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INFINITELY MANY SOLUTIONS FOR QUASILINEAR SCHRÖDINGER EQUATIONS UNDER BROKEN SYMMETRY SITUATION

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Abstract. In this paper, we study the existence of infinitely many solutions for the quasilinear Schrödinger equations

$$\begin{cases} -\Delta u - \Delta(|u|^\alpha)|u|^{\alpha-2}u = g(x,u) + h(x,u) & \text{for } x \in \Omega, \\ u = 0 & \text{for } x \in \partial\Omega, \end{cases}$$

where $\alpha \geq 2$, $g, h \in C(\Omega \times \mathbb{R}, \mathbb{R})$. When g is of superlinear growth at infinity in u and h is not odd in u, the existence of infinitely many solutions is proved in spite of the lack of the symmetry of this problem, by using the dual approach and Rabinowitz perturbation method. Our results generalize some known results and are new even in the symmetric situation.

1. Introduction and main results

Consider the following quasilinear Schrödinger equation:

$$\begin{cases} -\Delta u - \Delta(|u|^\alpha)|u|^{\alpha-2}u = g(x,u) + h(x,u) & \text{for } x \in \Omega, \\ u = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\alpha \geq 2$, $g, h \in C(\Omega \times \mathbb{R}, \mathbb{R})$, and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain.

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In recent years, the quasilinear Schrödinger equation has been involved in several models of mathematical physics (see [8], [9], [15]). Notice that equation (1.1) is the Euler–Lagrange equation associated with the energy functional $J\colon E \to \mathbb{R}$ given by

$$
(1.2)\quad J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2\alpha}\int_\Omega |\nabla(|u|^\alpha)|^2\,dx - \int_\Omega G(x,u)\,dx - \int_\Omega H(x,u)\,dx,
$$

where E denotes the Hilbert space $H_0^1(\Omega)$ equipped with the inner product

$$
(u,v) = \int_\Omega \nabla u \nabla v\,dx, \quad u,v \in E.
$$

By direct computation, we have

$$
(1.3)\qquad \frac{1}{2\alpha}\int_\Omega |\nabla(|u|^\alpha)|^2\,dx = \frac{\alpha}{2}\int_\Omega |u|^{2(\alpha-1)}|\nabla u|^2\,dx, \quad u \in E.
$$

In view of (1.2) and (1.3),

$$
J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\alpha}{2}\int_\Omega |u|^{2(\alpha-1)}|\nabla u|^2\,dx - \int_\Omega G(x,u)\,dx - \int_\Omega H(x,u)\,dx,
$$

for $u \in E$. By (1.4), the energy functional J could be naturally defined on

$$
X = \bigg\{ u \in H_0^1(\Omega) \;\bigg|\; \int_\Omega |u|^{2(\alpha-1)}|\nabla u|^2\,dx < \infty \bigg\},
$$

which is not a vector space. So there is no suitable space on which the energy functional J is well-defined. In recent years several methods have been developed to overcome this difficulty, such as the constrained minimization (see [10]), Nehari method (see [7], [11], [18]), change of variables (dual approach) (see [1], [7], [23], [25], [26]), perturbation method (see [12], [13], [24]). Recently, Liu and Zhao [14] considered the existence of infinitely many solutions for a more general quasilinear equation

$$
\begin{cases}
D_j\bigg(\displaystyle\sum_{i,j=1}^N a_{ij}(x,u)D_iu\bigg) - \dfrac{1}{2}\displaystyle\sum_{i,j=1}^N D_s a_{ij}(x,u)D_iuD_ju + |u|^{p-2}u + f = 0 & \\
 & \text{for } x \in \Omega, \\
u = 0 & \text{for } x \in \partial\Omega,
\end{cases}
$$

where $D_i := \frac{\partial}{\partial x_i}$, $i = 1,\ldots,N$, $D_s a_{ij}(x,s) = \frac{\partial}{\partial s}\,a_{ij}(x,s)$. They treated the case $f \neq 0$ as a perturbation from a symmetric equation. Under some suitable conditions, they showed the existence of infinitely many solutions for this quasilinear equation. Similar questions under symmetry breaking situation have been studied also for the problems of elliptic type, Hamiltonian systems and ordinary differential equations (see [3]–[6], [16], [19]–[22]).

But it should be noted that $|u|^{p-2}u$ is a special form of function, which satisfies the classical condition (AR) due to Ambrosetti and Rabinowitz. The condition (AR) is a convenient hypothesis since it achieves the mountain pass geometry as well as fulfils the Palais–Smale condition, but this condition is far too restrictive. There are many functions not satisfying (AR). For example, let

$$g(x,t) = 2\alpha\theta(x)|t|^{2(\alpha-1)}t\left[\ln(1+|t|^{2\alpha}) + \frac{|t|^{2\alpha}}{1+|t|^{2\alpha}}\right], \quad (x,t)\in\Omega\times\mathbb{R}, \tag{1.4}$$

where $\theta\colon\overline{\Omega}\to\mathbb{R}$ is a bounded continuous function with $\inf_{x\in\overline{\Omega}}\theta(x)>0$. To the best of our knowledge, the question whether infinitely many solutions persist for system (1.1) with functions not satisfying (AR) with broken symmetry is unsettled. In this paper, we give a positive answer to this question. But this question is different from the case discussed in [14], those methods cannot be applied directly to obtain our results. Our main tools are based on the dual approach and Rabinowitz perturbation method introduced in [16].

