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ON ABSTRACT DIFFERENTIAL EQUATIONS WITH NON INSTANTANEOUS IMPULSES

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Abstract. We introduce a class of abstract differential equation with non instantaneous impulses which extend and generalize some recent models considered in the literature. We study the existence of mild and classical solution and present some applications involving partial differential equations with non-instantaneous impulses.

1. Introduction

In this work we introduce and study a new model of abstract impulsive differential equations which improve substantially the theory on differential equations with non-instantaneous impulsive introduced recently by Hernandez and O'Regan in [8]. Specifically, we study a class of abstract differential equations with non-instantaneous impulses of the form

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in (s_i, t_{i+1}],\ i = 0, \ldots, N, \tag{1.1}$$

$$u(t) = h_i(t, u_{|_{I_i(t)}}), \quad t \in (t_i, s_i],\ i = 1, \ldots, N, \tag{1.2}$$

$$u(0) = x_0, \tag{1.3}$$

where $A\colon D(A) \subset X \to X$ is the generator of a C_0-semigroup of bounded linear operators $(T(t))_{t\geq 0}$ defined on a Banach space $(X, \|\cdot\|)$, $x_0 \in X$, $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \ldots t_N \leq s_N \leq t_{N+1} = a$ are pre-fixed numbers, the relation

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$t \to I_i(t)$ defines a $2^{[0,t]}$-set valued function, each function $h_i(t, \cdot\,)$ is a continuous function defined from a Banach space $\mathcal{C}_i(t)$ into X, the spaces $\mathcal{C}_i(t)$ are formed by function defined from $I_i(t)$ into X, the symbol $u_{|_I}$ denotes the restriction of $u(\,\cdot\,)$ to an interval $I \subset [0,a]$ and $f\colon [0,a] \times X \to X$ is a suitable function.

To explain our motivations and objectives, we include some comments related to the problem studied in [8]. In Hernández and O'Regan [8] the authors introduced a new class of differential equations with impulses (called differential equation with non-instantaneous impulses) described in the form

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in (s_i, t_{i+1}],\ i = 0, \ldots, N, \tag{1.4}$$

$$u(t) = g_i(t, u(t)), \quad t \in (t_i, s_i],\ i = 1, \ldots, N, \tag{1.5}$$

$$u(0) = x_0, \tag{1.6}$$

where A, f, x_0, s_i, t_i are as above and $g_i \in C((t_i, s_i] \times X; X)$ for all $i = 1, \ldots, N$.

The main results in [8], see Theorems 2.1 and 2.2, are proved via fixed point techniques and assuming that the functions g_i are globally Lipschitz. Specifically, the authors proved that the map $\Gamma\colon \mathcal{PC}(X) \to \mathcal{PC}(X)$ (see the definition of $\mathcal{PC}(X)$ below) given by $\Gamma u(0) = x_0$, $\Gamma u(t) = g_i(t, u(t))$ for $t \in (t_i, s_i]$ and

$$\begin{aligned} \Gamma u(t) &= T(t - s_i) g_i(s_i, u(s_i)) + \int_{s_i}^{t} T(t-s) f(s, u(s))\, ds, && t \in [s_i, t_{i+1}], \\ \Gamma u(t) &= \int_0^t T(t-s) f(s, u(s))\, ds, && t \in [0, t_1], \end{aligned}$$

have a fixed point $u \in \mathcal{PC}(X)$, which is called a mild solution of (1.4)–(1.5). A review of the proofs of the cited Theorems reveals that for each $i \in \{1, \ldots, N\}$, the map

$$\Gamma_i\colon \mathcal{PC}(X)_{|(t_i, s_i]} = \Big\{ u \in C((t_i, s_i]; X) : \lim_{t \downarrow t_i} u(t) \text{ exists} \Big\} \to \mathcal{PC}(X)_{|(t_i, s_i]},$$

given by $\Gamma_i u = g_i(\,\cdot\,, u(\,\cdot\,))_{|(t_i, s_i]}$ is a contraction on $\mathcal{PC}(X)_{|(t_i, s_i]}$ and there exists a unique function $v_i \in \mathcal{PC}(X)_{|(t_i, s_i]}$ such that $\Gamma_i v_i = v_i$. This can be a non-realistic situation and it is quite restrictive for a general theory. However we note that the ideas and analysis in [8] partly motivates the general theory presented below.

