Topological Methods in Nonlinear Analysis
Volume 49, No. 1, 2017, 273–298
DOI: 10.12775/TMNA.2016.076

HAUSDORFF PRODUCT MEASURES AND C^1-SOLUTION SETS OF ABSTRACT SEMILINEAR FUNCTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. A second order semilinear neutral functional differential inclusion with nonlocal conditions and multivalued impulse characteristics in a separable Banach space is considered. By developing appropriate computing techniques for the Hausdorff product measures of noncompactness, the topological structure of C^1-solution sets is established; and some interesting discussion is offered when the multivalued nonlinearity of the inclusion is a weakly upper semicontinuous map satisfying a condition expressed in terms of the Hausdorff measure.

1. Introduction

In this paper, we are concerned with the sets of C^1-solutions defined on a compact real interval for second order semilinear neutral functional differential inclusions with nonlocal conditions and multivalued impulse characteristics in a separable Banach space. More precisely, we will consider the following second

2010 *Mathematics Subject Classification.* 47H10, 34A60, 34A37, 34K30.
Key words and phrases. Impulsive functional inclusion; cosine family of operators; fixed point for multivalued map; measure of noncompactness.
This work is supported by the National Natural Science Foundation of China and Nanjing University of Information Science and Technology (2014X024).

order semilinear differential inclusions:

$$
\text{(FIP)}\qquad
\begin{cases}
\dfrac{d}{dt}\,[x'(t)-g(t,x_t)] \in Ax(t)+F(t,x_t) & \text{a.e. } t\in I\setminus\{t_1,\dots,t_m\},\\
x(t_k^+)-x(t_k^-)\in \varphi_k(x(t_k^-)) & \text{for } k=1,\dots,m,\\
x'(t_k^+)-x'(t_k^-)\in \psi_k(x(t_k^-)) & \text{for } k=1,\dots,m,\\
x(t)+h_1(x)=\phi(t), \quad x'(0)=h_2(x) & \text{for } t\in I_0,
\end{cases}
$$

where $I=[0,a]$, $I_0=[-r,0]$, $0<r,a<+\infty$ and $0=t_0<t_1<\ldots<t_m<t_{m+1}=a$. The linear operator $A\colon D(A)\subset X\to X$ is the infinitesimal generator of a strongly continuous cosine family $\{C(t)\}$ in a real separable Banach space X with the norm $\|\cdot\|$. The nonlinearity $F\colon I\times\Delta\multimap X$ is a multivalued map, $\Delta=\{u\colon I_0\to X : u$ is continuously differentiable everywhere except for a finite number of points at which $u(s^+),u(s^-),u'(s^+)$ and $u'(s^-)$ exist and $u(s)=u(s^-)\}$. The neutral item $g\colon I\times\Delta\to X$ is a single valued mapping such that $t\mapsto g(t,x_t)$ is absolutely continuous. For impulsive conditions, $\varphi_k,\psi_k\colon X\multimap X$ are all multivalued maps, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t=t_k$, respectively. For nonlocal conditions, h_1,h_2 are two single valued mappings such that $h_1(x),h_2(x)\in X$; $\phi\in\Delta$. For any function x defined on $[-r,a]$ and any $t\in I$, $x_t\in\Delta$ is defined by

$$x_t(\theta)=x(t+\theta),\quad \theta\in I_0=[-r,0].$$

Here $x_t(\,\cdot\,)$ represents the history of the state from $t-r$, up to the present time t.

Recently, the problems of existence of solutions and controllability for some abstract first order or second order semilinear functional differential inclusions, with or without impulsive conditions, have been studied by several researchers (see [1], [5], [6], [10], [14], [15], [19] and the references therein). By relying on the theory of semigroup or cosine families and fixed point theorems for multivalued maps, some existence and controllability results were obtained. Let us mention that some results often contain the assumption of compactness of the semigroup or cosine families generated by the linear part of the inclusion. It was pointed out in [17] that, in infinite-dimensional case, these hypotheses are in contradiction to each other.

In the present paper we assume that the linear part of the inclusion generates a cosine family which is not necessarily compact; and the multivalued nonlinearity of the inclusion is a weakly upper semicontinuous map satisfying a condition expressed in terms of the Hausdorff measure. At the same time, we consider nonlocal initial conditions and impulsive inclusions with multivalued jump operators. To the best of our knowledge, there are very few results for these aspects. Our goal in this paper is to establish the topological structure of the C^1-solution set for problem (FIP), by developing appropriate computing techniques for the Hausdorff product measures of noncompactness.

2. Preliminaries

Throughout this paper, $\mathbb{R}$ is the set of all real numbers and $\mathbb{Z}^+$ the set of all positive integers. Moreover, $\mathbb{R}^+ = [0, +\infty)$, $I = [0, a]$, $I_1 = [0, t_1]$, $I_k = (t_{k-1}, t_k]$, and $\overline{I_k} = [t_{k-1}, t_k]$, $k = 2, \ldots, m+1$. Let $(X, \|\cdot\|)$ be a real separable Banach space. For $U \subset X$, the notations $\overline{U}$ and $\operatorname{co} U$ stand for the closure and the convex hull, respectively. Let J_* be a compact interval in $\mathbb{R}$. Then $C(J_*, X)$ denotes the Banach space consisting of continuous functions from J_* into X with the norm

$$\|x\|_C = \sup_{t \in J_*} \|x(t)\|$$

and $C^1(J_*, X)$ denotes the Banach space of continuously differentiable functions from J_* into X with the norm

$$\|x\|_{C^1} = \sup_{t \in J_*} [\|x(t)\| + \|x'(t)\|].$$

Let $t_1, \ldots, t_m$ be fixed in I. We will consider the space of piecewise continuous functions

$$\begin{aligned} PC^1 = \big\{ x \colon I \to X : {} & x'(t) \text{ is continuous at } t \neq t_k, \\ & \text{and } x(t) \text{ is left continuous at } t = t_k, \\ & \text{and } x(t_k^+),\, x'(t_k^+),\, x'(t_k^-) \text{ exist},\ k = 1, \ldots, m \big\}. \end{aligned}$$

Endowed with the norm

$$\|x\|_\diamond = \max_{1 \le k \le m+1} \sup_{t \in I_k} [\|x(t)\| + \|x'(t)\|],$$

PC^1 is a Banach space. It is evident that

$$\|x\|_\diamond = \sup_{t \in I} [\|x(t)\| + \|x'(t)\|].$$

Note that for $x \in PC^1$ we have $x'_-(t_k) = x'(t_k^-)$, where $x'_-(t_k)$ is the left derivative of x at $t = t_k$. Hence we can think that x' is also left continuous at each $t = t_k$.

Set $J = I_0 \cup I = [-r, a]$, and $PC^1(J) = \{x \colon J \to X : x \in \Delta \cap PC^1\}$. For $u \in \Delta$, the norm of u is defined by

$$\|u\|_\Delta = \sup_{\theta \in I_0} [\|u(\theta)\| + \|u'(\theta)\|].$$

For $x \in PC^1(J)$, the norm of x is defined by

$$\|x\|_* = \sup_{t \in J} [\|x(t)\| + \|x'(t)\|] = \max \{\|x\|_\Delta, \|x\|_\diamond\}.$$

Δ and $PC^1(J)$ are Banach spaces. It is evident that if $\{x_n\}_{n=0}^\infty \subset PC^1(J)$, then $x_n \to x_0$ in $PC^1(J)$ if and only if $x_n \to x_0$ in Δ and in PC^1.

We denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X and put

$$\begin{aligned}\mathcal{P}_{\rm cl}(X) &= \{Z \in \mathcal{P}(X) : Z \text{ is closed}\},\\ \mathcal{P}_{\rm bd}(X) &= \{Z \in \mathcal{P}(X) : Z \text{ is bounded}\},\\ \mathcal{P}_{\rm cp}(X) &= \{Z \in \mathcal{P}(X) : Z \text{ is compact}\},\\ \mathcal{P}_{\rm cv}(X) &= \{Z \in \mathcal{P}(X) : Z \text{ is convex}\},\\ \mathcal{P}_{\rm wcp}(X) &= \{Z \in \mathcal{P}(X) : Z \text{ is weakly compact}\}.\end{aligned}$$

Let Y be a metric space. For a multivalued map $T\colon Y \multimap X$ we mean that it has at least nonempty values, i.e. $T\colon Y \to \mathcal{P}(X)$. A multivalued map T is said to have *convex* (*bounded, closed, compact, weakly compact*) *values* if $T(y)$ is convex (bounded, closed, compact, weakly compact) for every $y \in Y$. For $Z \in \mathcal{P}_{\rm cl,bd}(X)$ we mean that $Z \in \mathcal{P}_{\rm cl}(X) \cap \mathcal{P}_{\rm bd}(X)$. A point $u \in Y \subset X$ is called a *fixed point* of T if $u \in T(u)$. The fixed point set of T will be denoted by $\mathrm{Fix}(T)$. T is a *closed graph* map if the graph of T, i.e. $\mathrm{Gr}(T) = \{(y,u) \in Y \times X : u \in T(y)\}$ is closed in $Y \times X$. T is called (*weakly*) *upper semicontinuous* (u.s.c. for short) if for each nonempty (weakly) closed set $U \subset X$,

$$T^+(U) = \{y \in Y : T(y) \cap U \neq \emptyset\}$$

is closed in Y. Evidently, if T is u.s.c., then T is weakly u.s.c. If T has weakly compact and convex values, then T is weakly u.s.c. if and only if for each sequence $\{y_n\} \subset Y$ with $y_n \to y_0 \in Y$ and $z_n \in T(y_n)$ it follows that there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $\{z_{n_k}\}$ converges weakly to $z_0 \in T(y_0)$ (see [7]). T is said to be *quasicompact* if $T(B)$ is relatively compact in X for each relatively compact subset B of Y. If T is u.s.c. with closed values, then $\mathrm{Gr}(T)$ is closed (see [3], [18]). Conversely, the following assertion holds (see Theorem 1.1.12 in [12]).

Lemma 2.1. *If $T\colon Y \multimap X$ is a quasicompact map with closed graph, then T is a u.s.c. map.*

For $Z \subset Y$ and $y \in Y$, we denote $d(y,Z) = \inf_{z \in Z} d(y,z)$, where d is the metric function. For $B_1, B_2 \subset Y$, we denote by $H(B_1,B_2)$ the Hausdorff–Pompeiu distance between B_1 and B_2, that is

$$H(B_1,B_2) = \max\Big\{\sup_{x \in B_1} d(x,B_2), \sup_{y \in B_2} d(y,B_1)\Big\}.$$

$T\colon Y \multimap X$ is called an *L-Lipschitz map* if there exists $L > 0$ such that

$$H(Tx,Ty) \le L d(x,y), \quad \text{for all } x,y \in Y;$$

and is called a *contraction* if $L < 1$. Let β_H be the Hausdorff measure of noncompactness defined on a collection of nonempty, bounded subsets B of X

or Y by

$$\beta_H(B) = \inf\{\varepsilon > 0 : B \text{ has a finite } \varepsilon\text{-net}\}.$$

T is said to be β_H*-condensing* if, for each bounded nonrelatively-compact subset B of Y, $T(B)$ is bounded and satisfies $\beta_H(T(B)) < \beta_H(B)$. If T is an L-Lipschitz map with compact values, then $\beta_H(T(B)) \leq L\beta_H(B)$ for each bounded $B \subset Y$ (see [20]). The key tool in our approach is the following fixed point theorem.