Theorem 1.1. *Assume that g and h satisfy the following conditions*:

(g_1) $g\in C(\Omega\times\mathbb{R},\mathbb{R})$ *and there exists* $2\alpha<p<2^*\alpha$ *if* $N\ge 3$ *or* $2\alpha<p<\infty$ *if* $N=1,2$ *such that*

$$|g(x,t)|\le C_0(1+|t|^{p-1}),\quad (x,t)\in\Omega\times\mathbb{R};$$

(g_2) *there exists a positive constant* $r_0>0$ *such that*

$$G(x,t)\ge 0,\quad (x,t)\in\Omega\times\mathbb{R}\quad\text{and}\quad |t|\ge r_0,$$

and

$$\lim_{|t|\to\infty}\frac{G(x,t)}{|t|^{2\alpha}}=\infty,\quad \text{a.e. } x\in\Omega;$$

where $G(x,t):=\int_0^t g(x,s)\,ds$*;*

(g_3) *there exist constants* $C_1>0$ *and* $\kappa>\max\{1,N/2\}$ *such that*

$$|G(x,t)|^{\kappa}\le C_1|t|^{2\alpha\kappa}\overline{G}(x,t),\quad (x,t)\in\Omega\times\mathbb{R},\ |t|\ge r_0,$$

where $\overline{G}(x,t):=(2\alpha)^{-1}tg(x,t)-G(x,t)$*;*

(g_4) *there exists a positive constant* $C_2>0$ *such that*

$$\overline{G}(x,t)\ge C_2(|t|^{2\alpha}-1),\quad (x,t)\in\Omega\times\mathbb{R};$$

(g_5) $g(x,t)=-g(x,-t)$ *for* $(x,t)\in\Omega\times\mathbb{R}$*;*

(h_1) $h\in C(\Omega\times\mathbb{R},\mathbb{R})$ *and there exist constants* $C_3>0$ *and* $1<\sigma<2\alpha$ *such that*

$$|h(x,t)|\le C_3(1+|t|^{\sigma-1}),\quad (x,t)\in\Omega\times\mathbb{R};$$

(h_2) *the constants* p,α *and* σ *satisfy*

$$\frac{2\alpha N-p(N-2)}{N(p-2\alpha)}>\frac{2\alpha}{2\alpha-\sigma}.$$

Then system (1.1) *has an unbounded sequence of solutions.*

Corollary 1.2. *Assume that g and h satisfy* (g_1)–(g_5), (h_1) *and the following assumption*:

(h_3) $h(x,t) = -h(x,-t)$ *for* $(x,t) \in \Omega \times \mathbb{R}$.

Then there exists an unbounded sequence of solutions for system (1.1).

The plan of this paper is as follows. In Section 2 we provide some preliminary materials. We prove our main results by the use of the dual approach and Rabinowitz perturbation method in Section 3. In the last section an example is given to illustrate our results.

Notation. Throughout the paper, we denote by C_n various positive constants, which may vary from line to line and are not essential to the proof.

2. Preliminaries

For any $s \in [1, 2^*]$, $L^s(\Omega)$ is the usual Lebesgue space with the norm

$$\|u\|_s := \bigg(\int_\Omega |u|^s\,dx\bigg)^{1/s},$$

and $H_0^1(\Omega)$ is the usual Sobolev space with the norm

$$\|u\| := \bigg(\int_\Omega |\nabla u|^2\,dx\bigg)^{1/2},$$

we denote the Hilbert space $H_0^1(\Omega)$ by E. It is well-known that E is continuously embedded into $L^s(\Omega)$ for $s \in [1, 2^*]$, i.e. there exist constants $\tau_s > 0$ such that

$$\|u\|_s \le \tau_s \|u\|, \quad u \in E,\ s \in [1, 2^*].$$

Moreover, $E \hookrightarrow L^s(\Omega)$ is compact for $s \in [1, 2^*)$.

It is obvious that the second order differential operator with the Dirichlet boundary condition is a selfadjoint operator, and there exists a sequence of eigenvalues (counted with multiplicity) $\lambda_1 < \lambda_2 < \ldots \to \infty$, and the corresponding system of normalized eigenfunctions $\{e_n : n \in \mathbb{N}\}$ forming an orthogonal basis in E. Hereafter, let $E_n := \operatorname{span}\{e_1, \ldots, e_n\}$ and $E_n^\perp$ be the orthogonal complement of E_n in E.

Inspired by the transformation initially introduced in [9], the function f can be defined by

$$\begin{aligned} f'(t) &= \frac{1}{\sqrt{1+\alpha|f(t)|^{2(\alpha-1)}}} && \text{for } t \in [0,+\infty),\\ f(-t) &= -f(t) && \text{for } t \in (-\infty, 0]. \end{aligned}$$

Next we collect some useful properties of the function $f\colon \mathbb{R} \to \mathbb{R}$, which will be used frequently in the sequel of the paper. Proofs can be found in [1].

LEMMA 2.1. *The function f and its derivative have the following properties*:

(f_1) *f is a uniquely defined C^∞ function and it is invertible*;

(f_2) *$0 < f'(t) \le 1$ and $|f(t)| \le |t|$, for all $t \in \mathbb{R}$;*

(f_3) *$\lim_{t\to 0} |f(t)|/|t| = 1$ and $\lim_{t\to\infty} |f(t)|^\alpha/|t| = \sqrt{\alpha}$;*

(f_4) *there exists a positive constant C_0 such that*

$$|f(t)|^{\alpha-1} f'(t) \le C_0, \quad \text{for all } t \in \mathbb{R};$$

(f_5) *$f''(t)f(t) = (\alpha-1)(f'(t))^2((f'(t))^2-1)$, for all $t \in \mathbb{R}$.*

Therefore, after change of variables, we obtain the following functional:

$$I(v) := J(f(v)) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega G(x,f(v))\,dx - \int_\Omega H(x,f(v))\,dx. \tag{2.1}$$

Moreover, for any $v, w \in E$,

$$\langle I'(v), w\rangle = (v,w) - \int_\Omega g(x,f(v))f'(v)w\,dx - \int_\Omega h(x,f(v))f'(v)w\,dx. \tag{2.2}$$

By a standard argument which is similar to Lemma 2.6 *and Remark* 2.7 *in* [1], *if $v \in E$ is a critical point of the functional I, then $u = f(v) \in E$ and u is a weak solution of* (1.1).