Motivated by the above and by the fact that the above restriction arise from the abstract formulation of the problem (1.4)–(1.5), in this work we introduce a new abstract formulation for abstract differential equations with non-instantaneous impulses and we study the existence of mild and classical solutions for this type of problems.

Next we include some comments on the associated literature. The literature on abstract impulsive differential equations consider basically problems for which the impulses are abrupt and instantaneous. The literature on this type of problem is vast and different topics on the existence and qualitative properties of solutions are considered. On the general motivations, relevant developments and the current status of the theory we refer the reader to [1]–[7], [9]–[11], [13], [15], [18]–[20], [25] and the references therein.

The literature on abstract impulsive differential equations with non-instantaneous impulses is limited and recent. The theory was initiated by Hernández and O'Regan in [8]. In this paper the authors introduced the concepts of mild and classical solution, established some relations between these type of solutions and studied the existence and uniqueness of solutions. In [17] the authors studied the existence of mild solutions with values in fractional spaces. We also mention the recent papers [12], [21]–[24].

For motivations on the study of differential equations with non-instantaneous impulses, we include the example presented in [8] concerning the hemodynamical equilibrium of a person. In the case of a decompensation (for example, high or low levels of glucose) one can prescribe some intravenous drugs (insulin). Since the introduction of the drugs in the bloodstream and the consequent absorbtion for the body are gradual and continuous processes, we can interpret the above situation as an impulsive action which starts abruptly and stays active on a finite time interval.

Concerning the above example it is important to note the advantages of the new abstract model (1.1)–(1.3). In the formulation proposed in [8], we can think that the decision to administer glucose at the time t_i is independent of the "quantity of glucose" at the times $t < t_i$. On the other hand, in the current formulation, the introduction of drug at t_i depend on the level of glucose observed in time intervals previous to the time t_i.

We include now some notations and results. In work $A\colon D(A) \subset X \to X$ is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X and C_0, γ are positive constants such that $\|T(t)\|_{\mathcal{L}(X)} \leq C_0 e^{\gamma t}$ for all $t > 0$. Next, for sake of simplicity, we assume that $0 \in \rho(A)$ and we use the notation $\mathcal{D}$ for the domain of A endowed with the graph $\|x\|_{\mathcal{D}} = \|Ax\|$. For additional details on semigroup, we refer the reader to Pazy [16].

For Banach spaces $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$, we use the notation $\mathcal{L}(Z, W)$ for the space of bounded linear operators from Z into W endowed with the norm of operators denoted by $\|\cdot\|_{\mathcal{L}(Z,W)}$ and we write $\mathcal{L}(Z)$ and $\|\cdot\|_{\mathcal{L}(Z)}$ when $Z = W$. In addition, $C(Z; W)$ denotes the space of bounded continuous functions defined from Z into W endowed with the uniform norm denoted by $\|\cdot\|_{C(Z;W)}$ and we use the symbol $B_r(z, Z)$ for the closed ball with center at $z \in Z$ and radius r in Z.

As usual, $C(J, Z)$ (with $J \subset \mathbb{R}$) is the space formed by all the continuous bounded functions defined from J into Z endowed with the norm

$$\|u\|_{C(J,Z)} = \sup_{t\in J} \|u(t)\|_Z.$$

To treat the impulsive conditions, we consider the space $\mathcal{PC}(X)$ which is formed by all the functions $u\colon [0,a] \to X$ such that $u(\cdot)$ is continuous at $t \neq t_i$, $u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists for all $i = 1, \ldots, N$, endowed with the uniform norm on $[0,a]$ denoted by $\|u\|_{\mathcal{PC}(X)}$. It is easy to see that $\mathcal{PC}(X)$ is a Banach space.

This paper has three sections. In the next section we introduce the concept of mild and classical solution for the problem (1.1)–(1.2) and we study the existence of these type of solutions. In the same section we also discuss the existence of mild solution with values in fractional interpolation spaces. In the last section, some applications involving partial differential equation with non-instantaneous impulses are presented.

2. Existence of solution

In this section we study the existence and some qualitative properties of solutions for the problem (1.1)–(1.3). To begin, we introduce the following concepts of a solution.

