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SEMILINEAR INCLUSIONS WITH NONLOCAL CONDITIONS WITHOUT COMPACTNESS IN NON-REFLEXIVE SPACES

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Abstract. An existence result for an abstract nonlocal boundary value problem $x' \in A(t)x(t) + F(t, x(t))$, $Lx \in B(x)$, is given, where $A(t)$ determines a linear evolution operator, L is linear, and F and B are multivalued. To avoid compactness conditions, the weak topology is employed. The result applies also in nonreflexive spaces under a hypothesis concerning the De Blasi measure of noncompactness. Even in the case of initial value problems, the required condition is essentially milder than previously known results.

1. Introduction

We consider a nonlocal semilinear differential inclusion in a Banach space E

$$\begin{cases} x'(t) \in A(t)x(t) + F(t, x(t)) \quad (a < t \leq b), \\ Lx \in B(x) \end{cases} \tag{1.1}$$

where $A(t)$ $(t \in [a, b])$ is a family of linear not necessarily bounded operators, $F \colon [a, b] \times E \multimap E$, $B \colon C([a, b], E) \multimap E$, and $L \colon C([a, b], E) \to E$ is bounded and linear. (Here, $f \colon A \multimap B$ denotes a multivalued map, that is, $f(x)$ is a subset

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of B for every $x \in A$; we use the customary notation $f(A) = \bigcup_{x \in A} f(x)$ for such maps).

Although the above problem is very general, the results in this paper are new even in the single-valued case and for initial value problems ($Lx = x(a)$ and $B(x)$ being independent of x).

Multivalued equations in abstract spaces are motivated by the study of control problems for partial differential equations, by obstacle conditions (forcing "impulses"), or by a process known only up to some degree of uncertainty.

Nonlocal problems, on the other hand, have been studied in several contexts since the pioneering work of Byszewsky [8]. For instance, the multipoint boundary value problem

$$Ly = \sum_{i=1}^{n} L_i y(t_i) \tag{1.2}$$

with L_i being bounded linear operators in E and $t_i \in [a, b]$, allows measurements at various points $t = t_i$, rather than just at $t = a$, which is more suitable for some problems in physics than the classical initial problem. Moreover in many models of population dynamics, there is an integral condition

$$Ly = \int_a^b \widetilde{L}(t) y(t)\, d\varphi(t). \tag{1.3}$$

The existence of solutions for these problems is frequently studied with topological techniques based on fixed point theorems for a suitable solution operator. This requires strong compactness conditions, which are usually not satisfied in an infinite dimensional framework, if the evolution operator associated to $A(\,\cdot\,)$ fails to be compact.

The main aim of this paper is to obtain existence results in the lack of this compactness. Several techniques have previously been employed for this situation. One technique is based on the concept of measure of noncompactness (together with a corresponding degree theory or fixed point theorems), see e.g. [4]. Another technique makes use of weak topologies; for instance, in reflexive spaces the Ky Fan fixed point theorem has been used in the weak topology in [7]. Other techniques involve compactly embedded Gel'fand triples with a Hilbert space and Hartman-type conditions, see [5].

In this paper, we make use of a weak measure of noncompactness, thus avoiding hypotheses of compactness both on the semigroup generated by the linear part and on the nonlinear term F, as well as restrictions about compact Gel'fand triples. In particular, in contrast to [4], this approach allows us to treat a class of nonlinear maps F which are not necessarily compact-valued. Moreover, unlike [7], we can handle also nonreflexive Banach spaces.

In a normed space E, we use the notation $B_r(E)$ to denote the closed ball in E centered at 0 with radius r. We use the De Blasi measure of noncompactness introduced in [13] which for $M \subseteq E$ is defined as

$$\beta(M) = \inf\{\varepsilon \in [0,\infty] : M \subseteq W + B_\varepsilon(E) \text{ for some weakly compact } W \subseteq E\}.$$

As far as we know, all methods to find solutions of equations exploiting weak measure of noncompactness presented in literature require either the equicontinuity of the nonlinear term, or a uniform regularity with respect to a measure of noncompactness of the type

$$\beta(F([a,b] \times M)) \leq \beta(M) \quad (M \subseteq E), \tag{1.4}$$

see e.g. [1], [12], [18], [25]. Instead, we assume only a pointwise such condition, i.e. we require the existence of a function $\nu \in L_1([a,b],[0,\infty))$ such that

$$\beta(F(t,M)) \leq \nu(t)\beta(M) \quad (M \subseteq E). \tag{1.5}$$

(Actually, we will require only a weaker inequality involving countable sets; the latter is important in nonseparable spaces, as we shall see later.) Classes of maps satisfying (1.5) are described in the appendix and in the sections with examples.

This is not a trivial extension: While (1.4) trivially guarantees a "good" behaviour of β under a certain integration operation, this is not obvious for (1.5). We must make use of a deep result on the behaviour of β under integration, due to Kunze and Schlüchtermann, which holds only for countable sets: Hence, we must make a technical reduction to handle only countable sets.

We point out once more that our main result is a novelty even for the Cauchy initial value problem: The example in Section 7 is such a Cauchy problem for which our result applies, although an estimate of type (1.4) fails spectacularly. Moreover, in this example one cannot easily replace β by e.g. the Hausdorff measure of noncompactness, so that the well-known results for e.g. condensing maps (with the strong topology) do not apply.

In Sections 3 and 4, we show also some fixed point theorems and containment/selection results for the weak topology which we employ in the proof and the examples.

2. Main result

We assume the following hypotheses on the semilinear differential inclusion (1.1):

(U) For each $s \in [a,b]$ and each $u \in E$ the linear initial value problem $x'(t) = A(t)x(t)$, $x(s) = u$, has a solution $x(t) = U(t,s)u$ in some mild sense (see Remark 2.3) such that $U(t,s)$ $(a \leq s \leq t \leq b)$ are bounded linear operators on E which are strongly continuous, that is, $U(\,\cdot\,,\,\cdot\,)u$ is continuous on $\{(t,s) : a \leq s \leq t \leq b\}$ for every $u \in E$. Note that by the uniform boundedness principle, there is $D > 0$ with $\|U(t,s)\| \leq D$ for all $a \leq s \leq t \leq b$.

(F1) $F(t,u)$ is nonempty and convex for every $(t,u) \in [a,b] \times E$, and for every $u \in E$ the multivalued map $F(\,\cdot\,,u)\colon [a,b] \multimap E$ has a (strongly Bochner) measurable selection f_u.

(F2) $F(t,\,\cdot\,)\colon E \multimap E$ has a weakly sequentially closed graph for almost all $t \in [a,b]$.

(F3) For every n there exists $\varphi_n \in L_1([a,b],[0,\infty))$ with

$$\sup\{\|y\| : y \in F(t, B_n(E))\} \leq \varphi_n(t)$$

for almost all $t \in [a,b]$.

(F4) For every n there exists a function $\nu_n \in L_1([a,b],[0,\infty))$ such that, for almost all $t \in [a,b]$,

$$\beta(C_1) \leq \nu_n(t)\beta(C_0)$$

for all countable $C_0 \subseteq B_n(E)$, $C_1 \subseteq F(t, C_0)$ where β is the De Blasi measure of weak noncompactness.

(L1) $L\colon C([a,b],E) \to E$ is a bounded linear operator and has the property that for each n there is $\mu_n > 0$ such that

$$\beta(\{Lx : x \in C\}) \leq \mu_n \sup_{t\in[a,b]} \beta(C(t)) \tag{2.1}$$

for all countable $C \subseteq B_n(C([a,b],E))$, where we use the notation $C(t) := \{x(t) : x \in C\}$.

(L2) The operator $J\colon E \to E$ defined by $Jx := L(U(\,\cdot\,,a)x)$ is invertible.