In order to define a suitable modified functional, we prove the following lemma.

LEMMA 2.2. *Under the hypotheses of Theorem* 1.1, *there exists a positive constant A depending on α such that if v is a critical point of I,*

$$\int_\Omega |f(v)|^{2\alpha}\,dx \le A(I^2(v)+1)^{1/2}. \tag{2.3}$$

PROOF. Since v is a critical point of I, by (f_5), (g_4), (h_1), (2.1) and (2.2),

$$\begin{aligned} I(v) - \frac{1}{2\alpha}\left\langle I'(v), \frac{f(v)}{f'(v)}\right\rangle &> \int_\Omega \overline{G}(x,f(v))\,dx - C_4\bigg(\int_\Omega |f(v)|^\sigma\,dx + 1\bigg) \\ &> C_5\int_\Omega |f(v)|^{2\alpha}\,dx - C_6. \end{aligned} \tag{2.4}$$

Then (2.3) follows from (2.4) and the Young inequality. □

Next we introduce a cut-off function $\zeta \in C^\infty(\mathbb{R},\mathbb{R})$ such that

$$\begin{cases} \zeta(t) = 1 & \text{for } t \in (-\infty, 1], \\ 0 \le \zeta(t) \le 1 & \text{for } t \in (1,2), \\ \zeta(t) = 0 & \text{for } t \in [2,\infty), \\ |\zeta'(t)| \le 2 & \text{for } t \in \mathbb{R}. \end{cases} \tag{2.5}$$

With the help of this cut-off function, define

$$P(v) = 2A(I^2(v)+1)^{1/2}, \qquad \phi(v) = \zeta\bigg(P^{-1}(v)\int_\Omega |f(v)|^{2\alpha}\,dx\bigg). \tag{2.6}$$

If v is a critical point of I, by (2.3), (2.5) and (2.6), $\phi(v) = 1$. Set

$$\overline{I}(v) = \frac{1}{2}\|v\|^2 - \int_\Omega G(x, f(v))\,dx - \phi(v)\int_\Omega H(x, f(v))\,dx. \tag{2.7}$$

Since ζ is a smooth function, we have $\overline{I} \in C^1(E, \mathbb{R})$ and

$$\begin{aligned}\langle \overline{I}'(v), w\rangle = (v, w) &- \int_\Omega g(x, f(v))f'(v)w\,dx \\ &- \phi(v)\int_\Omega h(x, f(v))f'(v)w\,dx - \langle \phi'(v), w\rangle \int_\Omega H(x, f(v))\,dx,\end{aligned} \tag{2.8}$$

for $v,\ w \in E$. It is obvious that $I(v) = \overline{I}(v)$ if v is a critical point of I.

Lemma 2.3. *Assume all the hypotheses of Theorem* 1.1 *hold. Then*:

(H_1) *there is a positive constant C_7 such that*

$$|\overline{I}(v) - \overline{I}(-v)| \leq C_7(|\overline{I}(v)|^{\sigma/2\alpha} + 1), \quad v \in E;$$

(H_2) *there exists a positive constant M_1 such that if $\overline{I}(v) \geq M_1$ and $\overline{I}'(v) = 0$, then $\overline{I}(v) = I(v)$ and $I'(v) = 0$;*

(H_3) *there exists a positive constant $M_2 \geq M_1$ such that for any $c > M_2$, then $\overline{I}$ satisfies the $(C)_c$ condition at c.*

Proof. If $v \in \operatorname{supp}\phi$, by (2.5) and (2.6),

$$\int_\Omega |f(v)|^{2\alpha}\,dx \leq 4A(I^2(v) + 1)^{1/2} \leq 4A(|I(v)| + 1). \tag{2.9}$$

By (h_1) and direct computation, we have

$$\bigg|\int_\Omega H(x, f(v))\,dx\bigg| \leq C_8\bigg(\int_\Omega |f(v)|^{2\alpha}\,dx + 1\bigg)^{\sigma/2\alpha}. \tag{2.10}$$

It follows from (2.9) and (2.10) that

$$\bigg|\int_\Omega H(x, f(v))\,dx\bigg| \leq C_9(|I(v)|^{\sigma/2\alpha} + 1). \tag{2.11}$$

In view of (2.1), (2.7) and (2.11),

$$|I(v)| \leq |\overline{I}(v)| + \bigg|\int_\Omega H(x, f(v))\,dx\bigg| \leq |\overline{I}(v)| + 2C_9(|I(v)|^{\sigma/2\alpha} + 1). \tag{2.12}$$

In combination with (h_1) and (2.12),

$$|I(v)| \leq C_{10}(|\overline{I}(v)| + 1). \tag{2.13}$$

It follows from (2.11) and (2.13) that

$$\bigg|\int_\Omega H(x, f(v))\,dx\bigg| \leq C_{11}(|\overline{I}(v)|^{\sigma/2\alpha} + 1), \quad v \in \operatorname{supp}\phi. \tag{2.14}$$

By a similar estimate, we also have

$$\bigg|\int_\Omega H(x, f(-v))\,dx\bigg| \leq C_{12}(|\overline{I}(v)|^{\sigma/2\alpha} + 1), \quad -v \in \operatorname{supp}\phi. \tag{2.15}$$

It follows from (g_5), (2.7), (2.14) and (2.15) that (H_1) holds.