Definition 2.1. A function $u \in \mathcal{PC}(X)$ is said to be a mild solution of (1.1)–(1.3) if $u(0) = x_0$, $u(t) = h_i(t, u_{|_{I_i(t)}})$ for all $t \in (t_i, s_i]$ and each $i = 1, \ldots, N$, and

$$u(t) = T(t)x_0 + \int_0^t T(t-\tau) f(\tau, u(\tau))\, d\tau, \quad \text{for all } t \in [0, t_1],$$

$$u(t) = T(t-s_i) h_i(s_i, u_{|_{I_i(s_i)}}) + \int_{s_i}^t T(t-\tau) f(\tau, u(\tau))\, d\tau,$$

for all $t \in [s_i, t_{i+1}]$ and every $i = 1, \ldots, N$.

Definition 2.2. A function $u \in \mathcal{PC}(X)$ is said to be a classical solution of (1.1)–(1.3) if $u(0) = x_0$, $u(t) = h_i(t, u_{|_{I_i(t)}})$ for all $t \in (t_i, s_i]$ and $i \in \{1, \ldots, N\}$, the function $u_{|_{(s_i, t_{i+1}]}}$ belongs to $C((s_i, t_{i+1}]; \mathcal{D})$ for all $i = 1, \ldots, N$ and $u(\cdot)$ satisfies (1.1).

To prove the results of this section, we introduce the following conditions. Next we use the notation $(\mathcal{C}_i(t), \|\cdot\|_{\mathcal{C}_i(t)})$, with $t \in (t_i, s_i]$ and $i \in \{1, \ldots, N\}$, to represent an abstract Banach space formed by functions defined from $I_i(t) \subset [0, s_i]$ into X. In addition, for a set $I \subset [0,a]$, we use the notation $\mathcal{PC}(X)_{|_I}$ for the space $\mathcal{PC}(X)_{|_I} = \{u_{|_{I_i(t)}} : u \in \mathcal{PC}(X)\}$ endowed with the uniform norm. Similarly, we define the spaces $\mathcal{PC}(Z)$ and $\mathcal{PC}(Z)_{|_I}$ for a given Banach space Z.

(H$_1$) For all $i = 1, \ldots, N$ and each $t \in (t_i, s_i]$, the function $h_i(t, \cdot)$ belongs to $C(\mathcal{C}_i(t); X)$ and there is a bounded function $L_i \in C((t_i, s_i]; \mathbb{R}^+)$ such that

$$\|h_i(t,u) - h_i(t,v)\| \leq L_i(t) \|u - v\|_{\mathcal{C}_i(t)}, \quad \text{for all } u, v \in \mathcal{C}_i(t).$$

(H_2) The function $f(\,\cdot\,)$ belongs to $C([0,a]\times X;X)$ and there is a non-decreasing function $W_f \in C([0,\infty);\mathbb{R}^+)$ and a function $m_f \in L^p([0,a];\mathbb{R}^+)$, with $p \geq 1$, such that $\|f(t,x)\| \leq m_f(t)W_f(\|x\|)$ for all $(t,x)\in[0,a]\times X$.

(H_3) The function $f(\,\cdot\,)$ is continuous and there is a $L_f \in L^p([0,a];\mathbb{R}^+)$, with $p>1$, such that $\|f(t,x)-f(t,y)\| \leq L_f(t)\|x-y\|$ for all $x,y\in X$ and $t\in[0,a]$.

(H_4) For all $i\in\{1,\ldots,N\}$ and each $u\in\mathcal{PC}(X)$, the function $t\mapsto h_i(t,u_{|I_i(t)})$ belongs to $C((t_i,s_i];X)$ and $\lim\limits_{t\downarrow t_i} h_i(t,u_{|I_i(t)})$ exists.

