# ASYMPTOTICALLY ALMOST PERIODIC MOTIONS IN IMPULSIVE SEMIDYNAMICAL SYSTEMS

Everaldo M. Bonotto — Luciene P. Gimenes — Ginnara M. Souto

ABSTRACT. Recursive properties on impulsive semidynamical systems are considered. We obtain results about almost periodic motions and asymptotically almost periodic motions in the context of impulsive systems. The concept of asymptotic almost periodic motions is introduced via time reparametrizations. We also present asymptotic properties for impulsive systems and for their associated discrete systems.

## 1. Introduction

The theory of impulsive differential equations is an important tool to describe the evolution of systems where the continuous development of a process is interrupted by abrupt changes of state. An impulsive differential equation is modeled by a system that encompasses a differential equation, which describes the period of continuous variation of state, and additional conditions, which describe the discontinuities of the solutions of the differential equation or of their derivatives at the moments of impulses.

One of the branches of the theory of impulsive differential equations is the theory of impulsive dynamical systems. In recent years, a significant progress has been made in the study of discontinuous dynamical systems. Moreover,

this theory found application in many fields such as physics, pharmacokinetics, biotechnology, economics, chemical technology, population dynamics and others. The reader may consult, for instance, [5], [6], [9]–[11], [17].

The concepts of almost periodic and asymptotically almost periodic functions were introduced by Bohr in [2] and Fréchet in [18], respectively. Later, these concepts were developed in the context of dynamical systems by Bhatia and Szego in [1] and by Cheban in [12]. The existence of almost periodic and asymptotically almost periodic solutions is one of the most attractive topics in the qualitative theory of differential equations, see [12] and [19], for instance.

The goal of this paper is to consider almost periodic motions in the context of impulsive semidynamical systems. We shall give sufficient conditions to obtain the existence of asymptotic almost periodic motions in impulsive semidynamical systems. Since almost periodicity of motions is deeply connected with stability, we also investigate this connection on impulsive systems. The reader may consult some results about almost periodic motions on impulsive systems in [11].

In the next lines, we describe the organization of this paper. Section 2 deals with the basis of the theory of semidynamical systems with impulses. Section 3 concerns with the main results. This section is divided in four parts. In Subsection 3.1, we study some results about almost periodic motions. We show that all almost periodic points are positively Poisson stable in impulsive systems, see Theorem 3.9. In Subsection 3.2, we present the concepts of asymptotic almost periodic, stationary, periodic, recurrent and Poisson stable motions using time reparametrizations. Some topological properties for these motions are considered. We also use the concept of quasi stability of Zhukovskiĭ for impulsive systems to get asymptotic properties. In Subsection 3.3, we consider discrete systems in the sense of Kaul, [21], which are naturally associated to impulsive semidynamical systems. We study the concepts of almost periodicity and asymptotic almost periodicity for these systems. Asymptotic properties are obtained relating impulsive systems and their associated discrete systems. Finally, in Subsection 3.4, we present sufficient conditions to obtain Zhukovskiĭ quasi stability via Lyapunov stability.

## 2. Preliminaries

Let $(X, d)$ be a metric space, $\mathbb{R}_+$ be the set of non-negative real numbers, $\mathbb{Z}_+$ be the set of non-negative integers and $\mathbb{N} = \{1, 2, \ldots\}$ be the set of natural numbers. The triple $(X, \pi, \mathbb{R}_+)$ is called a *continuous semidynamical system* on $X$ if the mapping $\pi \colon X \times \mathbb{R}_+ \to X$ is continuous with $\pi(x, 0) = x$ and $\pi(\pi(x, t), s) = \pi(x, t + s)$, for all $x \in X$ and $t, s \geq 0$.

Along to this text, we shall denote the system $(X, \pi, \mathbb{R}_+)$ simply by $(X, \pi)$ and we will call it as a semidynamical system, that is, dropping the word continuous.