To prove (H_2), it suffices to show that v is a critical point of $\overline{I}$ with $\overline{I}(v) \geq M_1$, then

$$P^{-1}(v) \int_\Omega |f(v)|^{2\alpha} \, dx < 1. \tag{2.16}$$

Next we show that (2.16) holds. It follows from (2.8) that

$$\begin{aligned}\left\langle \overline{I}'(v), \frac{f(v)}{f'(v)} \right\rangle &= \alpha\|v\|^2 - (\alpha-1)\int_\Omega (f'(v))^2 |\nabla v|^2 \, dx \\ &\quad - \int_\Omega g(x, f(v)) f(v) \, dx - \phi(v) \int_\Omega h(x, f(v)) f(v) \, dx \\ &\quad - \left\langle \phi'(v), \frac{f(v)}{f'(v)} \right\rangle \int_\Omega H(x, f(v)) \, dx,\end{aligned} \tag{2.17}$$

where

$$\begin{aligned}\langle \phi'(v), \omega \rangle &= \zeta'(\theta(v)) P^{-2}(v) \\ &\quad \cdot \left[2\alpha P(v) \int_\Omega |f(v)|^{2(\alpha-1)} f(v) f'(v) \omega \, dx - (2A)^2 \theta(v) I(v) \langle I'(v), \omega \rangle \right],\end{aligned}$$

and

$$\theta(v) := P^{-1}(v) \int_\Omega |f(v)|^{2\alpha} \, dx. \tag{2.18}$$

If $v \notin \operatorname{supp} \phi$, $\phi(v) = \phi'(v) = 0$, then (2.17) reduces to

$$\begin{aligned}\left\langle \overline{I}'(v), \frac{f(v)}{f'(v)} \right\rangle = \alpha\|v\|^2 - (\alpha-1)\int_\Omega (f'(v))^2 |\nabla v|^2 \, dx& \\ - \int_\Omega g(x, f(v)) f(v) \, dx.&\end{aligned} \tag{2.19}$$

Moreover, if v is a critical point of $\overline{I}$, by (g_3), (2.1), (2.8) and (2.19),

$$\begin{aligned}I(v) - \frac{1}{2\alpha}\left\langle \overline{I}'(v), \frac{f(v)}{f'(v)} \right\rangle > \frac{\alpha-1}{2\alpha}\int_\Omega (f'(v))^2 |\nabla v|^2 \, dx& \\ + \int_\Omega \overline{G}(x, f(v)) \, dx - \int_\Omega H(x, f(v)) \, dx.&\end{aligned} \tag{2.20}$$

In view of (2.4) and (2.20), (2.16) holds. If $v \in \operatorname{supp} \phi$, we regroup terms in (2.17) yielding

$$\begin{aligned}\left\langle \overline{I}'(v), \frac{f(v)}{f'(v)} \right\rangle &= (1 + K_1(v)) \left[\alpha\|v\|^2 - (\alpha-1)\int_\Omega (f'(v))^2 |\nabla v|^2 \, dx \right] \\ &\quad - K_2(v) \int_\Omega H(x, f(v)) \, dx - (1 + K_1(v)) \int_\Omega g(x, f(v)) f(v) \, dx \\ &\quad - (\phi(v) + K_1(v)) \int_\Omega h(x, f(v)) f(v) \, dx,\end{aligned} \tag{2.21}$$

where

$$K_1(v) := (2A)^2 \zeta'(\theta(v)) P^{-2}(v) \theta(v) I(v) \int_\Omega H(x, f(v))\, dx, \tag{2.22}$$

$$K_2(v) := 2\alpha \zeta'(\theta(v))\theta(v). \tag{2.23}$$

Next we prove that

$$K_1(v) \to 0, \quad \text{as } M_1 \to \infty. \tag{2.24}$$

By (2.5), (2.6), (2.11) and (2.22),

$$|K_1(v)| \le 8C_9(|I(v)|^{\sigma/2\alpha} + 1)|I(v)|^{-1}. \tag{2.25}$$

In combination with (2.1) and (2.7),

$$I(v) \ge \overline{I}(v) - \bigg| \int_\Omega H(x, f(v))\, dx \bigg|. \tag{2.26}$$

In view of (2.11) and (2.26),

$$I(v) + C_9|I(v)|^{\sigma/2\alpha} \ge \overline{I}(v) - C_9 \ge \frac{M_1}{2}, \tag{2.27}$$

for M_1 large enough. If $I(v) \le 0$, by (2.27) and the Young inequality,

$$\frac{(2\alpha - \sigma){C_9}^{2\alpha/(2\alpha-\sigma)}}{2\alpha} + \frac{\sigma|I(v)|}{2\alpha} \ge \frac{M_1}{2} + |I(v)|. \tag{2.28}$$

But the above inequality is impossible if M_1 is large enough, e.g. $M_1 \ge (2\alpha - \sigma) \cdot {C_9}^{2\alpha/(2\alpha-\sigma)}/\alpha$. Therefore $I(v) > 0$. Hence it follows from (2.27) that

$$I(v) > \frac{M_1}{4} \quad \text{or} \quad I(v) > \left(\frac{M_1}{2C_9}\right)^{2\alpha/\sigma},$$

which implies that

$$I(v) \to +\infty, \quad \text{as } M_1 \to \infty, \tag{2.29}$$

which together with (2.25) shows that (2.24) holds. Moreover, it follows from (2.5), (2.6) and (2.23) that $|K_2(v)| \le 8\alpha$, $v \in E$.

If v is a critical point of $\overline{I}$ and M_1 is large enough such that $|K_1(v)| < 1/2$, it follows from (2.1) and (2.21) that

$$\begin{aligned} I(v) - \frac{1}{2\alpha(1 + K_1(v))} \left\langle \overline{I}'(v), \frac{f(v)}{f'(v)} \right\rangle &> \frac{\alpha - 1}{2\alpha} \int_\Omega (f'(v))^2 |\nabla v|^2\, dx \\ &+ \int_\Omega \overline{G}(x, f(v))\, dx - 8\alpha \int_\Omega \big[|h(x, f(v)) f(v)| + |H(x, f(v))|\big]\, dx. \end{aligned} \tag{2.30}$$

In view of (h_1), (2.4) and (2.30), we can replace A by a larger constant but smaller than $2A$ in (2.3), then (2.16) holds.