(B) $B\colon C([a,b],E) \multimap E$ has a weakly sequentially closed graph, $B(x)$ is nonempty and convex for every $x \in C([a,b],E)$, B maps bounded sets into weakly relatively compact sets, and we have the sublinear (for F and B) growth condition

$$\begin{aligned} D\,\|J^{-1}\| \limsup_{\|u\|\to\infty} \frac{\sup\{\|y\| : y \in B(u)\}}{\|u\|} \\ + (D^2\,\|L\|\,\|J^{-1}\| + D) \limsup_{n\to\infty} \frac{\int_0^T \varphi_n(t)\,dt}{n} < 1. \end{aligned} \tag{2.2}$$

Definition 2.1. A function $x\colon [a,b] \to E$ is a *mild solution* of problem (1.1) if there is a function $f \in L_1([a,b],E)$ with $f(t) \in F(t,x(t))$ for almost all $t \in [a,b]$ and

$$\begin{aligned} &x(t) = U(t,a)x(a) + \int_a^t U(t,s)f(s)\,ds \quad (t \in [a,b]), \\ &Lx \in B(x). \end{aligned} \tag{2.3}$$

Recall that E is *weakly compactly generated* if there is a weakly compact set $M \subseteq E$ with $E = \bigcup_{\lambda>0} \lambda M$. For instance, every separable space is weakly compactly generated. Also the space $C(M,\mathbb{R})$ is weakly compactly generated, if M is a weakly compact subset of a Banach space with the weak topology.

Now we can state the main result of the paper.

Theorem 2.2. *Let E be weakly compactly generated. Under assumptions* (U)*,* (F1)–(F4)*,* (L1)*,* (L2)*, and* (B)*, problem* (1.1) *has at least one mild solution, that is, there is some $x \in C([a,b],E)$ satisfying* (2.3)*.*

Before proceeding with the proof of the theorem, let us first discuss the above definitions and hypotheses.

Remark 2.3. The first equation in (2.3) implies that x is continuous, see Proposition 5.1. This equation means that x is a mild solution of the nonhomogeneous problem

$$x'(t) = A(t)x(t) + f(t), \tag{2.4}$$

that is, in the sense of the associated variation-of-constants formula. There is a vast literature on results in which sense such mild solutions actually satisfy (2.4) and in which sense it is needed for this that $x(t) = U(t,s)u$ satisfies the initial value problem mentioned in our hypothesis (U). We only point out some important special cases. For the case that $A(t) = A$ is a densely defined generator of a linear C_0-semigroup U_0 in E and $U(t,s) = U_0(t-s)$, it was shown in [3] that x satisfies the equation in (2.3) if and only if for each ℓ in the domain of the adjoint operator A^* the function $\ell \circ x$ is absolutely continuous and satisfies

$$(\ell \circ x)'(t) = (A^*\ell)x(t) + (\ell \circ f)(t)$$

for almost all $t \in [a,b]$. The more general case that the operators $A(t)$ are dependent of t, but that their domains $D(A(t))$ are independent of t and dense in E, was perhaps first studied systematically in the classical monograph [22, II, § 3], but nowadays also more general cases are important, e.g. that all $D(A(t))$ contain a common dense subspace, or even less restrictive requirements. To possibly treat such cases, we left the relation between U and A in hypothesis (U) intentionally vague. We can do this, because we study only mild solutions, and in their definition the operator A does not occur at all. In fact, our main result holds for every strongly continuous family U of bounded linear operators on E, as required in (U).

Remark 2.4. If $Lx = x(a)$, then J is the identity operator and thus (L2) holds. Moreover, if additionally $B(C([a,b],E))$ is bounded (e.g. if we consider the initial condition $x(a) = B$ with B independent of x), then Theorem 2.2 holds also if we drop (2.2) and instead replace (F3) by the linear growth condition

$$\sup\{\|y\| : y \in F(t,u)\} \leq \varphi(t)(1+\|u\|) \tag{2.5}$$

with some $\varphi \in L_1([a,b],\mathbb{R})$.

Remark 2.5. If Ω is a measure space, we call a measurable function $g\colon \Omega \to E$ a measurable selection of $G\colon \Omega \multimap E$ if $g(t) \in G(t)$ for almost all $t \in \Omega$. We emphasize that the corresponding exceptional null set in (F1) is allowed to depend on u.

In particular, if (F2) holds and $F(t,u) \subseteq [a,b] \times E$ is nonempty and convex for every $(t,u) \in [a,b] \times E$, and if $F(\,\cdot\,,u)$ is Bochner measurable as a multivalued map in the sense of [33] for every $u \in E$ then $F(\,\cdot\,,u)$ has a measurable selection, and so (F1) is satisfied: This is a variant of the classical Kuratowski–Ryll–Nardzewski selection theorem. From the vast literature of further selection theorems, we mention just the Aumann selection theorem for multimaps with a Borel measurable graph (see e.g. [11]) and the recent paper [10] for the nonseparable case. Concerning measurable multivalued maps, we refer the reader to [19].

Example 2.6. If $A(t) \equiv A$ is the abstract form of the Laplace operator then it generates a strongly continuous compact semigroup of contractions U_0 such that $U_0(b)x = x$ if and only if $x = 0$ (see e.g. [26] and [36]). According to the Fredholm alternative, $U_0(b) - I$ is invertible, where I denotes the identity operator. Hence, the associated periodic problem $Lx := x(b) - x(a)$ satisfies condition (L2).

We point out that the sublinear growth condition (2.2) in (F3) is weaker than the global boundedness condition often assumed in literature in connection with multivalued nonlocal boundary value problems.

Some notes on the apparently strange condition (2.1) are in order: If L has the form (1.2), then (2.1) holds automatically, because β is algebraically subadditive:

$$\beta\biggl(\sum_{k=1}^{n} M_k\biggr) \leq \sum_{k=1}^{n} \beta(M_k). \tag{2.6}$$

But actually also (1.3) satisfies condition (2.1) if E is weakly compactly generated as we show now.

Indeed, for weakly compactly generated E, we can use the following result which is a special case of [23] and which we will also use for the proof of Theorem 2.2.

Theorem 2.7. *Let E be a weakly compactly generated Banach space. Then for every countable uniformly integrable family C of vector functions $x\colon \Omega \to E$*

with Ω being a positive measure space, $\operatorname{mes}\Omega < \infty$*, the function $\beta(C(\,\cdot\,))$ is measurable, and*

$$\beta\biggl(\biggl\{\int_\Omega x(s)\,ds : x \in C\biggr\}\biggr) \le \int_\Omega \beta(C(s))\,ds.$$

Corollary 2.8. *Let E be a weakly compactly generated Banach space. Let $\varphi\colon [a,b] \to \mathbb{R}$ have bounded variation, and for $t \in [a,b]$, let $\widetilde{L}(t)\colon E \to E$ be bounded and linear with $\widetilde{L}(\,\cdot\,)u$ being Borel measurable for every $u \in E$ and $\|\widetilde{L}(t)\| \le \psi(t)$ with some Borel measurable $\psi\colon [a,b] \to [0,\infty)$ satisfying*

$$\mu := \int_a^b \psi(s)\,d(\operatorname{var}\varphi)(s) < \infty.$$

Then the operator (1.3) *is bounded from $C([a,b],E)$ into E and satisfies* (2.1).

Proof. Without loss of generality, we can assume that φ is monotone increasing; then $z(t) = \widetilde{L}(t)y(t)$ is Borel measurable for every $y \in C([a,b],E)$, see e.g. [28, Theorem A.1.1], and thus measurable with respect to the positive finite measure generated by φ. Hence, we have for every countable bounded $C \subseteq C([a,b],E)$ by Theorem 2.7 that

$$\beta(L(C)) \le \int_a^b \beta(\{\widetilde{L}(t)y(t) : y \in C\})\,d\varphi(t) \le \int_a^b \psi(t) \sup_{s\in[a,b]} \beta(C(s))\,d\varphi(t),$$

which implies (2.1) with $\mu_n = \mu$. $\square$

We build the proof of the above Theorem 2.2 on some preliminary fixed point theorems.