(H_5) For all $t\in(t_i,s_i]$ and $i\in\{1,\ldots,N\}$, the map

$$\Psi_i(t)\colon \mathcal{PC}(X)_{|_{I_i(t)}} = \{u_{|_{I_i(t)}} : u\in\mathcal{PC}(X)\} \to \mathcal{C}_i(t),$$

given by $\Psi_i(t)u = u_{|_{I_i(t)}}$ is a bounded linear operator and we always assume that the set of operators $\{\Psi_i(t) : t\in(t_i,s_i],\ i=1,\ldots,N\}$ is bounded. For convenience, next we use the notation

$$\widetilde{\Psi}_i(s) = \|\Psi_i(s)\|_{\mathcal{L}(\mathcal{PC}(X)_{|_{I_i(s)}},\mathcal{C}_i(s))}.$$

We establish now our first result on the existence of a mild solution for (1.1)–(1.3).

Theorem 2.3. *Let the conditions* (H_1), (H_3)–(H_5) *be holded and assume that any one of the following conditions is satisfied*:

(a) $\widehat{s}_i = \sup \bigcup\limits_{t\in(t_i,s_i]} I_i(t) < t_i$ *for all* $i\in\{1,\ldots,N\}$,

(b) $t_i < \widehat{s}_i < s_i$ *for all* $i \in 1,\ldots,N$,

$$\|L_{h_i}\|_{C((t_i,s_i];\mathbb{R})} e^{-\gamma(t_i-\widehat{s}_i)} \sup_{s\in(t_i,s_i]} \widetilde{\Psi}_i(s) \ < 1,$$
$$C_0(e^{\gamma(\widehat{s}_i-s_i)} L_{h_i}(s_i)\widetilde{\Psi}_i(s_i) + \|L_f\|_{L^1([s_i,t_{i+1}];\mathbb{R})}) \ < 1.$$

for all $i=1,\ldots,N$, *and* $C_0\|L_f\|_{L^1([0,t_1];\mathbb{R})} < 1$,

(c) $\widehat{s}_i = s_i$ *for all* i,

$$C_0 L_{h_i}(s_i)\widetilde{\Psi}_i(s_i) + C_0\|L_f\|_{L^1([s_i,t_{i+1}];\mathbb{R})} \ < 1,$$
$$\|L_{h_i}\|_{C((t_i,s_i];\mathbb{R})} e^{-\gamma(t_i-s_i)} \sup_{s\in(t_i,s_i]} \widetilde{\Psi}_i(s) \ < 1$$

and $C_0\|L_f\|_{L^1([0,t_1];\mathbb{R})} < 1$ *for all* $i=0,\ldots,N$.

Then there exists a unique mild solution of the problem (1.1)–(1.3).

Proof. Let $\beta\geq\gamma$ and $\mathcal{P}_\beta\mathcal{C}(X)$ be the set $\mathcal{PC}(X)$ endowed with the norm $\|\cdot\|_{\mathcal{P}_\beta\mathcal{C}(X)}$ given by $\|u\|_{\mathcal{P}_\beta\mathcal{C}(X)} = \sup\limits_{s\in[0,a]} e^{-\beta t}\|u(t)\|$. Let $\Gamma\colon \mathcal{P}_\beta\mathcal{C}(X)\to\mathcal{P}_\beta\mathcal{C}(X)$

be the map defined by $\Gamma u(0)=x_0$, $\Gamma u(t)=h_i(t,u_{|_{I_i(t)}})$ for $t\in(t_i,s_i]$ and

$$\Gamma u(t)=T(t-s_i)h_i(s_i,u_{|_{I_i(s_i)}})+\int_{s_i}^{t}T(t-s)f(s,u(s))\,ds,\quad t\in[s_i,t_{i+1}],$$

$$\Gamma u(t)=T(t)x_0+\int_0^t T(t-s)f(s,u(s))\,ds,\qquad t\in[0,t_1].$$

From the assumptions is easy to see that Γ is a well defined $\mathcal{P}_\beta\mathcal{C}(X)$-valued function. Next we prove that there exists $\beta\geq\gamma$ such that Γ is a contraction on $\mathcal{P}_\beta\mathcal{C}(X)$. In the remainder of this proof, we assume that $u,v\in\mathcal{PC}(X)$.