LEMMA 2.2 (see [12, Corollary 3.3.1]). *Let X be a Banach space, $\mathcal{D}$ a bounded convex closed subset of X and $T\colon \mathcal{D} \to \mathcal{P}_{\mathrm{cp,cv}}(\mathcal{D})$ a u.s.c. β_H-condensing multivalued map. Then* $\mathrm{Fix}(T)$ *is a nonempty compact set.*

Let $\mathcal{L}(I)$ be the Lebesgue σ-algebra of I. A multivalued map $T\colon I \multimap X$ is said to be *Lebesgue measurable* if for each closed set $U \subset X$, $T^+(U) \in \mathcal{L}(I)$. If T is measurable and has closed values, then T admits a measurable selector (see [2], [3]). By $L^1(I,X)$ we denote the Banach space of all Bochner integrable mappings from I into X with the norm

$$\|x\|_L = \int_0^a \|x(t)\|\, dt.$$

If $\alpha \in L^1(I,\mathbb{R}^+)$, then $\alpha(t) \geq 0$ for almost every $t \in I$ and

$$\|\alpha\|_L = \int_0^a \alpha(t)\, dt < +\infty.$$

The following lemma is a generalization of the Dunford–Pettis weak compactness criterion.

LEMMA 2.3 (see [22, Proposition 11] or [8, Corollary 2.6]). *Suppose the function $p_0 \in L^1(I,\mathbb{R}^+)$ and the sequence $\{f_n(t)\} \subset L^1(I,X)$ are such that*:

(a) *$\|f_n(t)\| \leq p_0(t)$ for almost every $t \in I$ and all $n \in \mathbb{Z}^+$;*
(b) *for almost every $t \in I$ the sequence $\{f_n(t)\}$ is relatively weakly compact in X.*

Then the sequence $\{f_n\}$ is relatively weakly compact in $L^1(I,X)$.

LEMMA 2.4 (Corollary to Mazur's Theorem, see [11]). *Suppose that $\{f_n\}_{n=1}^\infty$ is a sequence which converges weakly to f_0 in $L^1(I,X)$. Then there exists a sequence $\{g_n\}_{n=1}^\infty$ with $g_n \in \mathrm{co}\{f_i : i \geq n\}$ such that $\{g_n(t)\}$ converges to $f_0(t)$ in X for almost every $t \in I$.*

LEMMA 2.5 (see [12], [17]). *Let X be a separable Banach space and $T\colon I \multimap X$ an integrable map. If there exist $p_0, \alpha \in L^1(I,\mathbb{R}^+)$ such that $\sup_{z\in T(t)} \|z\| \leq p_0(t)$ and $\beta_H(T(t)) \leq \alpha(t)$ for almost every $t \in I$, then*

$$\beta_H\biggl(\int_0^t T(s)ds\biggr) \leq \int_0^t \alpha(s)\, ds, \quad \textit{for all } t \in I.$$

Let $F\colon I \times \Delta \multimap X$ be a multivalued map. F is said to be *locally integrably bounded* (or p_λ*-locally integrably bounded*) if for each $\lambda > 0$, there exists $p_\lambda \in L^1(I, \mathbb{R}^+)$ such that

$$\|u\|_\Delta \leq \lambda \ \Rightarrow\ \sup\{\|z\| : z \in F(t,u)\} \leq p_\lambda(t) \quad \text{for a.e. } t \in I.$$

For $x \in PC^1(J)$, we use the notation $S^1_F(x)$ to denote the set of integrable selectors (possibly empty), i.e.

$$S^1_F(x) = \big\{f \in L^1(I,X) : f(t) \in F(t,x_t) \text{ for a.e. } t \in I\big\}. \tag{2.1}$$

In what follows, $\{C(t) : t \in \mathbb{R}\}$ will denote a strongly continuous cosine family of bounded linear operators and $\{S(t) : t \in \mathbb{R}\}$ is the associated sine family defined by $S(t)x = \int_0^t C(\tau)x\,d\tau$, $x \in X$, $t \in \mathbb{R}$. The infinitesimal generator of $\{C(t) : t \in \mathbb{R}\}$ is the linear operator $A\colon D(A) \subset X \to X$, where $D(A) = \{x \in X : C(t)x \text{ is twice continuously differentiable in } t\}$. Also, E denotes the space $E = \{x \in X : C(t)x \text{ is once continuously differentiable in } t\}$; $B(E,X)$ stands for the Banach space of bounded linear operators from E into X with the norm $|\cdot|_*$, and we abbreviate this notation to $B(X)$ when $E = X$.

Lemma 2.6 (see [21]). *Let $\{C(t) : t \in \mathbb{R}\}$ be a strongly continuous cosine family in X with infinitesimal generator A. Then the following assertions are true.*

(a) *A is a closed linear operator in $D(A)$; $D(A) \subset E$; $D(A)$ is dense in X, i.e. $\overline{D(A)} = X$.*
(b) *$S(t+\tau) + S(t-\tau) = 2S(t)C(\tau)$, for all $t,\tau \in \mathbb{R}$.*
(c) *There exist $M_0 \geq 1$ and $\omega > 0$ such that for all $t \in \mathbb{R}$, $|C(t)|_* \leq M_0 e^{\omega|t|}$, $|S(t)|_* \leq M_0|t|e^{\omega|t|}$.*
(d) *$C(t+\tau) - C(t-\tau) = 2AS(t)S(\tau)$ for all $t,\tau \in \mathbb{R}$.*
(e) *$\frac{d}{dt}S(t)x = C(t)x$, for all $x \in X$ and $t \in \mathbb{R}$.*
(f) *$\frac{d}{dt}C(t)x = AS(t)x$, for all $x \in E$ and $t \in \mathbb{R}$.*
(g) *If $x \in E$, then $\lim_{t\to 0} AS(t)x = 0$.*

Lemma 2.7 (see [4], [14], [23]). *Each of the following conditions is equivalent to the norm-continuity* (*or uniform continuity*) *of $C(\,\cdot\,)$*:

(a) *$\lim_{t\to 0}|C(t) - I_X|_* = 0$, where I_X is the identity operator in X;*
(b) *the infinitesimal generator A is bounded* (*i.e. $D(A) = X$*).

It is known from Kisiński [13], that E endowed with the norm

$$\|x\|_E = \|x\| + \sup_{t\in I}\|AS(t)x\|$$

is a Banach space. From this definition it follows that $M_A = \sup_{t\in I}|AS(t)|_* \leq 1$ in $B(E,X)$. If an operator $W \in B(X)$, then we have $W \in B(E,X)$ and $|W|_{*B(E,X)} \leq |W|_{*B(X)}$ since $\|\cdot\| \leq \|\cdot\|_E$. Clearly, if $\{AS(t) : t \in I\} \subset B(X)$

and $M_A < +\infty$, then from Lemma 2.6 (c) (d) and Lemma 2.7 we see that A is bounded and $D(A) = E = X$. For the sake of simplicity, the space E mentioned in the sequel means $(E, \|\cdot\|_E)$. For $t, s \in I$, $\tau \in [0, \min\{t, s\}]$, from Lemma 2.6 we have

$$\begin{aligned} |S(t-\tau) - S(s-\tau)|_* &\leq 2M_0 e^{\omega a} |S((t-s)/2)|_* && \text{in } B(X), \\ |C(t-\tau) - C(s-\tau)|_* &\leq 2M_A |S((t-s)/2)|_* && \text{in } B(E, X). \end{aligned}$$

3. Several auxiliary results

LEMMA 3.1. *Let A be the infinitesimal generator of a strongly continuous cosine family $C(t)$ and let $f \in L^1(I, X)$. If $x \in PC^1(J)$ is a solution of the problem*

$$(\mathrm{FIP})^* \quad \begin{cases} \dfrac{d}{dt}\,[x'(t) - g(t, x_t)] = Ax(t) + f(t) & \textit{for a.e. } t \in I \setminus \{t_1, \ldots, t_m\}, \\ x(t_k^+) - x(t_k^-) = u_k & \textit{for } k = 1, \ldots, m, \\ x'(t_k^+) - x'(t_k^-) = v_k & \textit{for } k = 1, \ldots, m, \\ x(t) + h_1(x) = \phi(t), \quad x'(0) = h_2(x) & \textit{for } t \in I_0, \end{cases}$$

then it is given by $x(t) = \phi(t) - h_1(x)$ for $t \in I_0$; and

$$\begin{aligned} x(t) = {} & C(t)[\phi(0) - h_1(x)] + S(t)[h_2(x) - g(0, \phi)] \\ & + \sum_{0<t_k<t} [C(t - t_k)u_k + S(t - t_k)v_k] \\ & + \int_0^t C(t-\tau) g(\tau, x_\tau)\, d\tau + \int_0^t S(t-\tau) f(\tau)\, d\tau, \quad \textit{for } t \in I. \end{aligned} \tag{3.1}$$

PROOF. Suppose that $x \in PC^1(J)$ is a solution of problem $(\mathrm{FIP})^*$, $p(\tau) = C(t-\tau)x(\tau)$ and $q(\tau) = S(t-\tau)[x'(\tau) - g(\tau, x_\tau)]$ for fixed $t \in I$. Then $x(t) \in D(A)$. For $\tau \in I \setminus \{t_1, \ldots, t_m\}$, we have

$$p'(\tau) = -S(t-\tau)Ax(\tau) + C(t-\tau)x'(\tau); \tag{3.2}$$

$$q'(\tau) = -C(t-\tau)[x'(\tau) - g(\tau, x_\tau)] + S(t-\tau)\,\frac{d}{d\tau}\,[x'(\tau) - g(\tau, x_\tau)]. \tag{3.3}$$

For almost every $\tau \in I$, from (3.2) and (3.3), it follows that

$$\begin{aligned} p'(\tau) + q'(\tau) = {} & C(t-\tau) g(\tau, x_\tau) \\ & + S(t-\tau)\Big(- Ax(\tau) + \frac{d}{d\tau}\,[x'(\tau) - g(\tau, x_\tau)] \Big) \\ = {} & C(t-\tau) g(\tau, x_\tau) + S(t-\tau) f(\tau). \end{aligned} \tag{3.4}$$

Integrating equation (3.4), for $0 < t < t_1$, we have

$$\begin{aligned}\int_0^t C(t-\tau)g(\tau,x_\tau)\,d\tau + \int_0^t S(t-\tau)f(\tau)\,d\tau &= p(t)+q(t)-p(0)-q(0)\\ &= x(t)-C(t)[\phi(0)-h_1(x)]-S(t)[h_2(x)-g(0,\phi)].\end{aligned}$$