3. Fixed point theorems and condensing maps

Several fixed point theorems based on the De Blasi measure of noncompactness are known, see e.g. [9]. However, we need a variant where this measure occurs only indirectly in a countable form. To this end, we start with a rather general fixed point theorem.

Theorem 3.1 (Fundamental, Ky Fan). *Let K be a subset of a locally convex Hausdorff space E. Let $F\colon K \multimap E$ satisfy $\overline{\operatorname{conv}}(F(K) \cup \{x_0\}) \subseteq K$ with some $x_0 \in K$. Then there is a smallest closed convex set $M \subseteq K$ satisfying $F(M) \subseteq M$ and $x_0 \in M$. This set is simultaneously the smallest $M \subseteq K$ satisfying*

$$M = \overline{\operatorname{conv}}\bigl(F(M) \cup \{x_0\}\bigr). \tag{3.1}$$

If this set M is relatively compact in the locally convex topology, $F|_M$ has a closed graph in $M \times M$, and $F(x)$ is nonempty and convex for every $x \in M$, then F has a fixed point $x \in M \subseteq K$, that is, $x \in F(x)$.

Proof. We call a closed convex set $M_0 \subseteq K$ fundamental (with respect to x_0, K, and F), if $x_0 \in M_0$ and $F(M_0) \subseteq M_0$, Hence, M_0 is fundamental if and only if

$$\Phi(M_0) := \overline{\operatorname{conv}}(F(M_0) \cup \{x_0\}) \subseteq \overline{\operatorname{conv}}\, M_0 = M_0.$$

Our hypothesis implies that $\Phi(K) \subseteq K$, and so $\Phi(\Phi(K)) \subseteq \Phi(K)$. Hence, there is at least one fundamental set, namely $\Phi(K)$. Let M be the intersection of all fundamental sets. Then M is closed and convex, and for every fundamental set M_0, we have $F(M) \subseteq M_0$, hence $F(M) \subseteq M$. Thus M is fundamental, and so $F(\Phi(M)) \subseteq F(M) \subseteq \Phi(M)$. Hence, also $\Phi(M)$ is fundamental. Since M is the smallest fundamental set, we obtain $\Phi(M) \supseteq M$. Thus, M satisfies (3.1), and M is the smallest set with this property. Since M is fundamental and, by hypothesis, compact in the locally convex topology, and $F\colon M \multimap M$ has a closed graph and thus is upper semicontinuous by [34, Corollary 2.124], the existence of a fixed point follows from Ky Fan's fixed point theorem [16]. □

The above method of proof is well-known and was used in slightly different settings already in, e.g., [24], [31], [30], [34]. However, we cannot use the techniques from [31], [30], [34] directly to reduce the setting to countable subsets, which will be crucial for our approach.

Corollary 3.2 (Fundamental Sequential, Ky Fan). *Let E be a Banach space equipped with the weak topology. Then the hypothesis about the closed graph in Theorem* 3.1 *can equivalently be replaced by the hypothesis that the graph is weakly sequentially closed.*

Proof. Since $M \times M$ is weakly compact, the graph of $F|_M\colon M \multimap M$ is weakly closed if and only if it is weakly compact. By the Eberlein–Šmulian theorem, this holds if and only if the graph is weakly sequentially compact. This is the case if and only if the graph is weakly sequentially closed, because $M \times M$ is sequentially weakly compact by the Eberlein–Šmulian theorem. □

For a short proof of the strong variant of the Eberlein–Šmulian theorem for nonconvex sets which we used here, we refer to [38] and [35]. The latter contains also directly the assertion that for weakly (sequentially) relatively compact sets the weak closure and weak sequential closure coincide, so that we could even shortcut our above argument.

Let us first formulate a consequence of the above theorem in terms of so-called condensing maps.

Definition 3.3. A *measure of noncompactness* on a subset K of a locally convex Hausdorff space E is a map γ from the subsets of K into some set R with the property that $\gamma(\overline{\operatorname{conv}}\, M) = \gamma(M)$ for all $M \subseteq K$. If $x_0 \in M$, we say that γ is *x_0-stable* if $\gamma(M \cup \{x_0\}) = \gamma(M)$. If R is equipped with a partial order $\preceq$, we call γ *monotone* if $M \subseteq N \subseteq K$ implies $\gamma(M) \preceq \gamma(N)$.

Definition 3.4. Let $D_0 \subseteq K \subseteq E$. A map $F\colon D_0 \multimap E$ is

(a) x_0-*condensing* on $D_0 \subseteq K$ (with respect to K) if for each set $M \subseteq K$ for which $\overline{\operatorname{conv}}(F(M \cap D_0) \cup \{x_0\})$ is noncompact and contained in M, there is an x_0-stable monotone measure of noncompactness γ on M with $\gamma(F(M \cap D_0)) \not\geq \gamma(M)$.
(b) x_0-*unpreserving* on $D_0 \subseteq K$ (with respect to K) if for each set $M \subseteq K$ for which $\overline{\operatorname{conv}}(F(M \cap D_0) \cup \{x_0\}) = M$ is noncompact, there is an x_0-stable measure of noncompactness γ on M such that $\gamma(F(M)) \neq \gamma(M)$.

Our definition generalizes the notion of condensing maps (see e.g. [2], [27] and [34]) in three ways: First, the choice of γ can depend on M. Second, we require the inequality only if we know that $\overline{\operatorname{conv}}(F(M \cap D_0) \cup \{x_0\}) \subseteq M$. Third, for x_0-unpreserving maps, it is fine for us also if $\gamma(F(M)) \not\leq \gamma(M)$, and moreover, we have to verify this only if we have additionally the converse inclusion $M \subseteq \overline{\operatorname{conv}}(F(M \cap D_0) \cup \{x_0\})$.

In particular, each x_0-condensing map is trivially x_0-unpreserving. Except for the requirement of the inclusions $\overline{\operatorname{conv}}(F(M \cap D_0) \cup \{x_0\}) \subseteq M$ or $M \subseteq \overline{\operatorname{conv}}(F(M \cap D_0) \cup \{x_0\})$ all these generalizations have already been observed in [29] where, however, only a metrizable setting was considered. For this reason, we cannot use the techniques developed there to restrict ourselves to countable sets.

The notion of x_0-condensing maps is indeed appropriate if one wants to develop a degree theory (which we do not develop here and which would be overkill for our application) while the notion of x_0-unpreserving maps is the "right" notion if one is only interested in fixed point theorems of self-maps of closed convex sets K as we show now.

Remark 3.5. For the case that E is a Banach space, one can equip E either with the weak topology or with the norm topology. By Hahn–Banach, $\overline{\operatorname{conv}}\, M$ is the same for both topologies. For this reason, we do not have to distinguish in the definition of a measure of noncompactness which of the two topologies we consider: A measure of weak noncompactness is the same as a measure of noncompactness.

For the notion of x_0-condensing/unpreserving maps, however, we have to make precise which topology we consider: The corresponding notion of weakly condensing/unpreserving maps is obviously less restrictive, and since the aim of this paper is to relax the compactness requirements as far as possible, we will deal only with the latter in this paper.

Remark 3.6. By the Krejn–Šmuljan theorem [15, V.6.4], the set $\overline{\mathrm{conv}}\big(F(M\cap D_0)\cup\{x_0\}\big)$ is weakly compact if and only if $F(M\cap D_0)$ is weakly relatively compact, and so we obtain in particular that $F\colon D_0\to K$ is weakly x_0-unpreserving (weakly x_0-condensing) on K if the following holds:

Whenever $M\subseteq K$ is such that $F(M\cap D_0)\subseteq M$ and $\gamma(F(M\cap D_0))=\gamma(M)$ (or $\gamma(F(M\cap D_0))\succeq\gamma(M)$, respectively) for every x_0-stable (monotone) measure of noncompactness γ on M, then at least one of the sets $F(M\cap D_0)$ or M is weakly relatively compact.