To begin, for $i\in\{1,\ldots,N\}$ and $t\in[s_i,t_{i+1}]$ we note that

$$\begin{aligned}
&\|\Gamma u(t)-\Gamma v(t)\|\\
&\quad\leq\|T(t-s_i)h_i(s_i,u_{|_{I_i(s_i)}})-T(t-s_i)h_i(s_i,v_{|_{I_i(s_i)}})\|\\
&\qquad+C_0\int_{s_i}^{t}e^{\gamma(t-s)}e^{\beta s}L_f(s)e^{-\beta s}\|u(s)-v(s)\|\,ds\\
&\quad\leq C_0e^{\gamma(t-s_i)}L_{h_i}(s_i)\|u_{|_{I_i(s_i)}}-v_{|_{I_i(s_i)}}\|_{\mathcal{C}_i(s_i)}\\
&\qquad+C_0e^{\beta t}\int_{s_i}^{t}e^{(\gamma-\beta)(t-s)}L_f(s)\,ds\|u-v\|_{\mathcal{P}_\beta\mathcal{C}(X)}\\
&\quad\leq C_0e^{\gamma(t-s_i)}L_{h_i}(s_i)\|\Psi_i(s_i)\|_{\mathcal{L}(\mathcal{PC}(X)_{|_{I_i(s_i)}},\mathcal{C}_i(s_i))}\sup_{s\in I_i(s_i)}e^{\beta s}e^{-\beta s}\|u(s)-u(s)\|\\
&\qquad+C_0e^{\beta t}\int_{s_i}^{t}e^{(\gamma-\beta)(t-s)}L_f(s)\,ds\|u-v\|_{\mathcal{P}_\beta\mathcal{C}(X)},
\end{aligned}$$

so that,

$$\begin{aligned}
e^{-\beta t}\|\Gamma u(t)-\Gamma v(t)\|&\leq C_0e^{\gamma(t-s_i)}e^{\beta(\widehat{s}_i-t)}L_{h_i}(s_i)\widetilde{\Psi}_i(s_i)\|u-v\|_{\mathcal{P}_\beta\mathcal{C}(X)}\\
&\quad+C_0\int_{s_i}^{t}e^{(\gamma-\beta)(t-s)}L_f(s)\,ds\|u-v\|_{\mathcal{P}_\beta\mathcal{C}(X)},
\end{aligned}$$

for all $t\in[s_i,t_{i+1}]$. From the above, we obtain that

$$\begin{aligned}
(2.1)\qquad&\sup_{t\in[s_i,t_i+1]}e^{-\beta t}\|\Gamma u(t)-\Gamma v(t)\|\\
&\quad\leq C_0\sup_{t\in[s_i,t_i+1]}e^{(\gamma-\beta)(t-s_i)}e^{\beta(\widehat{s}_i-s_i)}L_{h_i}(s_i)\widetilde{\Psi}_i(s_i)\|u-v\|_{\mathcal{P}_\beta\mathcal{C}(X)}\\
&\qquad+C_0\sup_{t\in[s_i,t_i+1]}\int_{s_i}^{t}e^{(\gamma-\beta)(t-s)}L_f(s)ds\|u-v\|_{\mathcal{P}_\beta\mathcal{C}(X)}.
\end{aligned}$$

Moreover, from the above estimates it is easy to infer that

$$\begin{aligned}
(2.2)\qquad&\sup_{t\in[0,t_1]}e^{-\beta t}\|\Gamma u(t)-\Gamma v(t)\|\\
&\qquad\leq C_0\sup_{t\in[0,t_1]}\int_0^t e^{(\gamma-\beta)(t-s)}L_f(s)ds\|u-v\|_{\mathcal{P}_\beta\mathcal{C}(X)}.
\end{aligned}$$

On the other hand, for $t \in (t_i, s_i]$ we find that

$$\begin{aligned}\|\Gamma u(t) - \Gamma v(t)\| &\leq \|h_i(t, u_{|_{I_i(t)}}) - h_i(t, v_{|_{I_i(t)}})\| \\ &\leq L_{h_i}(t)\|\Psi_i(t)\|_{\mathcal{L}(\mathcal{PC}(X)_{|_{I_i(t)}}, \mathcal{C}_i(t))} \sup_{s\in I_i(t)} e^{\beta s} e^{-\beta s}\|u(s) - u(s)\| \\ &\leq L_{h_i}(t) \sup_{s\in(t_i,s_i]} \widetilde{\Psi}_i(s) e^{\beta \widehat{s}_i}\|u - v\|_{\mathcal{P}_\beta\mathcal{C}(X)},\end{aligned}$$

which implies that

$$\begin{aligned}\sup_{t\in[t_i,s_i]} & e^{-\beta t}\|\Gamma u(t) - \Gamma v(t)\| \\ &\leq \|L_{h_i}\|_{C((t_i,s_i];\mathbb{R})} \sup_{t\in(t_i,s_i]} e^{-\beta(t-\widehat{s}_i)} \sup_{s\in(t_i,s_i]} \widetilde{\Psi}_i(s)\|u - v\|_{\mathcal{P}_\beta\mathcal{C}(X)}.\end{aligned} \tag{2.3}$$