More generally, for $t_k < t < t_{k+1}$, we have

$$\begin{aligned}\int_0^t C(t-\tau)g(\tau,x_\tau)\,d\tau &+ \int_0^t S(t-\tau)f(\tau)\,d\tau\\ &= \sum_{j=1}^{k}\int_{t_{j-1}}^{t_j}[p'(\tau)+q'(\tau)]\,d\tau + \int_{t_k}^{t}[p'(\tau)+q'(\tau)]\,d\tau\\ &= x(t)-C(t)[\phi(0)-h_1(x)]-S(t)[h_2(x)-g(0,\phi)]\\ &\quad - \sum_{0<t_k<t}[C(t-t_k)u_k+S(t-t_k)v_k].\end{aligned}$$

i.e. (3.1) holds, which shows the lemma. □

Definition 3.2. A function $x \in PC^1(J)$ is said to be a C^1-*solution* of problem (FIP) if there exist $f \in L^1(I,X)$, $u_k \in \varphi_k(x(t_k))$ and $v_k \in \psi_k(x(t_k))$ $(k = 1,\ldots,m)$ such that $f(t) \in F(t,x_t)$ for almost every $t \in I$ and $x(t)$ is given by Lemma 3.1. Suppose that $x \in PC^1(J)$ and $f \in L^1(I,X)$. Let $\Gamma\colon L^1(I,X) \to PC^1(J)$ be a linear operator defined by

$$(\Gamma f)(t) = \begin{cases} 0 & \text{for } t \in I_0,\\ \displaystyle\int_0^t S(t-\tau)f(\tau)\,d\tau & \text{for } t \in I.\end{cases} \tag{3.5}$$

Let $\Lambda_0, \Lambda\colon PC^1(J) \to PC^1(J)$ be single valued mappings defined by

$$(\Lambda_0 x)(t) = \begin{cases} \phi(t)-h_1(x) & \text{for } t \in I_0,\\ C(t)[\phi(0)-h_1(x)]+S(t)[h_2(x)-g(0,\phi)] & \text{for } t \in I,\end{cases} \tag{3.6}$$

$$(\Lambda x)(t) = \begin{cases} 0 & \text{for } t \in I_0,\\ \displaystyle\int_0^t C(t-\tau)g(\tau,x_\tau)\,d\tau & \text{for } t \in I.\end{cases} \tag{3.7}$$

Let $\Psi\colon PC^1(J) \multimap PC^1(J)$ be a multivalued map defined by

$$\Psi(x) = \left\{\begin{aligned} &\eta \in PC^1(J):\\ &\qquad \eta(t) = \begin{cases} 0, & t \in I_0,\\ \displaystyle\sum_{0<t_k<t}[C(t-t_k)u_k+S(t-t_k)v_k], & t \in I,\end{cases}\\ &\qquad u_k \in \varphi_k(x(t_k)),\ v_k \in \psi_k(x(t_k)),\ k=1,\ldots,m.\end{aligned}\right\} \tag{3.8}$$

Let $F\colon I \times \Delta \multimap X$ be a multivalued map and $S_F^1(x) \neq \emptyset$ for all $x \in PC^1(J)$, where $S_F^1(x)$ is defined by (2.1). Now we define a multivalued map $T\colon PC^1(J) \multimap PC^1(J)$ by

$$(3.9)\quad T(x) = \{y \in PC^1(J) : y(t) = (\Lambda_0 x)(t) + (\Lambda x)(t) + \eta(t) + (\Gamma f)(t),\\ \eta \in \Psi(x),\ f \in S_F^1(x)\},$$

i.e. $T = \Lambda_0 + \Lambda + \Psi + \Gamma \circ S_F^1$.

It is clear that all C^1-solutions of problem (FIP) are fixed points of the multivalued map T in $PC^1(J)$.

Remark 3.3. The notion of C^1-solution is different from the one of mild solution in [10], [19], [16].

Lemma 3.4. *Suppose that $\varphi_k, \psi_k\colon X \multimap E$ are maps ($k = 1, \ldots, m$), $h_1, h_2\colon PC^1(J) \to E$ and $g\colon I \times \Delta \to E$ are mappings, $\phi(0) \in E$ and $S_F^1(x) \neq \emptyset$ for all $x \in PC^1(J)$. Then, for each $x \in PC^1(J)$, $T(x) \subset PC^1(J)$.*

Proof. Let $\Gamma, \Lambda_0, \Lambda, \Psi$ be defined by (3.5)–(3.8), respectively. Suppose that $x \in PC^1(J)$ and $f \in S_F^1(x)$. If $t \in I_0$, then it is clear that $\Lambda_0 x \in \Delta$ and $(\Lambda_0 x)'(t) = \phi'(t)$, everywhere except for a finite number of t. Suppose that $t \in I$. By the strong continuity of $S(t)$ and $C(t)$, we see that $\Gamma f, \Lambda_0(x), \Lambda(x) \in C(I, X)$. Suppose that

$$\overline{x}(t) = \begin{cases} x(t) & \text{if } t \in I_k, \\ x(t_{k-1}^+) & \text{if } t = t_{k-1}, \end{cases} \quad \text{and} \quad \overline{x}'(t) = \begin{cases} x'(t) & \text{if } t \in I_k, \\ x'(t_{k-1}^+) & \text{if } t = t_{k-1}. \end{cases}$$

It is clear that $x \in PC^1$ if and only if $\overline{x} \in C^1(\overline{I_k}, X)$ for $k = 1, \ldots, m+1$. Hence, from the strong continuity of $S(t)$ and $C(t)$, it is easy to check that $\eta(t)$ is continuous in each I_k and $\eta(t_{k-1}^+)$ exists, for each $\eta \in \Psi(x)$. Moreover, we have, for $t \in I$,

$$(3.10)\qquad (\Gamma f)'(t) = \int_0^t C(t-\tau) f(\tau)\, d\tau,$$

$$(3.11)\qquad (\Lambda_0 x)'(t) = AS(t)[\phi(0) - h_1(x)] + C(t)[h_2(x) - g(0, \phi)],$$

$$(3.12)\qquad (\Lambda x)'(t) = g(t, x_t) + \int_0^t AS(t-\tau) g(\tau, x_\tau)\, d\tau,$$

and for $t \in I \setminus \{t_1, \ldots, t_m\}$, $\eta \in \Phi(x)$,

$$(3.13)\quad \eta'(t) = \sum_{0 < t_k < t} [AS(t - t_k) u_k + C(t - t_k) v_k],\\ u_k \in \varphi_k(x(t_k)), \quad v_k \in \psi_k(x(t_k)), \quad k = 1, \ldots, m.$$

By Lemma 2.6 (b) (g), from the hypotheses it is easy to see that $(\Gamma f)'$, $(\Lambda_0 x)'$, $(\Lambda x)'$ in $C(I, X)$ and $\eta'(t)$ is continuous in each I_k and $\eta'(t_{k-1}^+)$ exists, for $\eta \in$

$\Psi(x)$. Hence, $\Gamma f, \Lambda_0 x, \Lambda x \in PC^1(J)$, and $\Psi(x) \subset PC^1(J)$. From (3.9) we see that $T(x) \subset PC^1(J)$. $\square$

Lemma 3.5. *Let Y be a metric space and X a Banach space. If $T_1, T_2 \colon Y \multimap X$ are all closed graph maps and T_1 is quasicompact, then $T_1 + T_2$ is a closed graph map.*

Proof. Suppose that $\{y_n\}_{n=1}^{\infty} \subset Y$, $y_n \to y_0$, $x_n \in (T_1+T_2)y_n$ and $x_n \to x_0$. Then, there exist $z_n \in T_1 y_n$ and $w_n \in T_2 y_n$ such that $x_n = z_n + w_n$ for all $n \in \mathbb{Z}^+$. Since $T_1\{y_n\}$ is relatively compact, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ which converges to z_0. Thus, $\{w_{n_k}\}$ converges to $x_0 - z_0$. Since $T_1, T_2 \colon Y \multimap X$ are all closed graph maps, we have $z_0 \in T_1 y_0$ and $x_0 - z_0 \in T_2 y_0$. This implies that $x_0 \in (T_1 + T_2)y_0$, and hence $T_1 + T_2$ is a closed graph map. $\square$

Lemma 3.6. *Let X_1, X_2 be two Banach spaces and the norm in $X_1 \times X_2$ be defined by*

$$\|(x_1, x_2)\| = \|x_1\| + \|x_2\|, \quad \text{for } (x_1, x_2) \in X_1 \times X_2.$$

Let $A_0 \colon X_1 \to X_2$ be a bounded linear operator.

(a) *If $B \subset X_1$ is bounded, then $\beta_H(A_0(B)) \leq |A_0|_* \beta_H(B)$.*
(b) *If $B_1 \subset X_1$ and $B_2 \subset X_2$ are bounded, then $\beta_H(B_1 \times B_2) \leq \beta_H(B_1) + \beta_H(B_2)$.*

The proof of Lemma 3.6 is easy, so we omit it.

Lemma 3.7. *Let J_* be a compact interval in $\mathbb{R}$, $B \subset C^1(J_*, X)$ and $t \in J_*$. Let $B(t), B(J_*), B'(t), B'(J_*)$ be subsets of X defined respectively by*

$$\begin{aligned} B(t) &= \{x(t) : x \in B\}, & B(J_*) &= \{x(t) : x \in B,\ t \in J_*\}, \\ B'(t) &= \{x'(t) : x \in B\}, & B'(J_*) &= \{x'(t) : x \in B,\ t \in J_*\}. \end{aligned}$$

If B is bounded in $C^1(J_*, X)$ and B' is equicontinuous in $C(J_*, X)$, then:

(a) *$\beta_H(B(J_*)) = \max_{t \in J_*} \beta_H(B(t))$ and $\beta_H(B'(J_*)) = \max_{t \in J_*} \beta_H(B'(t))$.*
(b) $\max\{\beta_H(B(J_*)), \beta_H(B'(J_*))\} \leq \beta_H(B) \leq \beta_H(B(J_*)) + \beta_H(B'(J_*))$.

Proof. Since B is bounded, there is $M > 0$ such that $\|x\|_{C^1} \leq M$ for all $x \in B$. This implies that $\|x(t)\| \leq M$ and $\|x'(t)\| \leq M$ for all $t \in J_*$. Thus, for $x \in B$ and $t_1, t_2 \in J_*$, we have

$$\|x(t_1) - x(t_2)\| \leq \sup_{s \in J_*} \|x'(s)\| |t_1 - t_2| \leq M|t_1 - t_2|,$$

which shows that B is also equicontinuous in $C(J_*, X)$. Suppose that $\varepsilon > 0$ is given and $J_* = [a_0, b_0]$. From equicontinuity it follows that there exist $\{s_j\}_{j=0}^N \subset J_*$, $a_0 = s_0 < s_1 < \ldots < s_{N-1} < s_N = b_0$ such that

$$\|x(t) - x(s)\| < \varepsilon, \qquad \|x'(t) - x'(s)\| < \varepsilon, \tag{3.14}$$

for all $x \in B$, all $s, t \in [s_{j-1}, s_j]$, $j = 1, \dots, N$.