Theorem 3.7 (Sadovskiĭ–Ky Fan). *Let K be a subset of a locally convex Hausdorff space E, $F\colon K\multimap E$ satisfy $\overline{\mathrm{conv}}(F(K)\cup\{x_0\})\subseteq K$ with some $x_0\in K$. Suppose that for each compact set $M\subseteq K$ with $F(M)\subseteq M$ the restriction $F|_M$ has a closed graph in $M\times M$, and $F(x)$ is nonempty and convex for every $x\in M$. If F is x_0-unpreserving on K, then F has a fixed point in K.*

Proof. This is a trivial consequence of Theorem 3.1, for if M satisfies (3.1), then M is compact. Indeed, if this would not be the case then M fails to be relatively compact (since it is closed by (3.1)), and so the hypothesis implies that there is an x_0-stable measure of noncompactness satisfying

$$\gamma(M)\neq\gamma(F(M))=\gamma\big(F(M)\cup\{x_0\}\big)=\gamma\big(\overline{\mathrm{conv}}(F(M)\cup\{x_0\})\big)=\gamma(M),$$

which is a contradiction. □

The same argument as in Corollary 3.2 implies that, when we speak about the weak topology in a Banach space, we can replace equivalently the hypothesis about a closed graph by a sequentially closed graph. We thus obtain:

Corollary 3.8 (Sequential Weak Sadovskiĭ–Ky Fan). *Let K be a subset of a Banach space E, $F\colon K\multimap E$ satisfy $\overline{\mathrm{conv}}(F(K)\cup\{x_0\})\subseteq K$ with some $x_0\in K$. Suppose that for each weakly compact set $M\subseteq K$ with $F(M)\subseteq M$ the restriction $F|_M$ has a weakly sequentially closed graph in $M\times M$, and $F(x)$ is nonempty and convex for every $x\in M$. If F is weakly x_0-unpreserving on K, then F has a fixed point in K.*

The term "sequential" in Corollaries 3.2 and 3.8 refers only to the closedness hypothesis, but unfortunately not to the compactness hypothesis. In view of Theorem 2.7, it is crucial for us to use a countable compactness assumption only. In order to obtain a corresponding result, we follow an idea of [20] and introduce the following definition:

Definition 3.9. Let E be a Banach space, $N\in\mathbb{R}$, and $M\subseteq C([a,b],E)$. We use the notation $M(t)=\{x(t):x\in M\}$ and define

$$\beta_N(M)=\sup_{C\subseteq M\text{ countable}}\ \sup_{t\in[a,b]}\beta(C(t))e^{-Nt},$$

where β denotes the De Blasi measure of noncompactness.

The crucial observation which eventually allows us the reduction to the countable case is the following.

Proposition 3.10. *β_N is a monotone measure of noncompactness on the space $C([a,b],E)$ (and x_0-stable for every $x_0 \in C([a,b],E)$). More general, an analogous assertion holds if β is replaced in Definition 3.9 by some monotone measure of noncompactness on E (which is u_0-stable for every $u_0 \in E$).*

Proof. By Remark 3.5, it suffices to consider the norm topologies on the spaces $C([a,b],E)$ and E. Since β is monotone, trivially also β_N is monotone. Thus, we are to show that $\beta_N(\overline{\operatorname{conv}}\, M) \le \beta_N(M)$ for every set $M \subseteq E$. Letting $\gamma < \beta_N(\overline{\operatorname{conv}}\, M)$ be arbitrary, it suffices to show that $\beta_N(M) > \gamma$. By definition of β_N, there is a countable $C \subseteq \overline{\operatorname{conv}}\, M$ and some $t \in [a,b]$ such that $\beta(C(t))e^{-Nt} > \gamma$. Since we consider the norm topology, and $C \subseteq \overline{\operatorname{conv}}\, M$ is countable and thus separable, we find by [34, Proposition 3.55] some countable $C_0 \subseteq M$ such that $C \subseteq \overline{\operatorname{conv}}\, C_0$. It follows that $C(t) \subseteq \overline{\operatorname{conv}}(C_0(t))$. Since β is a measure of noncompactness, it follows that

$$\gamma < \beta(C(t))e^{-Nt} < \beta(\overline{\operatorname{conv}}(C_0(t)))e^{-Nt} = \beta(C_0(t))e^{-Nt} \le \beta_N(M),$$

and so we are done. $\square$

4. Containment and selection results

Throughout this section, let Ω be a σ-finite measure space, and E be a Banach space. We need some results concerning weak convergence in $L_1(\Omega, E)$.

Lemma 4.1 (Containment Lemma). *Let $f_n, f\colon \Omega \to E$ be measurable and such that for each set $I \subseteq \Omega$ of finite measure the set $\{f_n(t) : n \in \mathbb{N}\}$ is weakly relatively compact for almost all $t \in \Omega$. Suppose that for every linear bounded functional ℓ on E and every subset $\Omega_0 \subseteq \Omega$ of positive measure there is a subset $I \subseteq \Omega_0$ of positive measure such that $\ell \circ f_n|_I$ and $\ell \circ f|_I$ are integrable and satisfy*

$$\lim_{n\to\infty} \int_I \ell(f_n(t))\, dt \to \int_I \ell(f(t))\, dt.$$

Then

$$f(t) \in \bigcap_{n=1}^{\infty} \overline{\operatorname{conv}}\,\{f_m(t) : m \ge n\} \quad \text{for almost all } t \in \Omega. \tag{4.1}$$

Proof. We can assume that $\bigcup_n f_n(\Omega) \cup f(\Omega)$ is separable, see e.g. [32, Corollary 1.1]. Hence, replacing E by the closed linear hull of this set, we can assume without loss of generality that E is separable and thus Suslin. Note that $M_n(t) := \overline{\operatorname{conv}}\,\{f_m(t) : m \ge n\}$ are weakly compact by the Krejn–Šmuljan theorem [15, V.6.4]. Hence, assuming without loss of generality that E is a real

Banach space, we obtain from [11, Proposition III.35]: If for every bounded linear functional ℓ on E

$$\ell(f(t)) \leq \sup_{v \in M_n(t)} \ell(v) \quad \text{for almost all } t \in \Omega, \tag{4.2}$$

then also $f(t) \in M_n(t)$ for almost all $t \in \Omega$. Since the union of the exceptional null sets for each n is a null set, we obtain: If we can show (4.2) for every n and ℓ then $f(t) \in \bigcap_n M_n(t)$ for almost all $t \in \Omega$, which is (4.1). Thus, let n and ℓ be fixed. Assume by contradiction that (4.2) fails. Note that both sides of (4.2) are measurable, because

$$s(t) := \sup_{v \in M_n(t)} \ell(v) = \sup_{v \in \{f_m(t) : m \geq n\}} \ell(v) = \sup_{m \geq n} \ell(f_m(t)). \tag{4.3}$$

Hence, there is a set $\Omega_1 \subseteq \Omega$ of positive measure such that (4.2) fails for every $t \in \Omega_1$. Since Ω_1 is the union of the sets $I_N := \{t \in \Omega_1 : \ell(f(t)) > s(t) + 1/N\}$, there is a natural number $N > 0$ such that $\Omega_0 := I_N$ has positive measure. Let $I \subseteq \Omega_0$ be as in the hypothesis. Then we have for every $m \geq n$ in view of (4.3) that

$$\int_I \ell(f(t))\,dt - \frac{\operatorname{mes} I}{N} \geq \int_I s(t)\,ds \geq \int_I \ell(f_m(t))\,dt \to \int_I \ell(f(t))\,dt$$

as $m \to \infty$ which is a contradiction. $\square$

Remark 4.2. The proof shows that we can relax the measurability of f_n and f in Lemma 4.1 to the hypothesis that each f_n and f assume almost all of their values in a separable subset of E.