From the inequalities (2.1), (2.2) and (2.3) it follows that

$$\|\Gamma u - \Gamma v\|_{\mathcal{P}_\beta\mathcal{C}(X)} \leq \max_{i=1,\ldots,N}\{(\Lambda_i(\beta) + \Xi_i(\beta)), \Theta_i(\beta), \Xi_0(\beta)\}\|u - v\|_{\mathcal{P}_\beta\mathcal{C}(X)}, \tag{2.4}$$

where

$$\begin{aligned}\Lambda_i(\beta) &= C_0 e^{\beta(\widehat{s}_i - s_i)} L_{h_i}(s_i)\widetilde{\Psi}_i(s_i), \\ \Theta_i(\beta) &= e^{-\beta(t_i - \widehat{s}_i)}\|L_{h_i}\|_{C((t_i,s_i];\mathbb{R})} \sup_{s\in(t_i,s_i]} \widetilde{\Psi}_i(s), \\ \Xi_i(\beta) &= C_0 \sup_{t\in[s_i,t_{i+1}]} \int_{s_i}^t e^{(\gamma-\beta)(t-s)} L_f(s)\, ds.\end{aligned}$$

Next, we consider the cases in which any of the conditions (a), (b) or (c) is satisfied.

Suppose that condition (a) is valid. In this case, for $\beta > \gamma$ we have that

$$\Xi_j(\beta) \leq C_0(p'(\beta - \gamma))^{-1/p'}\|L_f\|_{L^p([s_j,t_{j+1}];\mathbb{R})}, \quad j = 1, \ldots, N, \tag{2.5}$$

where $1/p + 1/p' = 1$, which implies $\Lambda_i(\beta) + \Theta_i(\beta) + \Xi_i(\beta) + \Xi_0(\beta) \to 0$ as $\beta \to 0$ for all $i \in \{1, \ldots, N\}$ since $\max\{\beta(\widehat{s}_i - s_i), -\beta(t_i - \widehat{s}_i)\} < 0$. Thus, for $\beta > \gamma$ large enough, the map $\Gamma(\,\cdot\,)$ is a contraction on $\mathcal{P}_\beta\mathcal{C}(X)$ which shows that there exists a unique mild solution of problem (1.1)–(1.3).

We assume now that condition (b) is satisfied. In this case, for $\beta = \gamma$ we get

$$\Xi_i(\gamma) \leq C_0\|L_f\|_{L^1([s_i,t_{i+1}];\mathbb{R})}.$$

From this inequality, (2.4) and condition (**b**) it follows that Γ is a contraction on $\mathcal{P}_\gamma\mathcal{C}(X)$ and there a unique mild solution of the problem (1.1)–(1.3).

To finish, we suppose that condition (c) is valid. In this case, for $\beta = \gamma$ we have that $\Lambda_i(\beta) = C_0 L_{h_i}(s_i)\widetilde{\Psi}_i(s_i)$, $\Xi_i(\beta) \leq \|L_f\|_{L^1([s_i,t_{i+1}];\mathbb{R})}$ and $\Theta_i(\beta) = \|L_{h_i}\|_{C((t_i,s_i];\mathbb{R})} e^{-\gamma(t_i - \widehat{s}_i)} \sup_{s\in(t_i,s_i]} \widetilde{\Psi}_i(s)$, which allows us to infer that Γ is a contraction and there exists a unique mild solution of (1.1)–(1.3). $\square$

In the next result we note that Theorem [8, Theorem 2.1] follows directly Theorem 2.3.

Corollary 2.4. *If the conditions in* [8, Theorem 2.1] *are satisfied, then there exists a unique mild solution* $u \in \mathcal{PC}(X)$ *of the problem* (1.4)–(1.6).

Proof. From [8, Theorem 2.1] we note that each function g_i is globally Lipschitz with Lipschitz constant L_{g_i} and $\gamma = 0$.

To use Theorem 2.3, for $i \in \{1, \ldots, N\}$ and $t \in (t_i, s_i]$ we consider the space $\mathcal{C}_i(t) = C(\{t\} : X)$ and the functions $h_i\colon (t_i, s_i] \times \mathcal{C}_i(t) \to X$, $\Psi_i(t)\colon \mathcal{PC}(X)_{|\{t\}} \to \mathcal{C}_i(t)$ given by $\Psi_i(t)u = u_{|\{t\}}$ and $h_i(t, v) = g_i(t, v(t))$. It is easy to see that the condition (H_1) is satisfied with $L_{h_i} = L_{g_i}$, the maps $\Psi_i(t)$ are bounded linear operator and $\widetilde{\Psi}_i(t) = 1$ for $t \in (t_i, s_i]$.

Finally, by noting that in [8, Theorem 2.1] it is assumed that

$$\Theta = C_0 \max\{L_{g_i} + \|L_f\|_{L^1([s_i, t_{i+1}])},\ \|L_f\|_{L^1([0,t_1])} : i = 1, \ldots N\} < 1,$$

so we have that condition (c) in Theorem 2.3 is satisfied, which implies that there exists a unique mild solution of problem (1.4)–(1.6). □

In the next theorem we prove the existence of a mild solution via a fixed point result for condensing operators.

Theorem 2.5. *Assume the conditions* (H_1), (H_2), (H_4) *and* (H_5) *are satisfied, the semigroup* $(T(t))_{t \geq 0}$ *is compact and*

$$\Theta = C_0 \varlimsup_{r \to \infty} \sup_{t \in [s_i, t_{i+1}]} r^{-1} W_f(r) \int_{s_i}^{t} e^{\gamma(t-\tau)} m_f(\tau)\, d\tau < 1, \tag{2.6}$$

for all $i \in \{0, 1, \ldots, N\}$. *If any one of the next conditions is verified,*

(a) $\widehat{s}_i = \sup \bigcup_{t \in (t_i, s_i]} I_i(t) < t_i$ *for all* $i \in \{1, \ldots, N\}$,

(b) $t_i < \widehat{s}_i < s_i$, $C_0 e^{\gamma(\widehat{s}_i - s_i)} L_{h_i}(s_i) \widetilde{\Psi}_i(s_i) < 1 - \Theta$ *and*

$$e^{-\gamma(t_i - \widehat{s}_i)} \|L_{h_i}\|_{C((t_i, s_i];\mathbb{R})} \sup_{s \in (t_i, s_i]} \widetilde{\Psi}_i(s) < 1 - \Theta$$

for all $i = 1, \ldots, N$,

(c) *for all* $i \in \{1, \ldots, N\}$, $\widehat{s}_i = s_i$, $e^{-\gamma(t_i - \widehat{s}_i)} \|L_{h_i}\|_{C((t_i, s_i];\mathbb{R})} \sup_{t \in (t_i, s_i]} \widetilde{\Psi}_i(t) < 1 - \Theta$ *and* $C_0 L_{h_i}(s_i) \widetilde{\Psi}_i(s_i) < 1 - \Theta$,

then there exists a mild solution of the problem (1.1)–(1.3).

Proof. Let Γ be the map introduced in the proof of Theorem 2.3 and consider the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ where

$$\begin{aligned} \Gamma^1 u(t) &= h_i(t, u_{|_{I_i(t)}}), && \text{for } t \in (t_i, s_i], \\ \Gamma^1 u(t) &= T(t - s_i) h_i(s_i, u_{|_{I_i(s_i)}}), && \text{for } t \in [s_i, t_{i+1}], \end{aligned}$$

$$\Gamma^2 u(t) = \int_{s_i}^{t} T(t-s)f(s,u(s))\,ds, \quad \text{for } t \in [s_i, t_{i+1}],$$

and $\Gamma^i u(t) = 0$ otherwise. Next, we prove that there exists $\beta \geq \gamma$ and $r > 0$ such that Γ is a condensing map from $B_r(0, \mathcal{P}_\beta\mathcal{C}(X))$ into $B_r(0, \mathcal{P}_\beta\mathcal{C}(X))$.