(a) From the continuity of $\beta_H(B(t))$ it follows that $\max_{t \in J_*} \beta_H(B(t))$ exists. Since $B(t) \subset B(J_*)$ for all $t \in J_*$, we have $\max_{t \in J_*} \beta_H(B(t)) \le \beta_H(B(J_*))$. Suppose that $B(s_j)$ has $(\beta_H(B(s_j)) + \varepsilon)$-net $\{x_{ij}(s_j)\}_{i=1}^{K_j}$, $j = 0, \dots, N$. Then for each $x(t) \in B(J_*)$ there exist j, i such that $t \in [s_{j-1}, s_j]$ and $\|x(s_j) - x_{ij}(s_j)\| < \beta_H(B(s_j)) + \varepsilon$. Thus, from (3.14) we have

$$\begin{aligned}\|x(t) - x_{ij}(s_j)\| &\le \|x(t) - x(s_j)\| + \|x(s_j) - x_{ij}(s_j)\| \\ &\le \beta_H(B(s_j)) + 2\varepsilon \le \max_{t \in J_*} \beta_H(B(t)) + 2\varepsilon,\end{aligned}$$

which shows that $\beta_H(B(J_*)) \le \max_{t \in J_*} \beta_H(B(t))$. Hence $\beta_H(B(J_*)) = \max_{t \in J_*} \beta_H(B(t))$. Similarly, we have $\beta_H(B'(J_*)) = \max_{t \in J_*} \beta_H(B'(t))$.

(b) Suppose that B has $(\beta_H(B)+\varepsilon)$-net $\{z_i\}_{i=1}^{K}$. We consider the set $\{z_i(s_j) : i = 1, \dots, K; j = 0, \dots, N\}$. For $x(t) \in B(J_*)$, there exist j, i such that $t \in [s_{j-1}, s_j]$ and $\|x - z_i\|_{C^1} < \beta_H(B) + \varepsilon$. Thus, from (3.14) we have

$$\begin{aligned}\|x(t) - z_i(s_j)\| &\le \|x(t) - x(s_j)\| + \|x(s_j) - z_i(s_j)\| \\ &\le \|x(t) - x(s_j)\| + \|x - z_i\|_{C^1} \le \beta_H(B) + 2\varepsilon,\end{aligned}$$

which shows that $\beta_H(B(J_*)) \le \beta_H(B)$. Similarly, we have $\beta_H(B'(J_*)) \le \beta_H(B)$. Hence $\max\{\beta_H(B(J_*)), \beta_H(B'(J_*))\} \le \beta_H(B)$. Set

$$\begin{aligned}G(t) &= \{(x(t), x'(t)) : x \in B\}, \quad t \in J_*, \\ G(J_*) &= \{(x(t), x'(t)) : x \in B,\ t \in J_*\}.\end{aligned}$$

Since $G(J_*) \subset B(J_*) \times B'(J_*)$, using Lemma 3.6 (b) we have $\beta_H(G(J_*)) \le \beta_H(B(J_*)) + \beta_H(B'(J_*))$. To prove $\beta_H(B) \le \beta_H(G(J_*))$, we suppose that $G(s_j)$ has a $(\beta_H(G(s_j)) + \varepsilon)$-net $\{(x_{ij}(s_j), x'_{ij}(s_j))\}_{i=1}^{K_j}$, $j = 0, \dots, N$. Then for each $x \in B$ and $t \in J_*$ there exist j, i such that $t \in [s_{j-1}, s_j]$ and $\|(x(s_j), x'(s_j)) - (x_{ij}(s_j), x'_{ij}(s_j))\| < \beta_H(G(s_j)) + \varepsilon$. From (3.14) it follows that

$$\begin{aligned}\|x(t) - x_{ij}(t)\| &+ \|x'(t) - x'_{ij}(t)\| \\ &\le \|x(t) - x(s_j)\| + \|x(s_j) - x_{ij}(s_j)\| + \|x_{ij}(s_j) - x_{ij}(t)\| \\ &\quad + \|x'(t) - x'(s_j)\| + \|x'(s_j) - x'_{ij}(s_j)\| + \|x'_{ij}(s_j) - x'_{ij}(t)\| \\ &\le \|(x(s_j), x'(s_j)) - (x_{ij}(s_j), x'_{ij}(s_j))\| + 4\varepsilon \\ &< \beta_H(G(s_j)) + 5\varepsilon \le \beta_H(G(J_*)) + 5\varepsilon.\end{aligned}$$

Hence $\|x - x_{ij}\|_{C^1} \le \beta_H(G(J_*)) + 5\varepsilon$, and so $\beta_H(B) \le \beta_H(G(J_*))$, which is the desired inequality. □

LEMMA 3.8. *Let $F \colon I \times \Delta \to \mathcal{P}_{\mathrm{wcp,cv}}(X)$ be a map such that $t \mapsto F(t, x_t)$ is measurable and $u \mapsto F(t, u)$ is weakly u.s.c. and locally integrably bounded. Then*:

(a) *the map* $S^1_F \colon PC^1(J) \multimap L^1(I,X)$ *has nonempty, closed, convex values,*
(b) *if* $\Gamma \colon L^1(I,X) \to PC^1(J)$ *is a continuous linear operator, then* $\Gamma \circ S^1_F \colon PC^1(J) \multimap PC^1(J)$ *is a closed graph map.*

Proof. (a) Suppose that $x \in PC^1(J)$ and $\|x\|_* = \lambda$. Then $\|x(t)\| \le \lambda$ for all $t \in J$, and so $\|x_t\|_\Delta \le \lambda$ for all $t \in I$. Since $t \mapsto F(t,x_t)$ is measurable and $F(t,x_t)$ is closed, there exists a measurable mapping $f_0 \colon I \to X$ satisfying $f_0(t) \in F(t,x_t)$. Since F is locally integrably bounded, there exists $p_\lambda \in L^1(I,\mathbb{R}^+)$ such that $\|f_0(t)\| \le p_\lambda(t)$, for almost every $t \in I$. This implies that $f_0(t) \in L^1(I,X)$, and so $f_0 \in S^1_F(x)$. Hence, $S^1_F(x) \ne \emptyset$. By the convexity and closedness of $F(t,x_t)$ for $x \in PC^1(J)$ it is easy to check that $S^1_F(x)$ is convex and closed.

(b) Suppose that $\{x_n\}_{n=1}^\infty \subset PC^1(J)$, $x_n \to x_0$, $y_n = \Gamma(S^1_F(x_n))$ and $y_n \to y_0$. Then, for each $n \in \mathbb{Z}^+$, there exist $f_n \in S^1_F(x_n)$ and $\lambda > 0$ such that $y_n = \Gamma f_n$ and

$$\sup\{\|x_0\|_*, \|x_n\|_* : n \in \mathbb{Z}^+\} \le \lambda.$$

Let $x_{nt}(\theta) = x_n(t+\theta)$ and $x_{0t}(\theta) = x_0(t+\theta)$, for $\theta \in I_0$. Then $f_n(t) \in F(t,t_{nt})$ for almost every $t \in I$ and $\sup\{\|x_{nt}\|_\Delta, \|x_{0t}\|_\Delta : n \in \mathbb{Z}^+\} \le \lambda$ for each $t \in I$. Since $\|x_n(t) - x_0(t)\| \to 0$ and $\|x'_n(t) - x'_0(t)\| \to 0$ are valid uniformly on J, we have $x_{nt} \to x_{0t}$ in Δ for $t \in I$. Since $u \mapsto F(t,u)$ is weakly u.s.c. for almost every $t \in I$ and F has weakly compact convex values, there exists a subsequence of $\{f_n(t)\}$ which converges weakly to a point in $F(t,x_{0t})$ for fixed $t \in I$. This means that $\{f_n(t)\}$ is weakly relatively compact for almost every $t \in I$; and also, for fixed $t \in I$, there exist a subsequence $\{f_{n_k}(t)\}$ of $\{f_n(t)\}$ and a sequence $\{g_k(t)\} \subset F(t,x_{0t})$ such that $\{f_{n_k}(t) - g_k(t)\}$ converges weakly to 0. Since F is locally integrably bounded, there exists $p_\lambda \in L^1(I,\mathbb{R}^+)$ such that $\|f_n(t)\| \le p_\lambda(t)$. From this fact and Lemma 2.3 it follows that $\{f_n\}$ is weakly relatively compact in $L^1(I,X)$, and so is $\{f_{n_k}\}$. Without loss of generality we suppose that $\{f_{n_k}\}$ converges weakly to f_0 in $L^1(I,X)$, and so $f_0(t)$ is measurable and integrable. In view of Lemma 2.4, there exists a sequence $\{P_k\}_{k=1}^\infty$ with $P_k \in \operatorname{co}\{f_{n_i} : i \ge k\}$ such that $\{P_k(t)\}$ converges to $f_0(t)$ for almost every $t \in I$. Since $\{f_{n_k}(t) - g_k(t)\}$ converges weakly to 0, there exists a corresponding sequence $\{Q_k\}_{k=1}^\infty$ with $Q_k \in \operatorname{co}\{g_i : i \ge k\}$ such that $\{Q_k(t)\}$ converges weakly to $f_0(t)$. Hence, from the convexity and weak closedness of $F(t,x_{0t})$ it follows that $f_0(t) \in F(t,x_{0t})$ for almost every $t \in I$, and so $f_0 \in S^1_F(x_0)$. Suppose that Γ^* is the adjoint operator of Γ and x^* is any bounded linear functional on $PC^1(J)$. Then we have

$$x^*(\Gamma f_{n_k}) = (\Gamma^* x^*) f_{n_k} \to (\Gamma^* x^*) f_0 = x^*(\Gamma f_0),$$

which shows that $\{\Gamma f_{n_k}\}$ converges weakly to Γf_0 in $PC^1(J)$. Letting $k \to \infty$ in $y_{n_k} = \Gamma f_{n_k}$ under weak topology, we obtain $y_0 = \Gamma f_0$, which means that $\Gamma \circ S^1_F$ is a closed graph map. $\square$

4. The solution sets

We first give an existence result for problem (FIP) when A is not necessarily bounded.

Theorem 4.1. *Suppose that the following conditions are satisfied*:

(H0) *A is an infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$; $\{S(t) : t \in \mathbb{R}\}$ is a sine family associated to the cosine family*; $\phi(0) \in E$; *and $M_A = \sup_{t\in I} |AS(t)|_*$ in $B(E, X)$.*

(H1) *$F\colon I \times \Delta \to \mathcal{P}_{\mathrm{wcp,cv}}(E)$ is a map such that $t \mapsto F(t, x_t)$ is measurable and $u \mapsto F(t, u)$ is weakly u.s.c. and it is p_λ-locally integrably bounded, and there exists a function $\alpha \in L^1(I, \mathbb{R}^+)$ such that*

$$\beta_H(F(t, \mathcal{B})) \le \alpha(t)\beta_H(\mathcal{B}), \quad \textit{for each } \mathcal{B} \in \mathcal{P}_{\mathrm{bd}}(\Delta) \textit{ and a.e. } t \in I.$$

(H2) *The maps $\varphi_k, \psi_k\colon X \to \mathcal{P}_{\mathrm{cp,cv}}(E)$ are a_k, b_k-Lipschitz $(k = 1, \dots, m)$.*

(H3) *The mapping $h_i\colon PC^1(J) \to E$ is σ_i-Lipschitz, where $i = 1, 2$.*

(H4) *The mapping $g\colon I \times \Delta \to E$ satisfies that $u \mapsto g(t, u)$ is l-Lipschitz for almost every $t \in I$.*

If $\xi + \gamma_0 < 1$ and $\limsup_{\lambda\to+\infty} \|p_\lambda\|_L/\lambda < (1 - \xi)/M$, then the set of C^1-solutions of problem (FIP) *is a nonempty and compact set, where*

$$\begin{aligned}
M &= \max\{M_0e^{\omega a} + M_A, M_0e^{\omega a}(a + 1)\};\\
\xi &= l(Ma + 1) + M(\sigma_1 + \sigma_2) + M\sum_{k=1}^{m}(a_k + b_k);\\
\gamma_0 &= M\|\alpha\|_L.
\end{aligned}$$

Next we consider the multivalued maps $\Gamma \circ S^1_F, \Psi, T$ and the single valued mappings Λ_0, Λ defined by Definition 3.2, respectively. To prove the result, we need the following lemmas.

