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HOMOCLINIC ORBITS OF FIRST ORDER NONLINEAR HAMILTONIAN SYSTEMS WITH ASYMPTOTICALLY LINEAR NONLINEARITIES AT INFINITY

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Abstract. By using variational methods and critical point theory, in particular, a generalized weak linking theorem, we study a first order nonlinear Hamiltonian system with asymptotically linear nonlinearity at infinity. We obtain the existence of ground state homoclinic orbits for this nonlinear Hamiltonian system. In particular, we obtain a *necessary and sufficient condition* for the existence of ground state homoclinic orbits. To the best of our knowledge, there is no published result focusing on necessary and sufficient conditions of the existence of ground state homoclinic orbits for this system.

1. Introduction and main results

In this paper, we consider the following first order nonlinear Hamiltonian system:

$$-J\dot{u}(t) - L(t)u = \nabla W(t, u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ denotes the standard symplectic matrix, $L(t)$ is a given continuous T-periodic and symmetric $2N \times 2N$-matrix-value function and $W(t, u) \in$

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$C^1(\mathbb{R}\times\mathbb{R}^{2N},\mathbb{R})$ is T-periodic in the t-variable and $\nabla W(t,u)$ denotes its gradient with respect to the u variable. Recall that a solution u of (1.1) is a homoclinic orbit if $u\neq 0$ and $u\to 0$ as $|t|\to\infty$.

In this paper, we are interested in the strongly indefinite case for (1.1), that is,

(A$_1$) $\underline{\Lambda} := \sup(\sigma(B)\cap(-\infty,0)) < 0 < \overline{\Lambda} := \inf(\sigma(B)\cap(0,\infty))$, where $B := -J\frac{d}{dt^2} - L(t)$ and $\sigma(B)$ denotes the spectrum of B. Clearly, $\sigma(B)$ is absolutely continuous.

REMARK 1.1. If (A_1) holds and the nonlinearity $W(t,u)$ of (1.1) is *superquadratic* at infinity, i.e.

$$\lim_{|u|\to\infty}\frac{W(t,u)}{|u|^2} = +\infty,$$

the authors of [2] have obtained the existence of *ground state* homoclinic orbits of (1.1) (i.e. solutions corresponding to the least energy of the action functional of (1.1)). Inspired by [2], we study the existence of ground state homoclinic orbits of (1.1) in the case where $W(t,u)$ is *asymptotically quadratic* at infinity, see Theorem 1.3. As we know, the asymptotically quadratic case is very different from the superquadratic case. In particular, we obtain a *necessary and sufficient* condition of the existence of ground state homoclinic orbits for (1.1), see Theorem 1.4.

REMARK 1.2. The *main novelty* of this paper is that we obtain a necessary and sufficient condition for the existence of ground state homoclinic orbits in the strongly indefinite case (A$_1$), see Theorem 1.4. In fact, for the positive definite case, i.e. $\inf\sigma(B) > 0$, we believe that the necessary and sufficient condition can also be obtained.

Let $\widetilde{W}(t,u) := (\nabla W(t,u),u)/2 - W(t,u)$, where $(\,\cdot\,,\,\cdot\,)$ denotes the standard inner product in $\mathbb{R}^{2N}$, and the associated norm is denoted by $|\cdot|$. We assume that

(W$_1$) $|\nabla W(t,u)| = o(|u|)$ as $|u|\to 0$ uniformly in $t\in\mathbb{R}$.

(W$_2$) $W(t,u)\geq 0$ for all $(t,u)\in\mathbb{R}\times\mathbb{R}^{2N}$ and $\widetilde{W}(t,u) > 0$ if $u\in\mathbb{R}^{2N}\setminus\{0\}$.

(W$_3$) $W(t,u) = V(t)|u|^2/2 + F(t,u)$, where $0 < V(t) < +\infty$ and

$$|\nabla F(t,u)| = o(|u|) \quad \text{as } |u|\to\infty \text{ uniformly in } t.$$

(W$_4$) $\widetilde{W}(t,u)\to+\infty$ as $|u|\to+\infty$, and there is a function $P(t)$ ($|P(t)| < +\infty$, for all $t\in\mathbb{R}$) such that

$$\limsup_{|u|\to 0}\frac{|\nabla W(t,u)|^2}{\widetilde{W}(t,u)} = P(t) \quad \text{uniformly in } t.$$

Now, our main results read as follows: