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MULTI-BUMP SOLUTIONS FOR A CLASS OF KIRCHHOFF TYPE PROBLEMS WITH CRITICAL GROWTH IN $\mathbb{R}^N$

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Abstract. Using variational methods, we establish existence of multi-bump solutions for a class of Kirchhoff type problems

$$-\bigg(1+b\int_{\mathbb{R}^N}|\nabla u|^p\,dx\bigg)\Delta_p u+(\lambda V(x)+Z(x))u^{p-1}=\alpha f(u)+u^{p^*-1},$$

where f is a continuous function, $V,Z\colon\mathbb{R}^N\to\mathbb{R}$ are continuous functions verifying some hypotheses. We show that if the zero set of V has several isolated connected components $\Omega_1,\ldots,\Omega_k$ such that the interior of Ω_i is not empty and $\partial\Omega_i$ is smooth, then for $\lambda>0$ large enough there exists, for any non-empty subset $\Gamma\subset\{1,\ldots,k\}$, a bump solution trapped in a neighbourhood of $\bigcup_{j\in\Gamma}\Omega_j$. The results are also new for the case $p=2$.

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1. Introduction

In this paper, we are concerned with the existence of multi-bump solutions of Kirchhoff type problems

$$
\begin{cases} -\left(1+b\displaystyle\int_{\mathbb{R}^N}\frac{1}{p}|\nabla u|^p\,dx\right)\Delta_p u+(\lambda V(x)+Z(x))u^{p-1}=\alpha f(u)+u^{p^*-1}, \\ \hfill x\in\mathbb{R}^N, \\ u\in W^{1,p}(\mathbb{R}^N), \quad u>0, \end{cases} \tag{1.1}
$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $p^* = Np/(N-p)$ is the so-called Sobolev critical exponent, $2 < p < N$, $N \geq 3$, b is a positive constant, $\lambda > 0$ is a real parameter and f is a continuous function, $V, Z\colon \mathbb{R}^N \to \mathbb{R}$ are continuous functions with $V(x) \geq 0$ for all $x \in \mathbb{R}^N$, $\Omega = \operatorname{int} V^{-1}(0)$ has k connected components denoted by Ω_j, $j \in \{1, \ldots, k\}$, $V^{-1}(\{0\}) = \overline{\Omega}$ and $\partial\Omega$ is smooth.

For the special case of problems (1.1), i.e. without $(\lambda V(x)+Z(x))u$, problem (1.1) reduces to the following Dirichlet problem of Kirchhoff type:

$$
\begin{cases} -\left(1+b\displaystyle\int_{\Omega}|\nabla u|^2\,dx\right)\Delta u=h(u), \quad x\in\Omega, \\ u|_{\partial\Omega}=0, \end{cases} \tag{1.2}
$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. Such problems are often referred to as being nonlocal because of the presence of the term $\int_\Omega |\nabla u|^2\,dx\Delta u$ which implies that the equation in (1.2) is no longer a pointwise identity. This phenomenon provokes some mathematical difficulties, which make the study of such a class of problems particularly interesting. On the other hand, we have its physical motivation. Indeed, this problem is a generalization of a model introduced by Kirchhoff [32]. More precisely, Kirchhoff proposed a model given by the equation

$$
\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.3}
$$

where ρ, ρ_0, h, E, L are constants, which extends the classical D'Alembert's wave equation, by considering the effects of changes in the length of strings during the vibrations. Equation (1.2) is related to the stationary analogue of problem (1.3). It received much attention only after Lions [33] proposed an abstract framework to the problem. Some important and interesting results can be found, for example, in [9], [16], [19], [20], [29], [37], [41]. In [9], Arosio and Panizzi studied the Cauchy–Dirichlet type problem related to (1.3) in the Hadamard sense as a special case of an abstract second-order Cauchy problem in a Hilbert space. Ma and Rivera [37] obtained positive solutions of such problems by using variational methods. Perera and Zhang [41] obtained a nontrivial solution of (1.2) via