Lemma 4.2. *The mapping $\Gamma\colon L^1(I, X) \to PC^1(J)$ is a continuous linear operator.*

Proof. For $f \in L^1(I, X)$, from (3.5), (3.10) and Lemma 2.6 (c) we have

$$\begin{aligned}
\|\Gamma f\|_* &= \|\Gamma f\|_\diamond = \sup_{t\in I}[\|(\Gamma f)(t)\| + \|(\Gamma f)'(t)\|]\\
&= \sup_{t\in I}\bigg[\bigg\|\int_0^t S(t-\tau)f(\tau)\,d\tau\bigg\| + \bigg\|\int_0^t C(t-\tau)f(\tau)\,d\tau\bigg\|\bigg]\\
&\le M_0e^{\omega a}(a+1)\|f\|_L \le M\|f\|_L,
\end{aligned}$$

which shows that Γ is bounded, i.e. Γ is a continuous linear operator. $\square$

Lemma 4.3. *$S^1_F(x) \neq \emptyset$ for each $x \in PC^1(J)$, and $\Gamma \circ S^1_F\colon PC^1(J) \multimap PC^1(J)$ is a closed graph map with closed, convex values.*

Proof. From (H1) and Lemma 3.8 (a) we see that the map $S_F^1 \colon PC^1(J) \multimap L^1(I,X)$ has nonempty, closed, convex values. Hence the assertion immediately follows from (H1), Lemmas 4.2 and 3.8 (b). $\square$

Lemma 4.4. *$\beta_H(\Gamma \circ S_F^1(B)) \leq \gamma_0 \beta_H(B)$, for each bounded subset $B \in PC^1(J)$.*

Proof. For each $\varepsilon > 0$, B has a finite $(\beta_0+\varepsilon)$-net $\{z_1,\ldots,z_k\}$, where $\beta_0 = \beta_H(B)$. Setting $B_\tau = \{x_\tau : x \in B\}$ for each $\tau \in I$, we first show that $\{z_{1\tau},\ldots,z_{k\tau}\}$ is a $(\beta_0+\varepsilon)$-net of B_τ, where $z_{i\tau}$ is an element of Δ such that $z_{i\tau}(\theta) = z_i(\tau+\theta)$ for $\theta \in I_0$. In fact, if $x_\tau \in B_\tau$, then $x \in B$, and so there exists z_i $(1 \leq i \leq k)$ such that $\|x - z_i\|_* < \beta_0 + \varepsilon$. Thus, we have

$$\|x_\tau - z_{i\tau}\|_\Delta \leq \|x - z_i\|_* < \beta_0 + \varepsilon.$$

This implies that $\beta_H(B_\tau) \leq \beta_0$. Observe that

$$\{f(\tau) : f \in S_F^1(B)\} \subset \{F(\tau, x_\tau) : x \in B\} \subset F(\tau, B_\tau).$$

From (H1) and Lemma 3.6 (a), it follows that

$$\begin{aligned}\beta_H(\{S(t-\tau)f(\tau) : f \in S_F^1(B)\}) &\leq |S(t-\tau)|_* \beta_H(\{f(\tau) : f \in S_F^1(B)\}) \\ &\leq |S(t-\tau)|_* \beta_H(F(\tau, B_\tau)) \\ &\leq |S(t-\tau)|_* \alpha(\tau)\beta_H(B_\tau) \leq M_0 e^{\omega(t-\tau)} a\alpha(\tau)\beta_0;\end{aligned} \tag{4.1}$$

$$\beta_H(\{C(t-\tau)f(\tau) : f \in S_F^1(B)\}) \leq M_0 e^{\omega(t-\tau)}\alpha(\tau)\beta_0. \tag{4.2}$$

In order to prove that $\{(\Gamma f)' : f \in S_F^1(B)\}$ is equicontinuous, we suppose that $f \in S_F^1(B)$, $t, s \in I$ and $0 \leq s < t \leq a$. Since B is bounded, there exists $\lambda_0 > 0$ such that $\|x\|_* \leq \lambda_0$ for all $x \in B$. For each $\varepsilon > 0$, from the uniform continuity of $S(t)$ and the absolutely integral continuity of p_{λ_0}, we see that there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|S((t-s)/2)|_* < \varepsilon \quad \text{and} \quad \int_s^t p_{\lambda_0}(\tau)\,d\tau < \varepsilon,$$

when $0 < t - s < \delta$. Thus, by (2.3), we have

$$\begin{aligned}\|(\Gamma f)'(t) &- (\Gamma f)'(s)\| \\ &\leq \bigg\| \int_0^t [C(t-\tau) - C(s-\tau)]f(\tau)\,d\tau \bigg\| + \bigg\| \int_s^t C(s-\tau)f(\tau)\,d\tau \bigg\| \\ &\leq 2M_A|S((t-s)/2)|_* \|p_{\lambda_0}\|_L + M_0 e^{\omega a}\int_s^t p_{\lambda_0}(\tau)\,d\tau \leq M(2\|p_{\lambda_0}\|_L + 1)\varepsilon.\end{aligned}$$

This shows that $\{(\Gamma f)' : f \in S^1_F(B)\}$ is equicontinuous in J. Hence, according to Lemma 3.7, from (4.1) and (4.2), we have

$$\begin{aligned}\beta_H(\Gamma \circ S^1_F(B)) &\le \max_{t\in I} \beta_H\bigg(\int_0^t S(t-\tau)f(\tau)\,d\tau : f \in S^1_F(B)\bigg)\\ &\quad + \max_{t\in I} \beta_H\bigg(\int_0^t C(t-\tau)f(\tau)\,d\tau : f \in S^1_F(B)\bigg)\\ &\le M_0 e^{\omega a}(a+1)\beta_0 \int_0^a \alpha(\tau)\,d\tau \le M\|\alpha\|_L\beta_0 = \gamma_0\beta_H(B). \qquad \square\end{aligned}$$

LEMMA 4.5. *Ψ is a γ-Lipschitz map with compact and convex values, where $\gamma = M \sum_{k=1}^{m} (a_k + b_k)$.*

PROOF. Since φ_k, ψ_k have convex values, and $S(t), C(t)$ are linear, it is easy to check that Ψ has convex values.

Suppose that $x \in PC^1(J)$ and $\{\eta_n\}_{n=1}^{\infty} \subset \Psi(x)$. Then there exist $u_{nk} \in \varphi_k(x(t_k))$ and $v_{nk} \in \psi_k(x(t_k))$ such that

$$\eta_n(t) = \sum_{0<t_k<t} [C(t-t_k)u_{nk} + S(t-t_k)v_{nk}]. \tag{4.3}$$

Since $\varphi_k(x(t_k))$ and $\psi_k(x(t_k))$ $(1 \le k \le m)$ are compact, without loss of generality we suppose that $\{u_{nk}\}$ converges to $u_{0k} \in \varphi_k(x(t_k))$ and $\{v_{nk}\}$ converges to $v_{0k} \in \psi_k(x(t_k))$, $k = 1, \ldots, m$. From the boundedness of $C(t-t_k)$ and $S(t-t_k)$ it follows that $\{C(t-t_k)u_{nk}\}$ converges to $C(t-t_k)u_{0k}$ and $\{S(t-t_k)v_{nk}\}$ converges to $S(t-t_k)v_{0k}$ as $n \to \infty$. Set

$$\eta_0(t) = \sum_{0<t_k<t} [C(t-t_k)u_{0k} + S(t-t_k)v_{0k}].$$

Then $\eta_0 \in \Psi(x)$. Letting $n \to \infty$ in (4.3) we see that $\{\eta_n\}$ converges to η_0, which shows that $\Psi(x)$ is compact.

Let $x_1, x_2 \in PC^1(J)$, $x_1 \ne x_2$, and $\eta_1 \in \Psi(x_1)$. Then from (3.8) we see that there exist $u_{1k} \in \varphi_k(x_1(t_k))$ and $v_{1k} \in \psi_k(x_1(t_k))$ such that for $t \in I$,

$$\eta_1(t) = \sum_{0<t_k<t} [C(t-t_k)u_{1k} + S(t-t_k)v_{1k}].$$

Let $\varepsilon > 0$ be arbitrarily given. From (H2) it follows that

$$\begin{aligned}d(u_{1k}, \varphi_k(x_2(t_k))) &\le H(\varphi_k(x_1(t_k)), \varphi_k(x_2(t_k)))\\ &< (1+\varepsilon)a_k\|x_1(t_k) - x_2(t_k)\| \le (1+\varepsilon)a_k\|x_1 - x_2\|_*.\end{aligned}$$

Thus, there exist $u_{2k} \in \varphi_k(x_2(t_k))$ and $v_{2k} \in \psi_k(x_2(t_k))$ such that

$$\|u_{1k} - u_{2k}\|_E \le (1+\varepsilon)a_k\|x_1 - x_2\|_*; \quad \|v_{1k} - v_{2k}\|_E \le (1+\varepsilon)b_k\|x_1 - x_2\|_*. \tag{4.4}$$

Suppose that for each $t \in I$,

$$\eta_2(t) = \sum_{0<t_k<t} [C(t-t_k)u_{2k} + S(t-t_k)v_{2k}].$$

Then, $\eta_2 \in \Psi(x_2)$, from (3.8), (3.13), (4.4) and Lemma 2.6 (c) we have

$$\begin{aligned}\|\eta_1 - \eta_2\|_\diamond &\le (1+\varepsilon)\|x_1 - x_2\|_* \sum_{k=1}^{m} [(M_0 e^{\omega a} + M_A)a_k + M_0 e^{\omega a}(a+1)b_k] \\ &\le (1+\varepsilon)\gamma\|x_1 - x_2\|_*.\end{aligned} \tag{4.5}$$