To prove (H_3), first we show that there exists $M_2 > M_1$ such that if $\{v_n\}_{n\in\mathbb{N}} \subset E$ is a sequence such that

$$\overline{I}(v_n) \to c \quad \text{and} \quad \|\overline{I}'(v_n)\|(1 + \|v_n\|) \to 0, \tag{2.31}$$

then (v_n) is bounded. To prove the boundedness of $\{v_n\}$, arguing by contradiction, suppose that $\|v_n\| \to \infty$. Let $w_n = v_n/\|v_n\|$. Then $\|w_n\| = 1$ and $\|v_n\|_s \le \tau_s \|v_n\| = \tau_s$ for $2 \le s < 2^*$. Passing to a subsequence, we assume that $w_n \rightharpoonup w$ in E, then $w_n \to w$ in $L^s(\Omega)$, $2 \le s < 2^*$. For $0 \le a < b$, let

$$\Omega_n(a,b) = \{x \in \Omega : a \le |f(v_n(x))| < b\}. \tag{2.32}$$

In view of (g_4), (h_1), (2.7), (2.8) and (2.21), for n large enough

$$\begin{aligned} c+1 &\ge \overline{I}(v_n) - \frac{1}{2\alpha(1+K_1(v_n))}\left\langle \overline{I}'(v_n), \frac{f(v_n)}{f'(v_n)} \right\rangle \\ &> \frac{1}{2}\int_{\Omega_n(r_0,\infty)} \overline{G}(x, f(v_n))\,dx - C_{13}, \end{aligned} \tag{2.33}$$

where C_{13} is a positive constant independent of n. By (f_3) and (h_1),

$$\int_\Omega |H(x, f(v))|\,dx \le C_{14}(\|v\|^{\sigma/\alpha} + 1). \tag{2.34}$$

It follows from (2.7), (2.31) and (2.34) that

$$\limsup_{n\to\infty} \int_\Omega \frac{|G(x, f(v_n))|}{\|v_n\|^2}\,dx \ge \frac{1}{2}. \tag{2.35}$$

If $w = 0$, then $w_n \to 0$ in $L^s(\Omega)$, $2 \le s < 2^*$, $w_n \to 0$ almost everywhere on Ω. Set $\kappa' = \kappa/(\kappa-1)$. Since $\kappa > \max\{1, N/2\}$, then $2\kappa' \in (2, 2^*)$. It follows from (g_1) and (2.33) that

$$\begin{aligned} &\int_{\Omega_n(r_0,\infty)} \frac{|G(x, f(v_n)|}{|v_n|^2}|w_n|^2\,dx \\ &\le \left[\int_{\Omega_n(r_0,\infty)} \left(\frac{|G(x, f(v_n)|}{f^{2\alpha}(v_n)}\right)^\kappa dx\right]^{1/\kappa} \left[\int_{\Omega_n(r_0,\infty)} |w_n|^{2\kappa'}dx\right]^{1/\kappa'} \\ &\le C_{15}\left[\int_{\Omega_n(r_0,\infty)} \overline{G}(x, f(v_n))\,dx\right]^{1/\kappa} \left(\int_{\Omega_n(r_0,\infty)} |w_n|^{2\kappa'}dx\right)^{1/\kappa'} \\ &\le C_{15}\left(\int_\Omega |w_n|^{2\kappa'}dx\right)^{1/\kappa'} \to 0. \end{aligned} \tag{2.36}$$

Combining (g_1) and (2.36), we have

$$\begin{aligned} \int_\Omega \frac{|G(x, f(v_n)|}{\|v_n\|^2}\,dx = \int_{\Omega_n(0,r_0)} &\frac{|G(x, f(v_n)|}{\|v_n\|^2}\,dx \\ &+ \int_{\Omega_n(r_0,\infty)} \frac{|G(x, f(v_n)|}{|v_n|^2}|w_n|^2\,dx \to 0, \end{aligned}$$

which contradicts (2.35).

Set $\Pi = \{x \in \Omega : w(x) \ne 0\}$. If $w \ne 0$, then $\mathrm{meas}(\Pi) > 0$. Moreover,

$$\lim_{n\to\infty} |v_n(x)| = \infty, \quad \text{a.e. } x \in \Pi. \tag{2.37}$$

It follows from (f_3) and (2.32) that $\Pi \subset \Omega_n(r_0, \infty)$ for large $n \in \mathbb{N}$. By (g_1), (h_1), (f_3), (2.34), (2.37) and Fatou's lemma,

$$\begin{aligned}
0 &= \limsup_{n\to\infty} \frac{\overline{I}(v_n)}{\|v_n\|^2} = \limsup_{n\to\infty} \bigg[\frac{1}{2} - \int_\Omega \frac{G(x, f(v_n))}{\|v_n\|^2}\,dx\bigg] \\
&= \limsup_{n\to\infty} \bigg[\frac{1}{2} - \int_{\Omega_n(0,r_0)} \frac{G(x, f(v_n))}{\|v_n\|^2}\,dx - \int_{\Omega_n(r_0,\infty)} \frac{G(x, f(v_n))}{|v_n|^2}\,|w_n|^2\,dx\bigg] \\
&\leq \limsup_{n\to\infty} \bigg[\frac{1}{2} + C_0(r_0 + r_0^p)\mathrm{meas}(\Omega)\|v_n\|^{-2} - \int_{\Omega_n(r_0,\infty)} \frac{G(x, f(v_n))}{|v_n|^2}\,|w_n|^2\,dx\bigg] \\
&\leq \frac{1}{2} - \liminf_{n\to\infty} \int_{\Omega_n(r_0,\infty)} \frac{G(x, f(v_n))}{f^{2\alpha}(v_n)} \cdot \frac{f^{2\alpha}(v_n)}{|v_n|^2}\,|w_n|^2\,dx \\
&= \frac{1}{2} - \liminf_{n\to\infty} \int_\Omega \frac{G(x, f(v_n))}{f^{2\alpha}(v_n)} \cdot \frac{f^{2\alpha}(v_n)}{|v_n|^2}\,|w_n|^2[\chi_{\Omega_n(r_0,\infty)}(x)]\,dx \\
&\leq \frac{1}{2} - \int_\Omega \liminf_{n\to\infty} \frac{G(x, f(v_n))}{f^{2\alpha}(v_n)} \cdot \frac{f^{2\alpha}(v_n)}{|v_n|^2}|w_n|^2[\chi_{\Omega_n(r_0,\infty)}(x)]\,dx = -\infty,
\end{aligned}$$

which is a contradiction. Thus $\{v_n\}$ is bounded in E.

