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NONAUTONOMOUS SUPERPOSITION OPERATORS IN THE SPACES OF FUNCTIONS OF BOUNDED VARIATION

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Abstract. The main goal of this paper is to give an answer to the main problem of the theory of nonautonomous superposition operators acting in the space of functions of bounded variation in the sense of Jordan. Namely, we give necessary and sufficient conditions which guarantee that nonautonomous superposition operators map that space into itself and are locally bounded. Moreover, special attention is drawn to nonautonomous superposition operators generated by locally bounded mappings as well as to superposition operators generated by functions with separable variables.

1. Introduction

The notion of a function of bounded variation is one of the basic notions of mathematical analysis. It was introduced by Camille Jordan (see [14]) in connection with his investigation on Fourier series.

It might seem that the problem of characterization of acting conditions for nonlinear superposition operators in the spaces of functions of bounded variation ([1]) has long been solved. It has been indeed, but only in the case of autonomous superposition operators (see [15]); the problem of stating necessary

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([1]) In this paper we will be interested only in bounded variation in the sense of Jordan, and therefore we will refer to it simply as 'bounded variation'.

and sufficient conditions for the nonautonomous superposition operator to map the space of functions of bounded variation into itself is still open.

In the monograph [4], on page 175 the authors write: "*As already mentioned, no general results on the acting, boundedness, or continuity of the superposition operator F are known in the nonautonomous case $f = f(t,u)$ (apart from trivial sufficient conditions, of course).*"

On page 174 of that monograph the authors quote and prove the following result coming originally from Ljamin's paper [19].

Theorem 1.1. *Assume that the function $f(t,\,\cdot\,)$ satisfies the Lipschitz condition on $\mathbb{R}$ uniformly in $t \in [0,1]$, and that the function $f(\,\cdot\,,u)$ is of bounded variation on the interval $[0,1]$ uniformly in $u \in \mathbb{R}$. Then the nonautonomous superposition operator F, generated by f, maps the space $\mathrm{BV}[0,1]$ into itself and is locally bounded, that is, it maps bounded sets into bounded ones.*

(In the above theorem $\mathrm{BV}[0,1]$ denotes the Banach space of all functions $x\colon [0,1] \to \mathbb{R}$ of bounded variation endowed with the norm $\|x\|_{\mathrm{BV}} = |x(0)| + \bigvee_0^1 x$; for more details see Section 2.)

In the paper [6], Bugajewska formulated the conjecture that Theorem 1.1 might not be true. Let us also add that the proof of Ljamin's theorem presented in the survey article [3] is false. One can find the suitable examples confirming its falsity in the review by Bugajewski for ZblMATH (Zbl 1255.47059). The conjecture from the paper [6] was confirmed by Maćkowiak (see [22]) who presented the following counterexample to Theorem 1.1.

Example 1.2 ([22]). Let the function $f\colon [0,1]\times\mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(t,u) = \begin{cases} 0 & \forall\, n \in \{2,3,\ldots\} : t \neq c_n \text{ or } u \notin I_n, \\ \dfrac{1}{n}\left(1 - \dfrac{|u-c_n|}{w_n}\right) & \exists\, n \in \{2,3,\ldots\} : t = c_n \text{ and } u \in I_n, \end{cases}$$

where $c_n = 1 - 1/n$, $w_n = 1/(2n)$ and $I_n = (c_n - w_n, c_n + w_n)$ for $n = 2,3,\ldots$ For an arbitrary $t \in [0,1]$, the function $f(t,\,\cdot\,)$ satisfies the Lipschitz condition (uniformly in the second variable) with a Lipschitz constant not greater than 2. Moreover, $\bigvee_0^1 f(\,\cdot\,,u) \leq 22$ for an arbitrary $u \in \mathbb{R}$. However, considering the functions $x(t) = t$ and $g(t) = f(t,x(t))$ for $t \in [0,1]$, one can easily be convinced that the nonautonomous superposition operator, generated by the function f, does not map the space $\mathrm{BV}[0,1]$ into itself.

In the introduction to the recently published monograph [2, p. 6], the authors stated three fundamental open problems of the theory of nonlinear superposition operators in the space of functions of bounded variation. The first problem