Thus, from (4.5) it follows that $d(\eta_1, \Psi(x_2)) \le (1+\varepsilon)\gamma\|x_1 - x_2\|_*$. Since ε is arbitrary, we have $d(\eta_1, \Psi(x_2)) \le \gamma\|x_1 - x_2\|_*$, and so

$$\sup_{\eta_1 \in \Psi(x_1)} d(\eta_1, \Psi(x_2)) \le \gamma\|x_1 - x_2\|_*.$$

Similarly, we can show that

$$\sup_{\eta_2 \in \Psi(x_2)} d(\eta_2, \Psi(x_1)) \le \gamma\|x_1 - x_2\|_*.$$

Combining with the two inequalities, we have

$$H(\Psi(x_1), \Psi(x_2)) \le \gamma\|x_1 - x_2\|_*. \qquad \square$$

Lemma 4.6. *Λ is a γ_1-Lipschitz mapping, where $\gamma_1 = l(Ma+1)$.*

Proof. Let $x_1, x_2 \in PC^1(J)$, $x_1 \ne x_2$. Let $x_{1t}(\theta) = x_1(t+\theta)$, $x_{2t}(\theta) = x_2(t+\theta)$, for $\theta \in I_0$. Then, for $\tau \in I$, by (H4), we have

$$\|g(\tau, x_{1\tau}) - g(\tau, x_{2\tau})\|_E \le l\|x_{1\tau} - x_{2\tau}\|_\Delta \le l\|x_1 - x_2\|_*.$$

From (3.7) and (3.12) it follows that

$$\begin{aligned}\|\Lambda x_1 - \Lambda x_2\|_\diamond \le \sup_{t\in I} \bigg\{ &\int_0^t |C(t-\tau)|_* \|g(\tau, x_{1\tau}) - g(\tau, x_{2\tau})\|_E \, d\tau \\ &+ \|g(t, x_{1t}) - g(t, x_{2t})\|_E \\ &+ \int_0^t |AS(t-\tau)|_* \|g(\tau, x_{1\tau}) - g(\tau, x_{2\tau})\|_E \, d\tau \bigg\} \\ \le\, & l\|x_1 - x_2\|_*(M_0 e^{\omega a} a + 1 + M_A a) \le \gamma_1\|x_1 - x_2\|_*,\end{aligned}$$

and so $\|\Lambda x_1 - \Lambda x_2\|_* = \|\Lambda x_1 - \Lambda x_2\|_\diamond \le \gamma_1\|x_1 - x_2\|_*$. $\square$

Lemma 4.7. *Λ_0 is a γ_2-Lipschitz mapping, where $\gamma_2 = M(\sigma_1 + \sigma_2)$.*

Proof. Let $x_1, x_2 \in PC^1(J)$. From (H3) we have

$$\begin{aligned}\|\Lambda_0 x_1 - \Lambda_0 x_2\|_\Delta &= \sup\big\{\|(\Lambda_0 x_1)(t) - (\Lambda_0 x_2)(t)\| + \|(\Lambda_0 x_1)'(t) - (\Lambda_0 x_2)'(t)\| : t \in I_0\big\} \\ &= \sup\{\|h_1(x_2) - h_1(x_1)\|_E : t \in I_0\} \le \sigma_1\|x_1 - x_2\|_*;\end{aligned} \tag{4.6}$$

$$
\begin{aligned}
(4.7)\quad &\|\Lambda_0 x_1 - \Lambda_0 x_2\|_\diamond \\
&= \sup\big\{\|(\Lambda_0 x_1)(t) - (\Lambda_0 x_2)(t)\| + \|(\Lambda_0 x_1)'(t) - (\Lambda_0 x_2)'(t)\| : t \in I\big\} \\
&\le \sup\big\{|C(t)|_*\|h_1(x_2) - h_1(x_1)\|_E + |S(t)|_*\|h_2(x_1) - h_2(x_2)\|_E \\
&\quad + |AS(t)|_*\|h_1(x_2) - h_1(x_1)\|_E + |C(t)|_*\|h_2(x_1) - h_2(x_2)\|_E : t \in I\big\} \\
&\le (M_0 e^{\omega a} + M_A)\sigma_1\|x_1 - x_2\|_* + M_0 e^{\omega a}(1+a)\sigma_2\|x_1 - x_2\|_*.
\end{aligned}
$$

Inequalities (4.6) and (4.7) yield

$$\|\Lambda_0 x_1 - \Lambda_0 x_2\|_* = \max\{\|\Lambda_0 x_1 - \Lambda_0 x_2\|_\Delta, \|\Lambda_0 x_1 - \Lambda_0 x_2\|_\diamond\} \le \gamma_2\|x_1 - x_2\|_*. \quad \square$$

PROOF OF THEOREM 4.1. From the assumptions and Lemma 3.4 we see that $T(x) \subset PC^1(J)$ for $x \in PC^1(J)$. We will prove that T is an u.s.c. β_H-condensing map with compact and convex values. For $x_1, x_2 \in PC^1(J)$, in view of Lemmas 4.5 and 4.6, we have

$$
\begin{aligned}
(4.8)\quad H((\Lambda + \Psi)(x_1), (\Lambda + \Psi)(x_2)) &\le \|\Lambda(x_1) - \Lambda(x_2)\|_* + H(\Psi(x_1), \Psi(x_2)) \\
&\le (\gamma + \gamma_1)\|x_1 - x_2\|_*.
\end{aligned}
$$

Hence, $\Lambda + \Psi$ is $(\gamma + \gamma_1)$-Lipschitz continuous. Now we show that T is a β_H-condensing multivalued map. Suppose that B is a bounded subset of $PC^1(J)$. Note that $\beta_H((\Lambda + \Psi)(B)) \le (\gamma + \gamma_1)\beta_H(B)$ due to (4.8). Hence, from Lemmas 4.4 and 4.7, we have

$$
\begin{aligned}
(4.9)\qquad \beta_H(T(B)) &= \beta_H((\Lambda + \Psi + \Lambda_0 + \Gamma \circ S_F^1)(B)) \\
&\le \beta_H((\Lambda + \Psi)(B)) + \beta_H(\Lambda_0 B) + \beta_H(\Gamma \circ S_F^1(B)) \\
&\le (\xi + \gamma_0)\beta_H(B),
\end{aligned}
$$

which shows that T is a β_H-condensing map due to $\xi + \gamma_0 < 1$.

Since Ψ has compact and convex values, and $\Gamma \circ S_F^1$ has closed and convex values, we infer that $\Psi + \Gamma \circ S_F^1$ has closed and convex values, and so does T. For each $x \in PC^1(J)$, from (4.9) we have

$$\beta_H(T(x) \le (\xi + \gamma_0)\beta_H(\{x\}) = 0,$$

i.e. $T(x)$ is relatively compact. Hence T has compact and convex values.

Next, we show that T is u.s.c. In fact, from Lemma 4.5 we see that Ψ is a u.s.c. map with close values. Thus, Ψ is a closed graph map. From Lemma 4.3 we see that $\Gamma \circ S_F^1$ is also a closed graph map. Let B_* be a relatively compact subset of $PC^1(J)$. Then by (4.9), Lemmas 4.4 and 4.5, we have

$$
\begin{aligned}
\beta_H(T(B_*)) &\le (\xi + \gamma_0)\beta_H(B_*) = 0, \\
\beta_H(\Gamma \circ S_F^1(B_*)) &\le \gamma_0\beta_H(B_*) = 0, \\
\beta_H(\Psi(B_*)) &\le \gamma\beta_H(B_*) = 0.
\end{aligned}
$$

This shows that $\Gamma\circ S_F^1$, Ψ and T are quasicompact, and so is $\Psi+\Gamma\circ S_F^1$. Using Lemma 3.5, $\Psi+\Gamma\circ S_F^1$ has closed graph. Since the single-valued mapping $\Lambda+\Lambda_0$ is continuous due to (H3) and Lemma 4.6, $\Lambda+\Lambda_0$ has closed graph. Using Lemma 3.5 again, we deduce that $T=(\Psi+\Gamma\circ S_F^1)+(\Lambda+\Lambda_0)$ is a closed graph map. Thus, the upper semicontinuity of T follows from Lemma 2.1.

Suppose that C_0, C_1, C_2, C_* are four constants given by

$$\begin{aligned} C_0 &= \sup_{t\in I}\|g(t,0)\|_E;\\ C_1 &= M(\|\phi\|_\Delta+\|g(0,\phi)\|_E+\|h_1(0)\|_E+\|h_2(0)\|_E);\\ C_2 &= M\sum_{k=1}^{m}[H(0,\varphi_k(0))+H(0,\psi_k(0))];\\ C_* &= C_0(Ma+1)+C_1+C_2. \end{aligned}$$

Since $\xi<1$, $M\geq 1$ and $\limsup\limits_{\lambda\to+\infty}\|p_\lambda\|_L/\lambda<(1-\xi)/M$, we take a constant ν such that

$$\limsup_{\lambda\to+\infty}\frac{\|p_\lambda\|_L}{\lambda}<\nu<\frac{1-\xi}{M}.$$

Thus, there exists a constant λ_* such that

$$\lambda_*>\frac{C_*}{1-\xi-M\nu}\quad\text{and}\quad\frac{\|p_{\lambda_*}\|_L}{\lambda_*}<\nu.$$

Set $\mathcal{D}=\{x\in PC^1(J):\|x\|_*\leq\lambda_*\}$. Then $\mathcal{D}$ is a bounded closed convex subset of $PC^1(J)$. We claim that $T(\mathcal{D})\subset\mathcal{D}$. In fact, if $x\in\mathcal{D}$ be any element and $y\in T(x)$, then there exist $\eta_x\in\Psi(x)$ and $f_x\in S_F^1(x)$ such that $y=\Lambda_0x+\Lambda x+\eta_x+\Gamma f_x$. From (H3) it follows that

$$\|h_1(x)\|_E\leq\sigma_1\|x\|_*+\|h_1(0)\|_E,\qquad\|h_2(x)\|_E\leq\sigma_2\|x\|_*+\|h_2(0)\|_E.$$

Thus, we obtain

$$\begin{aligned} \|\Lambda_0x\|_\Delta &\leq\sup\{\|\phi(t)-h_1(x)\|+\|\phi'(t)\|:t\in I_0\}\leq\|\phi\|_\Delta+\sigma_1\|x\|_*+\|h_1(0)\|_E;\\ \|\Lambda_0x\|_\diamond &=\sup\{\|(\Lambda_0x)(t)\|+\|(\Lambda_0x)'(t)\|:t\in I\}\\ &\leq(M_0e^{\omega a}+M_A)[\|\phi\|_\Delta+\sigma_1\|x\|_*+\|h_1(0)\|_E]\\ &\quad+M_0e^{\omega a}(1+a)[\|g(0,\phi)\|_E+\sigma_2\|x\|_*+\|h_2(0)\|_E]; \end{aligned}$$

and so

$$\|\Lambda_0x\|_*=\max\{\|\Lambda_0x\|_\Delta,\|\Lambda_0x\|_\diamond\}\leq C_1+\gamma_2\|x\|_*\leq C_1+\gamma_2\lambda_*. \tag{4.10}$$

If $t\in I$, then from (H4) it follows that

$$\|g(t,x_t)\|_E\leq\|g(t,x_t)-g(t,0)\|_E+\|g(t,0)\|_E\leq l\|x_t\|_\Delta+C_0.$$