Since E is a reflexive space, passing to a subsequence, also denoted by $\{v_n\}$, it can be assumed that $v_n \rightharpoonup v_0$, $n \to \infty$. By (f_3) in Lemma 2.1, there exists a positive constant M_3 such that

$$|f(t)| \leq C_{16}|t|^{1/\alpha}, \quad |t| \geq M_3. \tag{2.38}$$

For any $v, w \in E$, by (f_2), (f_4), (2.38) and the Hölder inequality,

$$\begin{aligned}
&\int_\Omega |f(v)|^{p-1} f'(v)|w|\,dx \\
&\quad = \int_{\Omega_0} |f(v)|^{p-1} f'(v)|w|\,dx + \int_{\Omega\setminus\Omega_0} |f(v)|^{p-1} f'(v)|w|\,dx \\
&\quad \leq C_0 C_{16} \int_{\Omega_0} |v|^{(p-\alpha)/\alpha}|w|\,dx + \int_{\Omega\setminus\Omega_0} |v|^{p-1}|w|\,dx \\
&\quad \leq C_0 C_{16}\|v\|_{p/\alpha}^{(p-\alpha)/\alpha}\|w\|_{p/\alpha} + M_3^{p-1}\|w\|_1,
\end{aligned} \tag{2.39}$$

where $\Omega_0 := \{x \in \Omega : |v(x)| \geq M_3\}$. By (2.8), we have

$$\begin{aligned}
\langle \overline{I}'(v_n), v_n - v_0\rangle = (v_n, v_n - v_0) &- \int_\Omega g(x, f(v_n))f'(v_n)(v_n - v_0)\,dx \\
&- \langle \phi'(v_n), v_n - v_0\rangle \int_\Omega H(x, f(v_n))\,dx \\
&- \phi(v_n) \int_\Omega h(x, f(v_n))f'(v_n)(v_n - v_0)\,dx,
\end{aligned}$$

where

$$\begin{aligned}(2.40)\quad \langle\phi'(v_n), v_n - v_0\rangle \int_\Omega H(x, f(v_n))\,dx \\ = 2\alpha\zeta'(\theta(v_n))P^{-1}(v_n)\int_\Omega |f(v_n)|^{2(\alpha-1)}f(v_n)f'(v_n)(v_n - v_0)\,dx \\ \cdot\int_\Omega H(x, f(v_n))\,dx - K_1(v_n)\langle I'(v_n), v_n - v_0\rangle.\end{aligned}$$

If $v_n \notin \operatorname{supp}\phi$, $\phi(v_n) = \phi'(v_n) = 0$. Then

$$\langle \overline{I}'(v_n), v_n - v_0\rangle = (v_n, v_n - v_0) - \int_\Omega g(x, f(v_n))f'(v_n)(v_n - v_0)\,dx.$$

Otherwise, $v_n \in \operatorname{supp}\phi$, in combination with (2.6) and (2.11), we have

$$(2.41)\quad \left|P^{-1}(v_n)\int_\Omega H(x, f(v_n))\,dx\right| \le (2A)^{-1}C_9(|I(v_n)|^{\sigma/2\alpha} + 1)|I(v_n)|^{-1}.$$

When M_2 is large enough, in view of (2.24), (2.29) and (2.41),

$$(2.42)\quad \left|P^{-1}(v_n)\int_\Omega H(x, f(v_n))\,dx\right| \le \frac{1}{16}, \quad |K_1(v_n)| \le \frac{1}{16}.$$

It follows from (g_1), (h_1) and (2.39) that

$$(2.43)\quad \int_\Omega |f(v_n)|^{2\alpha-1}f'(v_n)(v_n - v_0)\,dx \to 0,$$

and

$$(2.44)\quad \int_\Omega g(x, f(v_n))f'(v_n)(v_n - v_0)\,dx \to 0,$$

$$(2.45)\quad \int_\Omega h(x, f(v_n))f'(v_n)(v_n - v_0)\,dx \to 0.$$

In combination with (2.31), (2.40), (2.42)–(2.45), $v_n \to v_0$, $n \to \infty$. □

LEMMA 2.4. *Under assumptions* (g_1), (g_3) *and* (h_1), *for any finite dimensional subspace* $\widetilde{E} \subset E$,

$$\overline{I}(v) \to -\infty, \quad \|v\| \to \infty, \quad v \in \widetilde{E}.$$

PROOF. Arguing indirectly, assume that for some sequence $\{v_n\} \subset \widetilde{E}$ with $\|v_n\| \to \infty$, there is $M > 0$ such that $\overline{I}(v_n) \ge -M$ for all $n \in \mathbb{N}$. Set $w_n = v_n/\|v_n\|$, then $\|w_n\| = 1$. Passing to a subsequence, we can assume that $w_n \rightharpoonup w$ in E. Since $\widetilde{E}$ is a finite dimensional space, then $w_n \to w \in \widetilde{E}$ and $\|w\| = 1$. Hence, we can conclude a contradiction by a similar fashion as in Lemma 2.3.□