Another approach to prove the Containment Lemma 4.1 is by applying Mazur's convexity lemma. This approach was used in [6].

Lemma 4.3. *Let $G_n \subseteq E$ be a sequence of nonempty sets and $K \subseteq E$ closed and convex. Suppose that any subsequence of $u_n \in G_n$ contains a subsequence which converges weakly to some element of K. Then*

$$\bigcap_{n=1}^{\infty} \overline{\operatorname{conv}}\Big(\bigcup_{m=n}^{\infty} G_m\Big) \subseteq K. \tag{4.4}$$

Proof. Assume by contradiction that there is $v \in E \setminus K$ which belongs to the set on the left-hand side of (4.4). Assuming without loss of generality that E is a real Banach space, we find by the classical separation theorem a bounded linear functional ℓ on E such that $\ell(v) > \ell(u)$ for all $u \in K$. The choice of v implies that

$$\ell(v) \leq s := \limsup_{m \to \infty} \sup_{u \in G_m} \ell(u).$$

Hence, there are $m_k \to \infty$ and $u_k \in G_{m_k}$ with $\ell(u_k) \to s$. By hypothesis, we can assume that $u_k \rightharpoonup u$ for some $u \in K$. Then $\ell(u_k) \to \ell(u)$ implies $\ell(v) \leq s = \ell(u)$, a contradiction. $\square$

Theorem 4.4 (Containment Theorem). *Let $G_n, G\colon \Omega \multimap E$ be such that for almost all $t \in \Omega$ the following holds: Every subsequence of $u_n \in G_n(t)$ contains a subsequence which converges weakly to some element of $G(t)$. Let $y_n\colon \Omega \to E$ be uniformly integrable and such that $y_n(t) \in G_n(t)$ for almost all $t \in \Omega$. Then y_n contains a subsequence which converges weakly in $L_1(\Omega, E)$ to some y, and moreover, each such limit y satisfies $y(t) \in \overline{\operatorname{conv}}\, G(t)$ for almost all $t \in \Omega$.*

Proof. The hypothesis implies in particular that $\{y_n(t) : n \in \mathbb{N}\}$ is weakly sequentially relatively compact and thus weakly relatively compact by the theorem of Eberlein–Šmulian. The existence of a weakly convergent subsequence in $L_1(\Omega, E)$ thus follows from the Dunford–Pettis theorem for vector functions [14]. If $y_{n_k} \rightharpoonup y$, we have for almost all $t \in \Omega$ that

$$y(t) \in \bigcap_{k=1}^{\infty} \overline{\operatorname{conv}} \bigg(\bigcup_{j=k}^{\infty} G_{n_j}(t) \bigg) \subseteq \overline{\operatorname{conv}}\, G(t)$$

by Lemma 4.1 and Lemma 4.3, respectively. □

As a trivial application of the Containment Theorem 4.4, we formulate a selection result for multivalued superposition operators generated by functions which do not necessarily assume compact or separable (in the strong topology) values and which need not be measurable in any strong sense. This result generalizes [6, Proposition 2.1].

We emphasize that the exceptional null set in hypothesis (b) of this result is allowed to depend on u.

Proposition 4.5. *Let M be a metric space, and $F\colon \Omega \times M \multimap E$ have the following properties:*

(a) *For almost all $t \in \Omega$ the values $F(t, u)$ are convex for every $u \in M$.*
(b) *For every $u \in M$ the function $F(\,\cdot\,, u)$ has a measurable selection.*
(c) *For almost all $t \in \Omega$ the function $F(t, \cdot\,)$ has a sequentially closed graph in $M \times E$ with the weak topology in E.*
(d) *For almost all $t \in \Omega$ and every convergent sequence $u_n \in M$ the set $\bigcup_n F(t, u_n)$ is weakly compact.*
(e) *There is a measurable function $\varphi\colon \Omega \to [0, \infty)$ with*

$$\sup_{z \in F(t,M)} \|z\| \le \varphi(t) \quad \textit{for almost all } t \in \Omega.$$

Then, for every measurable $x\colon \Omega \to M$, there is a measurable $y\colon \Omega \to E$ with $y(t) \in F(t, x(t))$ for almost all $t \in \Omega$.

Proof. Let Γ denote the family of all sets of finite measure on which the function φ from (e) is bounded. Since Ω is σ-finite, every set of positive measure contains an element of Γ. It follows from the exhaustion theorem [32, Theorem 1.5] that Γ contains a sequence $I_1 \subseteq I_2 \subseteq \ldots$ such that $\bigcup_n I_n = \Omega$ (up to a null set). Trivially, it suffices to show the assertion for each I_n in place of Ω.

Since x is measurable, there is a sequence of simple functions x_n such that $x_n(t) \to x(t)$ for almost all $t \in I_n$. By (b), there are measurable y_n with $y_n(t) \in F(t, x_n(t))$ for almost all $t \in I_n$. Note that $|y_n(t)| \le \varphi(t)$ and $\varphi \in L_1(I_n, [0,\infty))$. Hence, using (c) and (d) and the Eberlein–Šmulian theorem, we find that x_n satisfies the hypotheses of the Containment Theorem 4.4 on the measure space I_n. Thus, a subsequence of y_n converges weakly in $L_1(I_n, E)$ to some y which satisfies $y(t) \in \overline{\operatorname{conv}}\, F(t, x(t)) = F(t, x(t))$ for almost all $t \in I_n$. The latter equality follows from (a) and (c), since $\operatorname{conv} F(t, x(t)) = F(t, x(t))$ is in particular sequentially closed with respect to the norm topology for almost all $t \in \Omega$. $\square$

5. Proof of the main result

We define an operator G by

$$(Gf)(t) := \int_a^t U(t,s) f(s)\, ds \quad (t \in [a,b]).$$

Proposition 5.1. *If U satisfies the continuity properties of* (U) *then the map $G\colon L_1([a,b],E) \to C([a,b],E)$ is linear and bounded with $\|G\| \le D$.*

Proof. Only the continuity of Gf requires a proof. However, this follows straightforwardly by observing that for all $a \le \tau - \varepsilon \le t \le \tau \le b$ there holds

$$\|(Gf)(t) - (Gf)(\tau)\| \le \left\| \int_a^{\tau-\varepsilon} (U(t,s) - U(\tau,s)) f(s)\, ds \right\| + 2D \int_{\tau-\varepsilon}^{\tau} \|f(s)\|\, ds,$$

applying Lebesgue's dominated convergence theorem twice. $\square$

Since J is invertible, we can rewrite (2.3) equivalently as the single equation

$$x(t) = U(t,a) J^{-1}(b - L(Gf)) + (Gf)(t) \quad \text{for all } t \in [a,b] \text{ with some } b \in B(x).$$

We intend to rewrite this in terms of a multivalued map. To this end, we introduce the superposition operator $S_F\colon C([a,b],E) \multimap L_1([a,b],E)$ by

$$S_F(q) := \{ f \in L_1([a,b],E) \mid f(t) \in F(t, q(t)) \text{ for almost all } t \in [a,b] \}.$$

Note that Proposition 4.5 implies under our hypotheses on F that this operator S_F assumes only nonempty values.

We define $T\colon C([a,b],E) \multimap C([a,b],E)$ by

$$T(q) := \bigcup_{\substack{b\in B(q)\\ f\in S_F(q)}} \left\{ x\in C([a,b],E) \;\middle|\; x(t) = U(t,a)J^{-1}(b - L(Gf)) + \int_a^t U(t,s)f(s)\,ds \right\}.$$

Then the fixed points of T are exactly the mild solutions of (1.1).

PROOF OF THEOREM 2.2. It suffices to show that T satisfies all hypotheses of Theorem 3.7 (more precisely, of Corollary 3.8) in an appropriate set $K \subseteq C([a,b],E)$. Note that, since F and B assume convex values, also T assumes convex values.