Thus, from (H4) and (H1) we have

$$\|(\Lambda x)(t)\|+\|(\Lambda x)'(t)\|\leq\int_0^t|C(t-\tau)|_*\|g(\tau,x_\tau)\|_E\,d\tau+\|g(t,x_t)\|_E$$

$$
\begin{aligned}
&+\int_0^t |AS(t-\tau)|_*\|g(\tau,x_\tau)\|_E\,d\tau \\
&\le C_0+l\|x_t\|_\Delta+(M_0e^{\omega a}+M_A)\int_0^t (C_0+l\|x_\tau\|_\Delta)\,d\tau \\
&\le C_0(Ma+1)+l\|x_t\|_\Delta+Ml\int_0^t \|x_\tau\|_\Delta\,d\tau,
\end{aligned}
$$

$$
\begin{aligned}
\|(\Gamma f_x)(t)\|+\|(\Gamma f_x)'(t)\| &\le \int_0^t \|S(t-\tau)f_x(\tau)\|d\tau+\int_0^t \|C(t-\tau)f_x(\tau)\|\,d\tau \\
&\le M_0e^{\omega a}(a+1)\int_0^t p_{\lambda_*}(\tau)d\tau \le M\int_0^t p_{\lambda_*}(\tau)\,d\tau,
\end{aligned}
$$

and so

$$
\|\Lambda x\|_*+\|\Gamma f_x\|_*=\|\Lambda x\|_\diamond+\|\Gamma f_x\|_\diamond\le C_0(Ma+1)+\gamma_1\lambda_*+M\|p_{\lambda_*}\|_L. \tag{4.11}
$$

For $u_k\in\varphi_k(x(t_k))$ and $v_k\in\psi_k(x(t_k))$, from (H2) it follows that

$$
\begin{aligned}
\|u_k\| &\le H(0,\varphi_k(x(t_k)))\le H(0,\varphi_k(0))+a_k\|x(t_k)\|,\\
\|v_k\| &\le H(0,\psi_k(x(t_k)))\le H(0,\psi_k(0))+b_k\|x(t_k)\|.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\|\eta_x\|_*=\|\eta_x\|_\diamond &\le \sup_{t\in I}\sum_{0<t_k<t}\big[|C(t-t_k)|_*\|u_k\|_E+|S(t-t_k)|_*\|v_k\|_E\big] \\
&\quad+\sup_{t\in I}\sum_{0<t_k<t}\big[|AS(t-t_k)|_*\|u_k\|_E+|C(t-t_k)|_*\|v_k\|_E\big] \\
&\le M\sum_{k=1}^m[H(0,\varphi_k(0))+a_k\|x(t_k)\|] \\
&\quad+M\sum_{k=1}^m[H(0,\psi_k(0))+b_k\|x(t_k)\|]\le C_2+\gamma\lambda_*.
\end{aligned} \tag{4.12}
$$

Combining with (4.10)–(4.12) we have

$$
\begin{aligned}
\|y\|_* &\le \|\Lambda_0x\|_*+\|\Lambda x\|_*+\|\Gamma f_x\|_*+\|\eta_x\|_* \\
&\le C_0(Ma+1)+C_1+C_2+(\gamma_1+\gamma_2+\gamma)\lambda_*+M\|p_{\lambda_*}\|_L \\
&= C_*+\xi\lambda_*+M\|p_{\lambda_*}\|_L \\
&< (1-\xi-M\nu)\lambda_*+\xi\lambda_*+M\nu\lambda_*=\lambda_*,
\end{aligned}
$$

which means that $T(\mathcal{D})\subset\mathcal{D}$.

As a consequence of Lemma 2.2 we deduce that $\mathrm{Fix}(T)$ is a nonempty and compact set. Therefore, the set of C^1-solutions of problem (FIP) is a nonempty and compact set. This completes the proof. □

If A is bounded, then we can obtain a existence result for problem (FIP) under some weak impulsive conditions and nonlocal conditions.

Theorem 4.8. *Suppose that the following conditions are satisfied*:

(h0) *A is a bounded infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$; $\{S(t) : t \in \mathbb{R}\}$ is a sine family associated to the cosine family.*

(h1) *The map $F\colon I \times \Delta \to \mathcal{P}_{\mathrm{wcp,cv}}(X)$ is a map such that $t \mapsto F(t, x_t)$ is measurable and $u \mapsto F(t, u)$ is weakly u.s.c. and it is p_λ-locally integrably bounded, and there exists a function $\alpha \in L^1(I, \mathbb{R}^+)$ such that*

$$\beta_H(F(t,\mathcal{B})) \le \alpha(t)\beta_H(\mathcal{B}), \quad \textit{for each } \mathcal{B} \in \mathcal{P}_{\mathrm{bd}}(\Delta) \textit{ and a.e. } t \in I.$$

(h2) *For $k = 1, \ldots, m$, the maps $\varphi_k, \psi_k\colon X \to \mathcal{P}_{\mathrm{cp,cv}}(X)$ are u.s.c.; $\varphi_k(X)$, $\psi_k(X) \in \mathcal{P}_{\mathrm{bd}}(X)$; and there exist nonnegative constants a_k, b_k such that $\beta_H(\varphi_k(D)) \le a_k\beta_H(D)$ and $\beta_H(\psi_k(D)) \le b_k\beta_H(D)$ for each $D \in \mathcal{P}_{\mathrm{bd}}(X)$.*

(h3) *The mappings $h_1, h_2\colon PC^1(J) \to X$ are continuous and there exist nonnegative constants σ_i, d_i such that $\beta_H(h_i(D)) \le \sigma_i\beta_H(D)$ for each bounded set $D \subset PC^1(J)$, and $\|h_i(x)\| \le d_i$ for each $x \in PC^1(J)$, where $i = 1, 2$.*

(h4) *The mapping $g\colon I \times \Delta \to X$ satisfies that $u \mapsto g(t, u)$ is l-Lipschitz for almost every $t \in I$.*

(h5) *ϕ' is continuous in I_0.*

If $\gamma_1 + \mu < 1$ and $\limsup_{\lambda\to+\infty} \|p_\lambda\|_L/\lambda < (1 - \gamma_1)/M$, then the set of C^1-solutions of problem (FIP) *is a nonempty and compact set, where*

$$\begin{aligned} M &= M_0 e^{\omega a}[a\max(|A|_*, 1) + 1]; \\ \gamma_1 &= l(Ma + 1); \\ \mu &= M\|\alpha\|_L + M(\sigma_1 + \sigma_2) + M\sum_{k=1}^{m}(a_k + b_k). \end{aligned}$$

To prove Theorem 4.8, we need the following lemmas. Since conditions (h1) and (H1) are identical, from Lemmas 4.3 and 4.4 we have the following Lemmas 4.9 and 4.10.

Lemma 4.9. *$\Gamma \circ S_F^1 : PC^1(J) \multimap PC^1(J)$ is a closed graph map with closed, convex values.*

Lemma 4.10. *$\beta_H(\Gamma \circ S_F^1(B)) \le \gamma_0\beta_H(B)$, for each bounded subset $B \in PC^1(J)$, where $\gamma_0 = M\|\alpha\|_L$.*

Since conditions (h4) and (H4) are identical, from Lemma 4.6 we have the following assertion.

Lemma 4.11. *Λ is a γ_1-Lipschitz mapping, where $\gamma_1 = l(Ma + 1)$.*

Lemma 4.12. *$\Psi(B)'$ is equicontinuous in I_{k+1}, where B is a bounded subset of $PC^1(J)$ and $k = 1, \ldots, m$.*

PROOF. Suppose that $t, s \in I_{k+1}$ and $t_k < s < t \leq t_{k+1}$. Since $\varphi_i(B), \psi_i(B)$ are all bounded by (h2), there exists $M_* > 0$ such that $\|u_i\| \leq M_*$ and $\|v_i\| \leq M_*$ for all $u_i \in \varphi_i(B)$, $v_i \in \psi_i(B)$, where $i = 1, \ldots, m$. For each $\varepsilon > 0$, from the uniform continuity of $S(t)$, we see that there exists $\delta = \delta(\varepsilon)$, $0 < \delta < \min\limits_{0 \leq k \leq m}(t_{k+1} - t_k)$ such that $|S((t-s)/2)|_* < \varepsilon$, when $0 < t - s < \delta$. Thus, by (2.2) and (2.3) we have, for $i = 1, \ldots, k$,

$$\begin{aligned} |C(t-t_i) - C(s-t_i)|_* &< 2a|A|_* M_0 e^{\omega a}\varepsilon, \\ |AS(t-t_i) - AS(s-t_i)|_* &< 2|A|_* M_0 e^{\omega a}\varepsilon. \end{aligned}$$

Hence, from (3.13), it follows that, for each $\eta \in \Psi(B)$,

$$\begin{aligned} &\|\eta'(t) - \eta'(s)\|_* \\ &\quad \leq \sum_{i=1}^{k} |AS(t-t_i) - AS(s-t_i)|_* \|u_i\| + |C(t-t_i) - C(s-t_i)|_* \|v_i\| \\ &\qquad\qquad \leq 2mM_*(a+1)|A|_* M_0 e^{\omega a}\varepsilon. \end{aligned}$$

This shows that $\Psi(B)'$ is equicontinuous in I_{k+1}. □

LEMMA 4.13. *Ψ is a closed graph map with compact and convex values and $\beta_H(\Psi(B)) \leq \gamma\beta_H(B)$, where B is a bounded subset of $PC^1(J)$ and*

$$\gamma = M\sum_{k=1}^{m}(a_k + b_k).$$

PROOF. Since the maps $\varphi_k, \psi_k\colon X \multimap X$ have all compact and convex values ($k = 1, \ldots, m$), in the same manner as Lemma 4.5, we can show that Ψ has compact and convex values. Since φ_k, ψ_k are u.s.c., they have closed graph. But (h2) implies that φ_k, ψ_k are quasicompact. Using Lemma 3.5 we deduce that Ψ is a closed graph map. Let B is a bounded subset of $PC^1(J)$. From (h2) and Lemma 3.7, it follows that

$$\begin{aligned} \beta_H\{\varphi_k(x(t_k)) : x \in B\} &\leq a_k\beta_H\{x(t_k) : x \in B\} \\ &\leq a_k\beta_H(B(I)) \leq a_k\beta_H(B); \end{aligned} \tag{4.13}$$

$$\begin{aligned} \beta_H\{\psi_k(x(t_k)) : x \in B\} &\leq b_k\beta_H\{x(t_k) : x \in B\} \\ &\leq b_k\beta_H(B(I)) \leq b_k\beta_H(B). \end{aligned} \tag{4.14}$$

Thus, from Lemma 4.12, inequalities (4.13), (4.14) and Lemmas 3.7 and 3.6 (a) we have

$$\begin{aligned} \beta_H(\Psi(B)) \leq \sup_{t\in I} &\sum_{0<t_k<t} \big[|C(t-t_k)|_* a_k\beta_H(B) + |S(t-t_k)|_* b_k\beta_H(B)\big] \\ &+ \sup_{t\in I}\sum_{0<t_k<t} \big[|AS(t-t_k)|_* a_k\beta_H(B) + |C(t-t_k)|_* b_k\beta_H(B)\big] \end{aligned}$$

$$\leq \beta_H(B)\sum_{k=1}^{m}\big[M_0e^{\omega a}(1+a|A|_*)a_k+M_0e^{\omega a}(a+1)b_k\big]\leq \gamma\beta_H(B). \ \square$$

