrete Contin. Dyn. Syst. Ser. B. 20 (9) (2015), 3165–3183) and also enlarge the parameter range $q>2/N-1$ (see Cieślak and Winkler (Nonlinearity 21 (2008), 1057–1076)).

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1. Introduction

In this paper, we consider the initial-boundary value problem for the two-species quasi-linear chemotaxis system with two chemicals

$$
\begin{cases}
u_t = \nabla \cdot (D_1(u)\nabla u) - \chi_1 \nabla \cdot (S_1(u)\nabla v), & x \in \Omega,\ t > 0,\\
\tau v_t = \Delta v - v + w, & x \in \Omega,\ t > 0,\\
w_t = \nabla \cdot (D_2(w)\nabla w) - \chi_2 \nabla \cdot (S_2(w)\nabla z), & x \in \Omega,\ t > 0,\\
\tau z_t = \Delta z - z + u, & x \in \Omega,\ t > 0,\\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0,\\
u(x,0) = u_0(x),\ w(x,0) = w_0(x), & x \in \Omega,
\end{cases}
\tag{1.1}
$$

where $\tau \in \{0,1\}$, Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$, $\Delta = \sum_{i=1}^{N} \partial^2/\partial x_i^2$, $\partial/\partial\nu$ denotes the outward normal derivative on $\partial\Omega$, $\chi_i \in \mathbb{R}$ $(i = 1,2)$ are parameters, which determine the attraction-repulsion case ($\chi_1 = 1$ and $\chi_2 = -1$), the repulsion-repulsion case ($\chi_1 = \chi_2 = -1$) and the attraction-attraction case ($\chi_1 = \chi_2 = 1$), respectively.

The first species, with density denoted by u, adapts its motion according to a chemical substance with concentration v, the latter being secreted by the second species, mathematically represented through its density w. The individuals of the second population themselves orient their movement along concentration gradients of a second signal with density z which in turn is produced by the first species. Moreover, we assume that

$$D_i, S_i \in C^2([0,\infty)) \quad \text{and} \quad S_i(u) \geq 0 \quad \text{for all } u \geq 0, \tag{1.2}$$

satisfy

$$D_i(u) \geq C_{D_i}(u+1)^{m_i - 1} \quad \text{for all } u \geq 0, \tag{1.3}$$

$$S_i(u) \leq C_{S_i} u^{q_i} \quad \text{for all } u \geq 0, \tag{1.4}$$

with $m_i \geq 1$, $q_i, C_{D_i}, C_{S_i} > 0$ $(i = 1,2)$.

System (1.1) may be viewed as a simplified variant of a fully parabolic two-species chemotaxis model with two chemicals, involving slightly more general crossdiffusion mechanisms, as it has been proposed in [22] to describe chemotaxis-driven processes of cell sorting.

During the past decades, the chemotaxis models have become one of the best study models in numerous biological and ecological contexts, and one of the main issues is under what conditions the solutions of chemotaxis system blow up or exist globally. In order to better understand problem (1.1), let us mention some previous contributions in this direction. When $w \equiv u$ and $v \equiv z$, PDE system