First, we show that the multioperator T has a weakly sequentially closed graph. Let $q_n, x_n \in C([a,b],E)$ satisfy $x_n \in T(q_n)$ for all n and $q_n \rightharpoonup q$, $x_n \rightharpoonup x$ in $C([a,b],E)$; we are to prove that $x \in T(q)$.

By the definition of the operator T, there exist sequences $f_n \in S_F(q_n)$ and $y_n \in B(q_n)$, such that

$$x_n(t) = U(t,a)J^{-1}(y_n - L(Gf_n)) + (Gf_n)(t).$$

Since $q_n \rightharpoonup q$ in $C([a,b],E)$, we obtain that the sequence q_n is uniformly bounded, and $q_n(t) \rightharpoonup q(t)$ for every $t \in [a,b]$. In view of (F2)–(F4), we can use the Containment Theorem 4.4 with $u_n = q_n$ and $y_n = f_n$ and thus find, passing to a subsequence if necessary, that $f_n \rightharpoonup f$ in $L_1([a,b],E)$ and $f \in S_F(q)$. Since G and L are bounded linear operators, we obtain $L(Gf_n) \rightharpoonup L(Gf)$.

Since B is weakly compact, we can assume by the Eberlein–Šmulian theorem that $y_n \rightharpoonup y$ is weakly convergent. Since $q_n \rightharpoonup q$ and B is weakly sequentially closed, we have $y \in B(q)$. Since J^{-1} and $U(t,s)$ are linear and bounded, we obtain

$$x_n(t) \rightharpoonup \widetilde{x}(t) := U(t,a)J^{-1}(y - L(Gf)) + \int_a^t U(t,s)f(s)\,ds$$

for every $t \in [a,b]$. Since $x_n \rightharpoonup x$ in $C([a,b],E)$ implies that $x_n(t) \to x(t)$ for every $t \in [a,b]$, we obtain that $x = \widetilde{x} \in T(q)$. Hence, we have proved that T has a weakly sequentially closed graph.

For every $x \in T(q)$, we find $f \in S_F(q)$ and $b \in B(q)$ with

$$\|x(t)\| \le D\,\|J^{-1}\|\,(\|b\| + \|LGf\|) + \|Gf(t)\|\,.$$

Thus, we obtain for $\|q\| \le n$ by (F3) the estimate

$$\|x(t)\| \le D\,\|J^{-1}\| \sup_{b\in B(q)} \|b\| + (D^2\,\|J^{-1}\|\,\|L\| + D)\int_a^b \varphi_n(t)\,dt.$$

Since B sends bounded sets into bounded sets, we thus obtain from (2.2) for sufficiently large n that $\|x\| \le n$ if $\|q\| \le n$. Hence, T maps the nonempty bounded closed convex set $K = B_n(C([a,b],E))$ into itself if n is sufficiently large.

It remains to show that $T\colon K \multimap K$ is weakly x_0-unpreserving for every x_0 in K. To this end, we make use of Remark 3.6. We will show that it actually suffices to consider the corresponding measure of noncompactness β_N with sufficiently small $N < 0$. Indeed, we obtain for every $M \subseteq K$ satisfying $\beta_N(M) = \beta(T(M))$ by (2.6) that

$$\begin{aligned}\beta_N(M) = \beta_N(\{U(\,\cdot\,,a)J^{-1}(B(q) - LGf) + Gf : q \in M,\ f \in S_F(q)\}) \\ \le \beta_N(U(\,\cdot\,,a)J^{-1}(B(M))) \\ + \sup_{C_1 \subseteq S_F(M) \text{ countable}} \sup_{t\in[a,b]} e^{Nt}\beta(\{U(t,a)J^{-1}L(Gf) + Gf : f \in C_1\})\end{aligned}$$

The first summand vanishes by (B). To estimate the second summand, we apply (2.6) and (2.1) to obtain that

$$\begin{aligned}\sup_{t\in[a,b]} \beta(\{U(t,a)J^{-1}L(Gf) + Gf(t) : f \in C_1\}) \\ \le (D\,\|J^{-1}\|\,\mu_n + 1) \sup_{t\in[a,b]} \beta(GC_1(t)).\end{aligned}$$

Applying Theorem 2.7 and (F4), we obtain further

$$\begin{aligned}\beta(GC_1(t)) &\le \int_a^t \beta(\{U(t,s)f(s) : f \in C_1\})\,ds \le \int_a^t D\nu_n(s)\beta(C_1(s))\,ds \\ &\le D\int_a^t \nu_n(s)e^{-Ns}e^{Ns}\beta(C_1(s))\,ds \le D\int_a^t \nu_n(s)e^{-Ns}\,ds\,\beta_N(C_1).\end{aligned}$$

Using the shortcut $D_n = (D^2\,\|J^{-1}\|\,\mu_n + D)$, we thus obtain by combining the above formulas that

$$\beta_N(M) \le D_n \sup_{t\in[a,b]} \int_a^t e^{N(t-s)}\nu_n(s)\,ds\,\beta_N(M).$$

Using the substitution $\sigma = t - s$ and splitting the integral, one finds that, for $N < 0$ small enough, the factor in front of $\beta_N(M)$ in this formula becomes strictly less than 1, and so we obtain $\beta_N(M) = 0$ for this N. By definition of $\beta_N(M)$, we obtain for every $t \in [a,b]$ that $M(t)$ is sequentially compact, and thus also $S_F(M)(t)$ is sequentially compact by (F4). Using again the Dunford–Pettis theorem, we obtain that $S_F(M)$ is weakly sequentially compact in $L_1([a,b],E)$. Hence, for any sequence $f_m \in S_F(M)$, we find a convergent subsequence in $L_1([a,b],E)$, without loss of generality $f_m \rightharpoonup f$. Since G is linear and bounded, we obtain that

$$Gf_m(t) = \int_a^t U(t,s)f_m(s)\,ds \rightharpoonup Gf(t) = \int_a^t U(t,s)f(s)\,ds$$

for every $t \in [a,b]$. Since $Gf \in C([a,b],E)$ and since Gf_m is uniformly bounded, we obtain that $Gf_m \rightharpoonup Gf$ in $C([a,b],E)$, see e.g. [21]. By condition (B), the set $B(M)$ is weakly relatively (sequentially) compact, and so $T(M)$ is weakly relatively (sequentially) compact by the Eberlein–Šmulian theorem. □

PROOF OF REMARK 2.4. Under the hypotheses of Remark 2.4, we cannot use the set $K = B_n(C([a,b],E))$, since in general T does not map K into itself. Instead, we use the nonempty closed bounded convex set

$$K := \{x \in C([a,b],E) : \|x(t)\|\, e^{Nt} \le n\}$$

with sufficiently large n and sufficiently small $N < 0$. Indeed, if B is bounded by $\gamma > 0$ then any $x \in T(K)$ satisfies the estimate

$$\|x(t)\| \le D\gamma + nDe^{-Nt} \int_0^t e^{N(t-s)}\varphi(s)\,ds.$$

Choosing $N < 0$, the last integral can be chosen as small as we want, by the same argument as in the proof given above. The rest of the proof remains unchanged. □

6. Application to a population growth model

In this section, we apply the result to show that for a certain class of age-population models subject to sublinear growth conditions there exists a periodic solution. The model which we consider has the form

$$\begin{aligned} &\frac{\partial u(a,s)}{\partial a} + \frac{\partial u(a,s)}{\partial s} = f(a,s,u(a,s)), \\ &u(0,s) = \int_0^T u(\alpha,s)\,d\varphi(\alpha), \end{aligned} \tag{6.1}$$

where $u(a,s)$ denotes the population density of age a at a time s. Here, f describes the rate of the decaying, arriving and leaving individuals, while $d\varphi$ describes the birth rate.

We shall assume that only $a \in [0,T]$ has to be considered. We will assume that f is T-periodic in s, and we will show that there is actually a T-periodic continuous solution with respect to s, that is, we work in the space

$$E = \{v \in C(\mathbb{R},\mathbb{R}) : v \text{ is } T\text{-periodic}\}.$$

Our hypotheses are the following:

(f1) $f\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous with $f(a,s+T,x) = f(a,s,x)$.

(f2) We have the sublinear condition

$$\lim_{n\to\infty} \sup\{|y|/n : y \in f([0,T]^2 \times [-n,n])\} = 0.$$

(f3) f satisfies a local Lipschitz condition with respect to the last argument.

(φ) φ is a function of bounded variation such that the Riemann–Stieltjes integral satisfies

$$\int_0^T d\varphi(\alpha) \neq 1. \tag{6.2}$$

In order to apply Theorem 2.2 (with $t = a$ on the interval $[0,T]$, $B(v) = \{0\}$, and $Lv(t) = v(0) - \int_0^T v(\alpha)\, d\varphi(\alpha)$), we put $D(A) = E \cap C^1([0,T])$, and $Av = -v'$. Then A is the generator of the C_0-semigroup $U_0(a)v(s) = v(s-a)$ on E, see e.g. [37, Problem 3.4], and so the problem is governed by the evolution operator $U(a,\xi) = U_0(a-\xi)$. Condition (6.2) means exactly that J from (L2) is invertible. We define $F(a,v)(s) = f(a,s,v(s))$. Then $F\colon [0,T] \times E \to E$ is continuous, and so (F1) is trivial. Moreover, since weak convergence in E means pointwise convergence and uniform boundedness, hypothesis (F2) is a consequence of (f1). Condition (f2) implies that the left-hand side of (2.2) vanishes. Condition (f3) implies that, on every set of the form $[0,T]^2 \times [-n,n]$ the function f satisfies a global Lipschitz with some constant ν_n with respect to the last argument. Hence, $F(a,\,\cdot\,)$ satisfies a Lipschitz condition on $B_n(E)$ with constant ν_n, and so condition (F4) follows from Proposition A.1 (e)(i) of the appendix.

In view of the local Lipschitz assumption, one could have probably also used the Hausdorff measure of noncompactness or even more elementary means to obtain the existence of a periodic solutions of (6.1). Therefore, we provide an example in the next section where this appears not to be possible.

7. An example in L_1

We consider the problem

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} + \frac{\partial u(t,x)}{\partial x} \in \Bigg[& f_1\bigg(t,x,\int_0^\infty k_1(t,\xi)u(t,\xi)\,d\xi\bigg), \\ & f_2\bigg(t,x,\int_0^\infty k_2(t,\xi)u(t,\xi)\,d\xi\bigg)\Bigg] + S(t)u(t,\,\cdot\,)(x),\end{aligned} \tag{7.1}$$

with $(t,x) \in [a,b] \times [0,\infty)$, subject to the Cauchy condition

$$u(t,0) = 0 \quad \text{and} \quad u(a,x) = B(x) \tag{7.2}$$

in the state space $E = L_1([0,\infty),\mathbb{R})$. Our hypotheses are the following:

(B) $B \in E = L_1([0,\infty),\mathbb{R})$.

(k) $k_i(t,\,\cdot\,) \in L_\infty([0,\infty),\mathbb{R})$ for almost all $t \in [a,b]$ and $i = 1,2$.

(S1) $S(t)\colon E \to E$ is bounded and linear for every $t \in [a,b]$, $S(\,\cdot\,)y$ is measurable for every $y \in E$, and $\|S(\,\cdot\,)\| \in L_1([a,b],\mathbb{R})$.

(f1) $f_1, f_2\colon [a,b] \times [0,\infty) \times \mathbb{R} \to \mathbb{R}$ are such that for each $y \in E$ there is a measurable $z\colon [a,b] \times [0,\infty) \to \mathbb{R}$ such that for almost all $t \in [a,b]$ the inequalities

$$f_1\bigg(t, x, \int_0^\infty k_1(t,\xi)y(\xi)\,d\xi\bigg) \le z(t,x) \le f_2\bigg(t, x, \int_0^\infty k_2(t,\xi)y(\xi)\,d\xi\bigg)$$

hold for almost all $x \in [0,\infty)$.

(f2) For almost all $t \in [a,b]$ there holds for almost all $x \in [0,\infty)$ that $f_1(t,x,\,\cdot\,)$ is lower semicontinuous and $f_2(t,x,\,\cdot\,)$ is upper semicontinuous, that is

$$f_1(t,x,u) \le \liminf_{v\to u} f_1(t,x,v), \qquad f_2(t,x,u) \ge \limsup_{v\to u} f_2(t,x,v)$$

for every $u \in \mathbb{R}$.

(f3) There exists $\varphi \in L_1([a,b],[0,\infty))$ such that for almost all $t \in [a,b]$ and each $r > 0$ there is $\psi_{t,r} \in L_1([0,\infty),[0,\infty))$ with

$$\sup_{|s|\le\|k_i(t,\,\cdot\,)\|r} |f_i(t,x,s)| \le \psi_{t,r}(x)$$

for almost all $x \in [0,\infty)$ and $i = 1,2$ such that

$$\int_0^\infty \psi_{t,r}(x)\,dx \le \varphi(t)(1+r) \quad \text{for } i = 1,2.$$

It is somewhat surprising that we do not have to assume any measurability of $f_i(\,\cdot\,,\,\cdot\,,s)$ except for the rather mild assumption (f1).

Remark 7.1. From [28, Theorem 4.4.2], we obtain that if $z\colon [a,b]\times[0,\infty) \to \mathbb{R}$ is measurable with $z(t,\,\cdot\,) \in E$ for almost all $t \in E$ then $t \mapsto z(t,\,\cdot\,)$ is measurable as a function from $[a,b]$ into E. Moreover, modifying $z(t,\,\cdot\,)$ on appropriate null sets if necessary, we have also the converse.

Remark 7.2. The measurability of $\|S(\,\cdot\,)\|$ is automatic, because E is separable: There exists a dense $x_n \in B_1(E)$, and so $\|S(\,\cdot\,)\|$ is the supremum of the countable family of measurable functions $\|S(\,\cdot\,)x_n\|$.

In order to apply Theorem 2.2 (more precisely, Remark 2.4) to (7.1), we note that $Ay = -y'$ with domain $D(A) = \{y \in W^{1,1}([0,\infty)) : y(0) = 0\}$ generates in E a C_0-semigroup U_0 of translations, see e.g. [17, p. 420]. Our evolution is thus governed by the evolution operator $U(t,s) = U_0(t-s)$.

For $y \in E$, we define $F(t,y)$ as the set of all functions of the form $f = g + S(t)y$ with $g \in E$ satisfying

$$g(x) \in \bigg[f_1\bigg(t, x, \int_0^\infty k_1(t,\xi)y(\xi)\,d\xi\bigg), f_2\bigg(t, x, \int_0^\infty k_2(t,\xi)y(\xi)\,d\xi\bigg)\bigg] \tag{7.3}$$

for almost all $x \in [0,\infty)$. With this definition of F, we can rewrite (7.1), (7.2) in the abstract form (1.1) (with $Lx = x(0)$, and with $B(x)$ denoting the single element consisting of the above function B).

We verify now that all hypotheses of Theorem 2.2/Remark 2.4 are satisfied. The linear growth estimate (2.5) is immediate from our growth assumptions, and (F1) follows in view of Remark 7.1 from (f1). The only hypotheses which are not so obvious are (F2) and (F4).