Lemma 4.14. *$\beta_H(\Lambda_0(B))\leq\gamma_2\beta_H(B)$, where B is a bounded subset of $PC^1(J)$ and $\gamma_2=M(\sigma_1+\sigma_2)$.*

Proof. From (h3) and Lemma 3.6 (a) we have, for $t\in I_0$,

$$\begin{aligned}\beta_H(\{(\Lambda_0x)(t):x\in B\})&=\beta_H(\phi(t)-h_1(B))\leq\sigma_1\beta_H(B);\\ \beta_H(\{(\Lambda_0x)'(t):x\in B\})&=0;\end{aligned}\tag{4.15}$$

and for $t\in I$,

$$\begin{aligned}\beta_H(\{(\Lambda_0x)(t):x\in B\})&\\ \leq\beta_H(C(t)[\phi(0)-h_1(B)]+S(t)[h_2(B)-g(0,\phi)])&\\ \leq(|C(t)|_*\sigma_1+|S(t)|_*\sigma_2)\beta_H(B);&\end{aligned}\tag{4.16}$$

$$\begin{aligned}\beta_H(\{(\Lambda_0x)'(t):x\in B\})&\\ \leq\beta_H(\{AS(t)[\phi(0)-h_1(B)]+C(t)[h_2(B)-g(0,\phi)])&\\ \leq(|AS(t)|_*\sigma_1+|C(t)|_*\sigma_2)\beta_H(B).&\end{aligned}\tag{4.17}$$

On the other hand, for $t,s\in I_0$ and $x\in B$, we obtain

$$\|(\Lambda_0x)'(t)-(\Lambda_0x)'(s)\|=\|\varphi'(t)-\varphi'(s)\|;$$

for $t,s\in I$ and $x\in B$, from (2.2) and (2.3), we obtain

$$\begin{aligned}\|(\Lambda_0x)'(t)&-(\Lambda_0x)'(s)\|\\ &\leq|AS(t)-AS(s)|_*\|\phi(0)-h_1(x)\|+|C(t)-C(s)|_*\|h_2(x)-g(0,\phi)\|\\ &\leq2\max(1,a)|A|_*M_0e^{\omega a}(\|\phi\|_\Delta+d_1+d_2+\|g(0,\phi)\|)|S((t-r)/2)|_*.\end{aligned}$$

This implies that $\{(\Lambda_0x)':x\in B\}$ is equicontinuous by (h5) and the uniform continuity of $S(t)$. Thus, from (4.15)–(4.17) and Lemma 3.7, it follows that

$$\begin{aligned}\beta_H(\Lambda_0B)&\leq\max_{t\in J}\beta_H(\{(\Lambda_0x)(t):x\in B\})+\max_{t\in J}\beta_H(\{(\Lambda_0x)'(t):x\in B\})\\ &\leq[M_0e^{\omega a}(1+a|A|_*)\sigma_1+M_0e^{\omega a}(1+a)\sigma_2]\beta_H(B)\\ &\leq M(\sigma_1+\sigma_2)\beta_H(B)=\gamma_2\beta_H(B).\end{aligned}\qquad\square$$

Proof of Theorem 4.8. From Lemma 3.4 we see that $T(x)\subset PC^1(J)$ for $x\in PC^1(J)$. We will prove that T is a u.s.c. β_H-condensing map with compact and convex values. Suppose that B is a bounded subset of $PC^1(J)$. Note that $\beta_H((\Lambda+\Lambda_0)(B))\leq(\gamma_1+\gamma_2)\beta_H(B)$ due to Lemmas 4.11 and 4.14. Hence, from

Lemmas 4.10 and 4.13 we have

$$\begin{aligned}\beta_H(T(B)) &= \beta_H((\Lambda+\Psi+\Lambda_0+\Gamma\circ S_F^1)(B))\\ &\le \beta_H((\Lambda+\Lambda_0)(B))+\beta_H(\Psi(B))+\beta_H(\Gamma\circ S_F^1(B))\\ &\le (\gamma_1+\gamma_2+\gamma+\gamma_0)\beta_H(B)=(\gamma_1+\mu)\beta_H(B),\end{aligned}$$

which shows that T is a β_H-condensing map due to $\gamma_1+\mu<1$. In the same manner as the proof of Theorem 4.1, from Lemmas 4.9, 4.13, 3.5 and 2.1 we can show that T is a u.s.c. map with compact and convex values.

Suppose that C_0, G_1, G_2, G_0 are four constants given by

$$\begin{aligned}C_0 &= \sup_{t\in I}\|g(t,0)\|;\\ G_1 &= M(\|\phi\|_\Delta+d_1+d_2+\|g(0,\phi)\|);\\ G_2 &= 2mM\sup\Big\{\|y\| : y\in\bigcup_{k=1}^m[\varphi_k(X)\cup\psi_k(X)]\Big\};\\ G_0 &= C_0(Ma+1)+G_1+G_2.\end{aligned}$$

Since $\gamma_1<1$, $M\ge 1$ and $\limsup_{\lambda\to+\infty}\|p_\lambda\|_L/\lambda<(1-\gamma_1)/M$, we take a constant ρ such that

$$\limsup_{\lambda\to+\infty}\frac{\|p_\lambda\|_L}{\lambda}<\rho<\frac{1-\gamma_1}{M}.$$

Thus, there exists a constant λ_0 such that $\lambda_0>G_0/(1-\gamma_1-M\rho)$, $\|p_{\lambda_0}\|_L/\lambda_0<\rho$. Set

$$\mathcal{D}=\{x\in PC^1(J):\|x\|_*\le\lambda_0\}.$$

Then $\mathcal{D}$ is a bounded closed convex subset of $PC^1(J)$. Next we prove that $T(\mathcal{D})\subset\mathcal{D}$. Let $x\in\mathcal{D}$ be any element and $y\in T(x)$. Then there exist $\eta_x\in\Psi(x)$ and $f_x\in S_F^1(x)$ such that $y=\Lambda_0x+\Lambda x+\eta_x+\Gamma f_x$. By (h3) we obtain

$$\begin{aligned}\|\Lambda_0x\|_\Delta &\le \sup\{\|\phi(t)-h_1(x)\|+\|\phi'(t)\| : t\in I_0\}\\ &\le \sup\{\|\phi(t)\|+\|\phi'(t)\|+\|h_1(x)\| : t\in I_0\}\le\|\phi\|_\Delta+d_1,\\ \|\Lambda_0x\|_\diamond &= \sup\{\|(\Lambda_0x)(t)\|+\|(\Lambda_0x)'(t)\| : t\in I\}\\ &\le M_0e^{\omega a}(1+a|A|_*)(\|\phi\|_\Delta+d_1)+M_0e^{\omega a}(1+a)(d_2+\|g(0,\phi)\|);\end{aligned}$$

and so

$$\|\Lambda_0x\|_*=\max\{\|\Lambda_0x\|_\Delta,\|\Lambda_0x\|_\diamond\}\le G_1. \tag{4.18}$$

Since conditions (h1) and (H1) are identical, (h4) and (H4) are identical, from (4.11), we have

$$\|\Lambda x\|_*+\|\Gamma f_x\|_*=\|\Lambda x\|_\diamond+\|\Gamma f_x\|_\diamond\le C_0(Ma+1)+\gamma_1\lambda_0+M\|p_{\lambda_0}\|_L. \tag{4.19}$$

From (h2), we have

$$
\begin{aligned}
(4.20)\qquad \|\eta_x\|_* = \|\eta_x\|_\diamond &= \sup_{t\in I}[\|\eta_x(t)\| + \|\eta_x'(t)\|] \\
&\leq \sup_{t\in I}\sum_{0<t_k<t}[|C(t-t_k)|_*\|u_k\| + |S(t-t_k)|_*\|v_k\|] \\
&\quad + \sup_{t\in I}\sum_{0<t_k<t}[|AS(t-t_k)|_*\|u_k\| + |C(t-t_k)|_*\|v_k\|] \leq G_2.
\end{aligned}
$$

Combining with (4.18)–(4.20), we have

$$
\begin{aligned}
\|y\|_* &\leq \|\Lambda_0 x\|_* + \|\Lambda x\|_* + \|\Gamma f_x\|_* + \|\eta_x\|_* \\
&\leq C_0(Ma+1) + G_1 + G_2 + \gamma_1\lambda_0 + M\|p_{\lambda_0}\|_L = G_0 + \gamma_1\lambda_0 + M\|p_{\lambda_0}\|_L \\
&< (1-\gamma_1 - M\rho)\lambda_0 + \gamma_1\lambda_0 + M\rho\lambda_0 = \lambda_0,
\end{aligned}
$$

which means that $T(\mathcal{D}) \subset \mathcal{D}$.

Using Lemma 2.2 we deduce that $\mathrm{Fix}(T)$ is a nonempty and compact set. Hence, the set of C^1-solutions of problem (FIP) is a nonempty and compact set. $\square$

Example 4.15. As an application of our result, we consider the impulsive neutral partial differential inclusion of the following form:

$$
(\mathrm{P})\qquad
\begin{cases}
\dfrac{\partial^2}{\partial t^2}\, y(t,s) - \dfrac{\partial}{\partial t}\, g(t, y(t-r,s)) - \dfrac{\partial^2}{\partial s^2}\, y(t,s) \in F(t, y(t-r,s)) & \\
 & \text{a.e. } t \in I \setminus \{t_1,\ldots,t_m\}, \\
y(t,0) = y(t,\pi) = 0, & t \in I, \\
y(t_k^+, s) - y(t_k^-, s) \in \varphi_k(y(t_k^-, s)), & k = 1,\ldots,m, \\
\dfrac{\partial}{\partial t}\, y(t_k^+, s) - \dfrac{\partial}{\partial t}\, y(t_k^-, s) \in \psi_k(y(t_k^-, s)), & k = 1,\ldots,m, \\
y(t,s) + h_1(y(0,s)) = \phi(t,s), & t \in I_0, \\
\dfrac{\partial}{\partial t}\, y(0,s) = h_2(y(0,s)), & t \in I_0,
\end{cases}
$$

where $s \in [0,\pi]$. Let $X = L^2[0,\pi]$, $\phi(t,\,\cdot\,) = \phi(t)(\,\cdot\,)$ and $y(t,\,\cdot\,) = x(t)$. Then we have $x(t) \in X$. Define $A\colon D(A) \to X$ by $Ax = x''$ with the domain

$$
\begin{aligned}
D(A) = \big\{x \in X : x \text{ and } x' &\text{ are absolutely continuous,} \\
&x'' \in X \text{ and } x(0) = x(\pi) = 0\big\},
\end{aligned}
$$

then $\frac{\partial^2}{\partial s^2}\, y(t,s) = Ax(t)$, and it is well known that (see [9, 13] for more details)

$$
E = \big\{x \in X : x \text{ are absolutely continuous, } x' \in X \text{ and } x(0) = x(\pi) = 0\big\}.
$$

Thus, problem (FIP) is an abstract formulation of problem (P). From Theorem 4.1 we can establish the topological structure of C^1-solution sets for problem (P).

Acknowledgments. The authors are grateful to the referees for their valuable comments and suggestions which helped to improve the manuscript.

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Manuscript received November 28, 2015

accepted September 24, 2016

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