3. Construction of minimax sequences and proof of Theorem 1.1

By Lemma 2.4, there exists a strictly increasing sequence of numbers R_n such that $\overline{I}(v) \le 0$ for $v \in E_n \setminus B_{R_n}$, where B_{R_n} denotes the open ball of radius R_n centred at 0 in E, and $\overline{B}_{R_n}$ denotes the closure of B_{R_n} in E. Next we introduce some continuous maps in E. Set

$$\Gamma_n = \big\{\zeta \in C(D_n, E) : \zeta \text{ is odd and } \zeta = \mathrm{id} \text{ on } \partial B_{R_n} \cap E_n\big\}, \tag{3.1}$$

where $D_n := \overline{B}_{R_n} \cap E_n$, and

$$\begin{aligned}\Lambda_n := \big\{\gamma \in C(U_n, E) : \gamma|_{D_n} \in \Gamma_n \text{ and } \gamma = \mathrm{id}\\ \text{for } v \in Q_n := (\partial B_{R_{n+1}} \cap E_{n+1}) \cup ((B_{R_{n+1}} \setminus \overline{B}_{R_n}) \cap E_n)\big\},\end{aligned} \tag{3.2}$$

where

$$U_n := \big\{v = te_{n+1} + \omega : t \in [0, R_{n+1}],\ \omega \in \overline{B}_{R_{n+1}} \cap E_n,\ \|v\| \le R_{n+1}\big\}. \tag{3.3}$$

With the help of these continuous maps, we define two sequences of minimax values

$$b_n = \inf_{\zeta \in \Gamma_n} \max_{v \in D_n} \overline{I}(\zeta(v)), \qquad c_n = \inf_{\gamma \in \Lambda_n} \max_{v \in U_n} \overline{I}(\gamma(v)). \tag{3.4}$$

It is obvious that $c_n \ge b_n$. For the sake of getting the lower bound of the above minimax values, we give an intersection property which has been proved by Rabinowitz in Lemma 1.44 of [16].

Lemma 3.1. *$\zeta(D_n) \cap \partial B_\rho \cap E_{n-1}^\perp \ne \emptyset$ for any $n \in \mathbb{N}$, $\rho < R_n$ and $\zeta \in \Gamma_n$.*

Next we give the lower bounds for b_n.

Lemma 3.2. *There are a positive constant C_{17} and $n_0 \in \mathbb{N}$ such that*

$$b_n \ge C_{17} n^{2\alpha N - p(N-2)/(N(p-2\alpha))}, \quad n \ge n_0. \tag{3.5}$$

Proof. By Lemma 3.1, for any $\zeta \in \Gamma_n$ and $\rho < R_n$, there exists $v_n \in \zeta(D_n) \cap \partial B_\rho \cap E_{n-1}^\perp$, then

$$\max_{v \in D_n} \overline{I}(\zeta(v)) \ge \overline{I}(v_n) \ge \inf_{v \in \partial B_\rho \cap E_{n-1}^\perp} \overline{I}(v). \tag{3.6}$$

In view of (g_1), (g_3) and (f_3) in Lemma 2.1, we have

$$\int_\Omega |G(x, f(v))|\, dx \le C_{18}(\|v\|_{p/\alpha}^{p/\alpha} + 1), \quad v \in E, \tag{3.7}$$

$$\int_\Omega |H(x, f(v))|\, dx \le C_{19}(\|v\|_{\sigma/\alpha}^{\sigma/\alpha} + 1), \quad v \in E. \tag{3.8}$$

In view of (2.7), (3.7) and (3.8),

$$\overline{I}(v) \ge \frac{1}{4}\|v\|^2 - C_{20}(\|v\|_{p/\alpha}^{p/\alpha} + 1). \tag{3.9}$$

By the Gagliardo–Nirenberg inequality, we have

$$\|v\|_{p/\alpha} \le \tau \|v\|^s \|v\|_2^{1-s}, \tag{3.10}$$

where τ is a positive constant and $s = (2p)^{-1}N(p-2\alpha)$. If $v \in E_{n-1}^{\perp}$,

$$\|v\|_2^2 \le \lambda_n^{-1} \|v\|^2. \tag{3.11}$$

By (3.9), (3.10) and (3.11), if $v \in \partial B_\rho \cap E_{n-1}^{\perp}$,

$$\overline{I}(v) \ge \rho^2 \left(\frac{1}{2} - C_{20} \lambda_n^{(s-1)p/(2\alpha)} \rho^{(p-2\alpha)/\alpha} \right) - C_{20}. \tag{3.12}$$

In view of (3.12), choose $\rho_n = (4C_{20})^{\alpha/(2\alpha-p)} \lambda_n^{(1-s)p/(2(p-2\alpha))}$, then

$$\overline{I}(v) \ge \frac{1}{4}\rho_n^2 - C_{20}. \tag{3.13}$$

It follows from (3.4), (3.6) and (3.13) that (3.5) holds. □

By (H_1) in Lemma 2.3 and a similar fashion as in the proof of Proposition 10.46 in [17], we have

Lemma 3.3. *If $c_n = b_n$ for all $n \ge n_0$, where n_0 is a positive integer, there exists a positive constant C_{21} such that*

$$b_n \le C_{21} n^{2\alpha/(2\alpha-\sigma)}. \tag{3.14}$$

In view of (h_2), (3.5) and (3.14), it is impossible that $c_n = b_n$ for all large n. Next we can construct critical values of $\overline{I}$ as follows.

Lemma 3.4. *Suppose $c_n > b_n \ge M_2$ for any n large enough. For any $\delta \in (0, c_n - b_n)$, define*

$$\Lambda_n(\delta) = \{\gamma \in \Lambda_n : \overline{I}(\gamma(v)) \le b_n + \delta \text{ for } v \in D_n\} \tag{3.15}$$

and

$$c_n(\delta) = \inf_{\gamma \in \Lambda_n(\delta)} \max_{v \in U_n} \overline{I}(\gamma(v)). \tag{3.16}$$

Then $c_n(\delta)$ is a critical value of $\overline{I}$.

