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# SIGN-CHANGING SOLUTIONS FOR $p$-LAPLACIAN EQUATIONS WITH JUMPING NONLINEARITY AND THE FUČIK SPECTRUM

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Abstract. We study the existence of sign-changing solutions for the $p$-Laplacian equation

$$-\Delta_p u + \lambda g(x)|u|^{p-2}u = f(u), \quad x \in \mathbb{R}^N,$$

where $\lambda$ is a positive parameter and the nonlinear term $f$ has jumping nonlinearity at infinity and is superlinear at zero. The Fučik spectrum plays an important role in the proof. We give sufficient conditions for the existence of nontrivial Fučik spectrum.

## 1. Introduction

In this paper we are concerned with the following $p$-Laplacian equations in $\mathbb{R}^N$ with jumping nonlinearity:

$$\text{(P)} \qquad -\Delta_p u + \lambda g(x)|u|^{p-2}u = f(u), \quad x \in \mathbb{R}^N,$$

where $1 < p < N$, $\lambda > 0$ and $\Delta_p u = \operatorname{div}(|u|^{p-2}u)$ is the $p$-Laplacian operator.

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In the weak form of problem (P) the solution $u \in W^{1,p}(\mathbb{R}^N)$ satisfies

$$\text{(P)} \qquad \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla\varphi\,dx + \lambda\int_{\mathbb{R}^N} g(x)|u|^{p-2}u\varphi\,dx = \int_{\mathbb{R}^N} f(u)\varphi\,dx,$$

for all $\varphi \in W^{1,p}(\mathbb{R}^N)$. Problem (P) has a variational structure given by the functional

$$I(u) = \frac{1}{p}\int_{\mathbb{R}^N}(|\nabla u|^p + \lambda g(x)|u|^p)\,dx - \int_{\mathbb{R}^N} F(u)\,dx, \quad u \in W^{1,p}(\mathbb{R}^N),$$

where $F(t) = \int_0^t f(s)\,ds$. Assume that the potential function $g$ satisfies the conditions:

- ($g_1$) $g \in L^\infty(\mathbb{R}^N)$, $0 \le g \le 1 = \lim_{|x|\to\infty} g(x)$, and $g(x) \not\equiv 1$.
- ($g_2$) There exist $c, m, R_0 > 0$ such that $1 - g(x) \ge c/|x|^m$, for $|x| \ge R_0$.

For the nonlinear term $f$ we make the following assumptions:

- ($f_1$) $f \in C(\mathbb{R}^N,\mathbb{R})$, $f(t)t > 0$, for $t \ne 0$.
- ($f_2$) $\displaystyle\lim_{|t|\to 0}\frac{f(t)}{|t|^{p-2}t} = 0$.
- ($f_3$) There exist $a, b$ such that $\displaystyle\lim_{t\to+\infty}\frac{f(t)}{t^{p-1}} = a$, $\displaystyle\lim_{t\to-\infty}\frac{f(t)}{|t|^{p-2}t} = b$.
- ($f_4$) $\dfrac{f(t)}{|t|^{p-2}t}$ is nondecreasing in $t > 0$ and nonincreasing in $t < 0$.

The condition ($f_3$) means that $f$ has a "jumping" nonlinearity at the infinity.

Let $L_\lambda\colon W^{1,p}(\mathbb{R}^N) \ni u \mapsto -\Delta_p u + \lambda g(x)|u|^{p-2}u \in (W^{1,p}(\mathbb{R}^N))'$, where $(W^{1,p}(\mathbb{R}^N))'$ is the dual space of $W^{1,p}(\mathbb{R}^N)$. By Cuesta et al. [4], the Fučik spectrum of $L_\lambda$ is defined as the set $\sigma(L_\lambda)$ of those $(a,b) \in \mathbb{R}^2$ for which equation (1.1) (or (1.2) in the weak form) has a nontrivial solution $u$:

$$\text{(1.1)} \qquad -\Delta_p u + \lambda g(x)|u|^{p-2}u = a(u^+)^{p-1} - b(u^-)^{p-1}$$

or $u \in W^{1,p}(\mathbb{R}^N)$ satisfies

$$\text{(1.2)} \qquad \int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\nabla\varphi\,dx + \lambda\int_{\mathbb{R}^N} g(x)|u|^{p-2}u\varphi\,dx = \int_{\mathbb{R}^N}(a(u^+)^{p-1} - b(u^-)^{p-1})\varphi\,dx$$

for all $\varphi \in W^{1,p}(\mathbb{R}^N)$, where $u^\pm = \max(\pm u, 0)$. Given $\theta \in (0, \pi/2)$, we define

$$\text{(1.3)} \qquad \rho(\theta) = \inf_{u\in\Sigma}\int_{\mathbb{R}^N}(|\nabla u|^p + \lambda g(x)|u|^p)\,dx,$$

where

$$\Sigma = \{u \mid u \in W^{1,p}(\mathbb{R}^N),\ u = u^+ - u^-, u^\pm \ne 0, \text{ and (1.4), (1.5) hold}\},$$

$$\text{(1.4)} \qquad \int_{\mathbb{R}^N}(\cos\theta(u^+)^p + \sin\theta(u^-)^p)\,dx = 1,$$