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NONLINEAR, NONHOMOGENEOUS PARAMETRIC NEUMANN PROBLEMS

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Abstract. We consider a parametric nonlinear Neumann problem driven by a nonlinear nonhomogeneous differential operator, with a Carathéodory reaction f which is p-superlinear in the second variable, but not necessarily satisfying the usual in such cases Ambrosetti–Rabinowitz condition. We prove a bifurcation type result describing the dependence of positive solutions on the parameter $\lambda > 0$, show the existence of a smallest positive solution $\overline{u}_\lambda$ and investigate properties of the map $\lambda \mapsto \overline{u}_\lambda$. Finally, we show the existence of nodal solutions.

1. Introduction

In this paper we study the following nonlinear parametric Neumann problem:

$$(\mathrm{P}_\lambda) \qquad \begin{cases} -\operatorname{div} a(Du(z)) + \lambda|u(z)|^{p-2}u(z) = f(z,u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \ \lambda > 0, \end{cases}$$

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$1 < p < \infty$. Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$. The map $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is continuous, strictly monotone and satisfies certain regularity conditions which are listed in hypotheses H(a) (see Section 2). These hypotheses are general enough to incorporate in our framework many differential operators of interest, such as the p-Laplacian. Also $\lambda > 0$ is a parameter and f is a Carathéodory function (i.e. for all $x \in \mathbb{R}$, $z \mapsto f(z,x)$ is measurable and for almost all $z \in \Omega$, $x \mapsto f(z,x)$ is continuous) which exhibits a $(p-1)$-superlinear growth in the second variable, but not necessarily satisfying the usual in such cases Ambrosetti–Rabinowitz condition (AR-condition for short).

Our work is motivated by a recent paper of Motreanu, Motreanu and Papageorgiou [18], who produced constant sign and nodal solutions. Our results complement and improve those of [18]. More precisely, the authors in [18] produced positive solutions for problem (P_λ) but did not give the precise dependence of the set of positive solutions on the parameter $\lambda > 0$. Here, we prove a bifurcation-type theorem for large values of λ, which gives a complete picture of the set of positive solutions as the parameter varies. Moreover, in [18] nodal (that is, sign-changing) solutions were produced only for the particular case of equations driven by the p-Laplacian. In contrast, here we generate nodal solutions for the general case. We stress that the p-Laplacian differential operator is homogeneous, while the differential operator in (P_λ) is not. Hence, the methods and techniques used in [18] fail in the present setting, and so a new approach is needed. Finally, we mention that a bifurcation near infinity for a different class of p-Laplacian Dirichlet problems was recently produced by Gasinski and Papageorgiou [12].

In the next section, we review the main mathematical tools which will be used in this paper. We also present the hypotheses on the map $y \mapsto a(y)$ and state some useful consequences of them.

2. Mathematical background

Let $(X, \|\cdot\|)$ be a Banach space and X^* be its topological dual. By $\langle\,\cdot\,,\cdot\,\rangle$ we denote the duality brackets for the pair (X^*, X) and $\xrightarrow{\mathrm{w}}$ will designate the weak convergence.

Let $\varphi \in C^1(X)$. We say that $x^* \in X$ is a critical point of φ if $\varphi'(x^*) = 0$. If $x^* \in X$ is a critical point of of φ then $c = \varphi(x^*)$ is a critical value of φ. We say that φ satisfies the "Palais–Smale condition" (PS-condition for short), if the following holds:

> "Every sequence $\{u_n\}_{n\geq1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq1}$ is bounded in $\mathbb{R}$ and $\varphi'(u_n) \to 0$ in X^* as $n \to \infty$ admits a strongly convergent subsequence."