To prove (F2), let $y_n \rightharpoonup y$ and $f_n \rightharpoonup f$ in E satisfy $f_n \in F(t, y_n)$. We claim that $f \in F(t, y)$ whenever $t \in [a, b]$ is such that $f_1(t, x, \cdot\,)$ is lower semicontinuous and $f_2(t, x, \cdot\,)$ is upper semicontinuous for almost all $x \in [0, \infty)$. Indeed, noting that $g_n := f_n - S(t)y_n \rightharpoonup g := f - S(t)y \in E$, this means that we have to prove (7.3). Since

$$f_n(x) \in \bigg[f_1\bigg(t, x, \int_0^\infty k_1(t, \xi)y_n(\xi)\,d\xi\bigg), f_2\bigg(t, x, \int_0^\infty k_2(t, \xi)y_n(\xi)\,d\xi\bigg)\bigg],$$

this follows by the Containment Theorem 4.4, because

$$\ell_i(z) = \int_0^\infty k_i(t, \xi)z(\xi)\,d\xi$$

are bounded linear functionals on E and thus satisfy $\ell_i(y_n) \to \ell_i(y)$ for $i = 1, 2$.

To prove (F4), let $M \subseteq E$ be bounded, and let $M_t \subseteq E$ denote the family consisting of all $g \in E$ for which there is some $y \in M$ with (7.3). Then M_t is dominated by some integrable function and thus is uniformly integrable. By the Dunford–Pettis theorem, it follows that M_t is weakly relatively compact, hence $\beta(M_t) = 0$. Since $F(t, M) \subseteq M_t + S(t)M$ and $S(t)M \subseteq F(t, M) - M_t$, we obtain from (2.6) that

$$\beta(F(t, M)) = \beta(S(t)M) \le \|S(t)\|\,\beta(M), \tag{7.4}$$

where we used Proposition A.1 for the last inequality.

Remark 7.3. Even in case $M = \{y\}$ the set M_t fails to be relatively compact in E, in general. Hence, $F(t, y)$ is not relatively compact, in general, and thus one cannot prove an analogous estimate to (1.5) when β is replaced by the Hausdorff measure of noncompactness. In particular, none of the well-known existence results employing the Hausdorff or Kuratowski measure of noncompactness does apply directly.

Remark 7.4. The first equality in (7.4) implies that the operator $F(t, \cdot\,)$ is weakly compact if and only if $S(t)$ is weakly compact. In particular, if we would have only considered weakly compact maps in our main theorem instead of dealing with the De Blasi measure of noncompactness, the most natural choices for $S(t)$ (like multiplication operators) would have been excluded.

Remark 7.5. If the function $\|S(\,\cdot\,)\|$ is unbounded then also the set $F([a, b] \times B_1(E))$ is unbounded. It follows that (1.4) fails spectacularly, because the left-hand is not even finite. Thus, our relaxation of (1.4) to (1.5) (actually even to (F4)) is not only a theoretical improvement. In fact, if we would have required (1.4) in our main theorem, we could not even have used it in case $\|S(t)\| = 1/\sqrt{t-a}$ for $t > a$ to obtain a local solution in some interval $[a, a+\varepsilon]$.

Appendix A. The condition (1.5)

Let E_1 and E_2 be Banach spaces, and $F\colon B_R(E_1) \multimap E_2$. In this section, we discuss sufficient conditions for the estimate

$$\beta(F(M)) \le \nu\beta(M) \quad \text{for all } M \subseteq B_r(E_1),$$

where $0 \le r \le R \le \infty$. If such an estimate holds, we denote the smallest number $\nu \ge 0$ with this property by $[F]_r$; otherwise, we put $[F]_r = \infty$.

Recall that if β is replaced by the Kuratowski measure of noncompactness (or in case $r \le R/2$ by the Hausdorff measure of noncompactness), then one has an estimate of such a type if F is single-valued and a compact perturbation of a Lipschitz map. We show now that one obtains a similar result also for multivalued maps. In contrast to the Kuratowski/Hausdorff measure of noncompactness, it suffices for β that the values $F(u)$ are *weakly* relatively compact.

Recall that the Hausdorff distance of two nonempty sets $M_1, M_2 \subseteq E_i$ is defined by

$$d_H(M_1, M_2) := \max\Big\{ \sup_{u\in M_1} \operatorname{dist}(u, M_2), \sup_{v\in M_2} \operatorname{dist}(v, M_1) \Big\}.$$

We call $F|_M$ weakly sequentially upper semicontinuous in the uniform sense at $u \in M \subseteq E_1$ if for every sequence $u_n \in M$ with $u_n \rightharpoonup u \in M$ and every sequence $v_n \in F(u_n)$ there is a sequence $w_n \in F(u)$ with $v_n - w_n \rightharpoonup 0$.

Proposition A.1.

(a) *If $F(B_r(E_1))$ is weakly relatively compact then $[F]_r = 0$.*
(b) $[F+G]_r \le [F]_r + [G]_r$.
(c) $[F \cup G]_r = \max\{[F]_r, [G]_r\}$.
(d) $[\lambda F]_r = |\lambda|\,[F]_r$ *for every $\lambda \in \mathbb{R}$ (put $0\cdot\infty := 0$).*
(e) *Assume that $F(u)$ is nonempty and weakly relatively compact for every $u \in B_r(E_1)$, and that at least one of the following holds:*
  (i) *$F|_{B_r(E_1)}$ is weakly sequentially upper semicontinuous in the uniform sense at every point of $B_r(E_1)$.*
  (ii) *F sends weakly compact subsets of $B_r(E_1)$ into weakly relatively compact sets.*

  If F satisfies the Lipschitz type condition

  $$d_H(F(u), F(v)) \le \nu\,\|u-v\| \quad \text{for all } u, v \in B_r(E_1)$$

  then $[F]_{r/2} \le \nu$.

In particular, if $F = F_1 + F_2$ with F_1 as in (a) *and F_2 as in* (e), *then $[F]_r \le \nu$ for every $r \in (0, R/2)$.*

Proof. Assertion (a) follows from the fact that $\beta(M) = 0$ if M is weakly relatively compact. Assertions (b)–(d) follow from $\beta(M_1+M_2) \le \beta(M_1)+\beta(M_2)$, $\beta(M_1 \cup M_2) = \max\{\beta(M_1), \beta(M_2)\}$, and $\beta(\lambda M) = |\lambda|\,\beta(M)$, respectively.

For the proof of the last assertion, we first show that in both cases $F(W)$ is weakly relatively compact if $W \subseteq B_r(E_1)$ is weakly compact. Indeed, let $v_n \in F(W)$, say $v_n \in F(u_n)$. Passing to a subsequence if necessary, we can assume by the Eberlein–Šmulian theorem that $u_n \rightharpoonup u$, and so there is a sequence $w_n \in F(u)$ with $v_n - w_n \rightharpoonup 0$. Since $F(u)$ is weakly relatively compact, we can assume that $w_n \rightharpoonup w$, and so $v_n \rightharpoonup v$. Hence, $F(W)$ is weakly sequentially relatively compact and thus relatively compact by the Eberlein–Šmulian theorem.

Now let $M \subseteq B_{r/2}(E)$. Then $\beta(M) \le r/2$. Let $\varepsilon \in (\beta(M), r/2]$ be arbitrary; in case $\beta(M) = r/2$ put $\varepsilon := r/2$. Then there is some weakly compact $W \subseteq E_1$ such that $M \subseteq W + B_\varepsilon(E_1)$. In view of $\varepsilon \le r/2$ and $M \subseteq B_{r/2}(E_1)$, this is still true when we replace W by $W \cap B_r(E)$. Hence, it is no loss of generality to assume that $W \subseteq B_r(E)$. Since $W_0 = F(W)$ is weakly relatively compact, its weak closure W_1 is weakly compact. The Lipschitz type condition implies that $F(M) \subseteq W_0 + B_{\nu\varepsilon}(E_2) \subseteq W_1 + B_{\nu\varepsilon}(E_2)$, and so $\beta(F(M)) \le \nu\varepsilon$. □

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