Proof. First the definition of $\Lambda_n(\delta)$ implies that this set is nonempty. By (3.2) and (3.15),

$$\Lambda_n(\delta) \subset \Lambda_n, \quad c_n \le c_n(\delta). \tag{3.17}$$

By (H_3) in Lemma 2.3, the Deformation Theorem also holds (see [2]). Suppose $c_n(\delta)$ is not a critical value of $\overline{I}$, choose $\overline{\varepsilon} := (c_n - b_n - \delta)/2$, there exists $\varepsilon \in (0, \overline{\varepsilon})$ and $\eta \in C([0,1] \times E, E)$ such that

$$\eta(1, v) = v, \quad \overline{I}(v) \notin [c_n(\delta) - \overline{\varepsilon}, c_n(\delta) + \overline{\varepsilon}], \tag{3.18}$$

$$\eta(1, \overline{I}_{c_n(\delta)+\varepsilon}) \subset \overline{I}_{c_n(\delta)-\varepsilon}. \tag{3.19}$$

By (3.16), there exists $\gamma \in \Lambda_n(\delta)$ such that

$$\max_{v \in U_n} \overline{I}(\gamma(v)) < c_n(\delta) + \varepsilon. \tag{3.20}$$

Define

$$\overline{\gamma}(\,\cdot\,) = \eta(1, \gamma(\,\cdot\,)). \tag{3.21}$$

Next we prove $\overline{\gamma} \in \Lambda_n(\delta)$. It is obvious that $\overline{\gamma} \in C(U_n, E)$. By (3.15) and (3.17),

$$\overline{I}(\gamma(v)) \leq b_n + \delta < c_n - \overline{\varepsilon} \leq c_n(\delta) - \overline{\varepsilon}, \quad v \in D_n,$$

which implies that

$$\overline{I}(\gamma(v)) < c_n(\delta) - \overline{\varepsilon}, \quad v \in D_n. \tag{3.22}$$

In combination with (3.18), (3.21) and (3.22),

$$\overline{\gamma}(v) = \eta(1, \gamma(v)) = \gamma(v), \quad v \in D_n,$$

which yields that

$$\overline{\gamma}|_{D_n} \in \Gamma_n \quad \text{and} \quad \overline{I}(\overline{\gamma}(v)) = \overline{I}(\gamma(v)) \leq b_n + \delta, \quad v \in D_n. \tag{3.23}$$

In view of $\gamma \in \Lambda_n(\delta)$ and the definitions of R_n and R_{n+1},

$$\gamma(v) = v \quad \text{and} \quad \overline{I}(\gamma(v)) \leq 0, \quad v \in Q_n. \tag{3.24}$$

Since $b_n \geq M_2 > 0$ and $c_n(\delta) \geq c_n > b_n$, then $c_n(\delta) > \overline{\varepsilon}$. It follows from (3.18) and (3.24) that

$$\overline{\gamma}(v) = \eta(1, \gamma(v)) = \gamma(v) = v, \quad v \in Q_n. \tag{3.25}$$

In view of (3.23) and (3.25), $\overline{\gamma} \in \Lambda_n(\delta)$. Moreover, by (3.19)–(3.21),

$$\max_{v \in U_n} \overline{I}(\overline{\gamma}(v)) = \max_{v \in U_n} \overline{I}(\eta(1, \gamma(v))) \leq c_n(\delta) - \varepsilon,$$

which is a contradiction to (3.16). □

Proof of Theorem 1.1. Since it is impossible that $c_n = b_n$ for all large n, then we can choose a subsequence $\{n_k\} \subset \mathbb{N}$ such that $c_{n_k} > b_{n_k}$. In view of Lemma 3.2, $c_{n_k} > b_{n_k} > M_2$, when n_k is large enough. It follows from (H_2) in Lemmas 2.3, 3.2 and 3.4 that $\overline{I}$ has an unbounded sequence of critical values which yields infinitely many solutions for system (1.1). □

Proof of Corollary 1.2. First, it follows from (g_5), (h_3) and (2.1) that I is an even functional. Arguing as in (H_3) in Lemma 2.3, we can prove that the functional I satisfies the $(C)_c$ condition. Moreover, by a similar fashion as in the proof of Lemma 3.1, there exists a strictly increasing sequence of numbers R'_n such that $I(v) \leq 0$ for $v \in E_n \setminus B_{R'_n}$. Define

$$\Gamma'_n = \big\{h \in C(D'_n, E) : h \text{ is odd and } h = \mathrm{id} \text{ on } \partial B_{R'_n} \cap E_n\big\},$$

where $D'_n := \overline{B}_{R'_n} \cap E_n$ and $b'_n := \inf_{h\in\Gamma'_n} \max_{v\in D'_n} I(h(v))$. Arguing as in Lemma 3.2, we have $b'_n \to \infty$, as $n \to \infty$. Then there exists $n_0 \in \mathbb{N}$ such that $b'_n > 0$, $n \geq n_0$. If $n \geq n_0$, by a standard argument and the Deformation Theorem, we can also prove b'_n are unbounded critical values of I. $\square$

4. Example

In this section, we give one example to illustrate our result.

Example 4.1. In system (1.1), let Ω be a bounded smooth domain in $\mathbb{R}^4$ and $\alpha = 3$. Let g is given by (1.5) and $h(x,t) = t^2$. Thus all conditions of Theorem 1.1 are satisfied with $N = 4$, $\kappa = 3$, $\sigma = 7/2$, $p = 13/2$. By Theorem 1.1, system (1.1) has an unbounded sequence of infinitely many solutions. But the results in [14] cannot be applied to this example.

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