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ON THE ASYMPTOTIC RELATION OF TOPOLOGICAL AMENABLE GROUP ACTIONS

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Abstract. For a topological action Φ of a countable amenable orderable group G on a compact metric space we introduce a concept of the asymptotic relation $\mathbf{A}(\Phi)$ and we show that $\mathbf{A}(\Phi)$ is non-trivial if the topological entropy $h(\Phi)$ is positive. It is also proved that if the Pinsker σ-algebra $\pi_\mu(\Phi)$ is trivial, where μ is an invariant measure with full support, then $\mathbf{A}(\Phi)$ is dense. These results are generalizations of those of Blanchard, Host and Ruette ([3]) that concern the asymptotic relation for $\mathbb{Z}$-actions. We give an example of an expansive G-action ($G = \mathbb{Z}^2$) with $\mathbf{A}(\Phi)$ trivial which shows that the Bryant–Walters classical result ([3]) fails to be true in general case.

1. Introduction

One of important characteristics of topological dynamical systems with $\mathbb{Z}$ as the group of time is the asymptotic relation. Let $\mathbf{A}(T)$ denote the asymptotic relation of a dynamical system (X,T). It is known ([10]) that $\mathbf{A}(T)$ is trivial (i.e. equals the diagonal relation Δ) for deterministic systems in the sense of [10], in particular for distal systems. On the other hand, $\mathbf{A}(T)$ is non-trivial for expansive T (cf. [3]) and also for systems with positive topological entropy $h(T)$ (cf. [2]). An interesting characterization of systems with zero topological entropy by use of $\mathbf{A}(T)$ is given in [6].

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If T admits an invariant probability measure μ with full support such that T is a K-automorphism with respect to μ, then $\mathbf{A}(T)$ is dense in $X \times X$ ([2], [10]).

The aim of this paper is to extend the concept of asymptoticity to topological actions of countable amenable orderable groups.

First we show that if the topological entropy $h(\Phi)$ is positive then $\mathbf{A}(\Phi)$ is non-trivial (Corollary 4.5).

Next we prove that if Φ satisfies a stronger condition, namely if the Pinsker σ-algebra $\pi_\mu(\Phi)$ is trivial for an invariant measure μ with full support then $\mathbf{A}(\Phi)$ is dense in $X \times X$ (Proposition 4.6).

In order to show these results we first prove that for any invariant measure μ there exists a measurable partition η with properties analogous to those of the Rokhlin extreme partitions (cf. [16]) and such that any pair of points from the same atom of η belongs to $\mathbf{A}(\Phi)$.

We also give an example of an expansive $\mathbb{Z}^2$-action ($\mathbb{Z}^2$ is equipped with the lexicographical order) with $\mathbf{A}(\Phi)$ trivial.

2. Preliminaries

Let (X, d) be a compact metric space and suppose μ is a Borel probability measure on X.

We assume X is equipped with the σ-algebra $\mathcal{B}$ being the completion of the Borel σ-algebra with respect to μ. The extension of μ to $\mathcal{B}$ will be also denoted by μ.

We denote by $\mathcal{M}(X)$ the lattice of measurable partitions of $(X, \mathcal{B}, \mu)$. For the definition and basic properties of $\mathcal{M}(X)$ we refer the reader to [16] (see also [12]).

Let $\mathcal{F}(X) \subset \mathcal{M}(X)$ denote the set of finite partitions.

For any $\xi \in \mathcal{M}(X)$ we denote by $R_\xi \subset X \times X$ the equivalence relation determined by ξ and by $\widehat{\xi}$ the σ-algebra of ξ-sets, i.e. measurable unions of elements of ξ. We denote by $\mathcal{N}$ the σ-algebra corresponding to the trivial partition ν_X of X.

Let $\xi, \eta \in \mathcal{M}(X)$. The relation $\xi \prec \eta$ means that any atom of η is included in some atom of ξ.

If $\xi \prec \eta$ then obviously $\widehat{\xi} \subset \widehat{\eta}$.

For a countable family $\{\xi_t;\ t \in T\} \subset \mathcal{M}(X)$ we denote by $\bigvee_{t\in T} \xi_t$ its join. It is known ([16]) that $\bigvee_{t\in T} \xi_t \in \mathcal{M}(X)$. Moreover, if the elements of ξ_t, $t \in T$, are Borel sets then the elements of $\bigvee_{t\in T} \xi_t$ are so.

Let $\langle G, \cdot\rangle$ be a countable amenable group equipped with a set $\Gamma \subset G$ called an algebraic past satisfying the following conditions:

- $\Gamma \cap \Gamma^{-1} = \emptyset$,
- $\Gamma \cup \Gamma^{-1} \cup \{e\} = G$,