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MULTI-BUMP SOLUTIONS FOR SINGULARLY PERTURBED SCHRÖDINGER EQUATIONS IN $\mathbb{R}^2$ WITH GENERAL NONLINEARITIES

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Abstract. We are concerned with the following equation:

$$-\varepsilon^2\Delta u + V(x)u = f(u), \quad u(x) > 0 \quad \text{in } \mathbb{R}^2.$$

By a variational approach, we construct a solution u_ε which concentrates, as $\varepsilon \to 0$, around arbitrarily given isolated local minima of the confining potential V: here the nonlinearity f has a quite general Moser's critical growth, as in particular we do not require the *monotonicity* of $f(s)/s$ nor the *Ambrosetti–Rabinowitz* condition.

1. Introduction

We are concerned with the existence of positive solutions to the ε-perturbed Schrödinger equation

$$-\varepsilon^2\Delta u + V(x)u = f(u), \quad u > 0, \quad x \in \mathbb{R}^2, \tag{1.1}$$

where $\varepsilon > 0$ and $V \in C(\mathbb{R}^2, \mathbb{R})$. In the past decades, a lot of literature has been devoted to bound states of (1.1) in $\mathbb{R}^N$. From the physical point of view, these solutions represent semi-classical states for small $\varepsilon > 0$, living on the interface between classical and quantum mechanics: for the physics aspects and related

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topics we refer to [4], [7], [14]–[16], [26], [28], [39]–[41], and references therein. In the pioneering work [33], Floer and Weinstein considered problem (1.1) in dimension one and $f(s) = s^3$ and constructed a single-peak solution around any given non-degenerate critical point of V. Motivated by [33], Oh [42] obtained a similar result in the higher dimensional case. A key ingredient of [33] and [42] is a reduction method and a non-degeneracy condition for ground states to the limiting problem with constant potential. To overcome non-degeneracy conditions, Rabinowitz [43] exploited the variational approach which has become an important tool in studying semiclassical states of (1.1). In more recent years, there have been further developments to cover more general nonlinearities, see [48], [24]–[27]. In [24], Del Pino and Felmer used a penalization technique to construct a single-peak solution around a local minimum point of V, with some restrictions on the nonlinearity such as the monotonicity of $f(t)/t$ which is required to be nondecreasing in $(0,\infty)$ as well as the Ambrosetti–Rabinowitz condition. More recently, Byeon and Jeanjean [8] introduced a new penalization approach to show that the Berestycki–Lions conditions, see [5], are almost optimal to get spike solutions around the local minima of V. Closely related results can be found in [12], [13], [22], [49]. In [20], with the Berestycki–Lions conditions, Cingolani, Jeanjean and Tanaka considered the multiplicity of solutions to (1.1) concentrating around the local minima of V in $\mathbb{R}^N$ for $N \geq 3$. Moreover, the authors established the number of solutions related to the topology of the set of minima of V. An interesting class of solutions to (1.1) are semi-classical states which have a spike shape concentrated around some point in $\mathbb{R}^2$, as $\varepsilon \to 0$. In this paper, we focus on localized bound states of (1.1), namely solutions which develop multi bumps around the local minima of V. In the sequel, we assume that V satisfies the following assumptions:

- (V1) $\inf\limits_{x\in\mathbb{R}^2} V(x) = V_0 > 0$;
- (V2) there exist k bounded disjoint open sets O^i, $i = 1,\ldots,k$, such that

$$0 < m_i = \inf_{x\in O^i} V(x) < \min_{x\in\partial O^i} V(x), \quad i = 1,\ldots,k.$$

In 2008, Byeon, Jeanjean and Tanaka [11] constructed a single-spike solution of (1.1) exploiting the Berestycki–Lions conditions. Precisely, the authors assumed $k = 1$ and $f \in C(\mathbb{R}^+,\mathbb{R}^+)$ satisfies:

- (f_1) $\lim\limits_{t\to 0} f(t)/t = 0$;
- (f_2) for any $\alpha > 0$, there exists $C_\alpha > 0$ such that $|f(t)| \leq C_\alpha \exp(\alpha t^2)$ for $t \geq 0$;
- (f_3) there exists $T > 0$ such that $T^2 m < 2F(T)$, where $m = m_1$ and $F(s) := \int_0^s f(t)\,dt$.