Let $\beta \geq \gamma$. From the assumptions, we select $r_0 > 0$ and $\Theta \in (0,1)$ such that

$$C_0 \frac{W_f(e^{\beta s_i} s)}{e^{\beta s_i} s} \sup_{s \in [s_i, t_{i+1}]} \int_{s_i}^{s} e^{\gamma(s-\tau)} m_f(\tau)\,d\tau < \Theta, \tag{2.7}$$

for all $s \geq r_0$ and for all $i \in \{0, \ldots, N\}$. Proceeding as in the proof of Theorem 2.3, for $r \geq r_0$, $u \in B_r(0, \mathcal{P}_\beta\mathcal{C}(X))$ and $i \geq 1$ we infer that

$$\begin{aligned}\sup_{t \in [s_i, t_i+1]} e^{-\beta t}\|\Gamma^1 u(t)\| \leq{} & C_0 \sup_{t \in [s_i, t_i+1]} e^{(\gamma-\beta)(t-s_i)} e^{\beta(\widehat{s}_i - s_i)} L_{h_i}(s_i) \widetilde{\Psi}_i(s_i) \|u\|_{\mathcal{P}_\beta\mathcal{C}(X)} \\ & + C_0 \sup_{t \in [s_i, t_i+1]} e^{\gamma(t-s_i)-\beta t} \|h_i(s_i, 0)\|,\end{aligned}$$

and hence,

$$\begin{aligned}\sup_{t \in [s_i, t_i+1]} e^{-\beta t}\|\Gamma^1 u(t)\| \leq{} & C_0 \sup_{t \in [s_i, t_i+1]} e^{(\gamma-\beta)(t-s_i)} e^{\beta(\widehat{s}_i - s_i)} L_{h_i}(s_i) \widetilde{\Psi}_i(s_i) r \\ & + C_0 \sup_{t \in [s_i, t_i+1]} e^{\gamma(t-s_i)-\beta t} \|h_i(s_i, 0)\|.\end{aligned} \tag{2.8}$$

Similarly, from the proof of Theorem 2.3 we also infer that

$$\begin{aligned}&\sup_{t \in [t_i, s_i]} e^{-\beta t}\|\Gamma^1 u(t)\| \\ &\leq \|L_{h_i}\|_{C((t_i, s_i]; \mathbb{R})} \sup_{t \in (t_i, s_i]} e^{-\beta(t-\widehat{s}_i)} \sup_{s \in (t_i, s_i]} \widetilde{\Psi}_i(s) r + e^{-\beta t_i} \|h_i(\,\cdot\,, 0)\|_{C((t_i, s_i]; X)}.\end{aligned} \tag{2.9}$$

On the other hand, for $t \in [s_i, t_{i+1}]$ and $i \geq 0$ we have that

$$\begin{aligned} e^{-\beta t}\|\Gamma^2 u(t)\| &\leq C_0 \int_{s_i}^{t} e^{-\beta t} e^{\gamma(t-s)} W_f(e^{\beta s} e^{-\beta s}\|u(s)\|) m_f(s)\,ds \\ &\leq C_0 r \int_{s_i}^{t} e^{\gamma(t-s)} \frac{W_f(e^{\beta s} r)}{e^{\beta s} r} m_f(s)\,ds \\ &\leq r C_0 \frac{W_f(e^{\beta s_i} r)}{e^{\beta s_i} r} \sup_{t \in [s_i, t_i+1]} \int_{s_i}^{t} e^{\gamma(t-s)} m_f(s)\,ds, \end{aligned}$$

which implies via (2.7) that

$$\sup_{t \in [s_i, t_i+1]} e^{-\beta t}\|\Gamma^2 u(t)\| \leq r\Theta, \quad \text{for all } i = 0, 1, \ldots, N. \tag{2.10}$$

From the estimates (2.8)–(2.10) we obtain that

$$\|\Gamma u\|_{\mathcal{P}_\beta\mathcal{C}(X)} \leq \max_{i=1,\ldots,N} \{\Lambda_i(\beta) r + \widetilde{\Lambda}_i(\beta) + r\Theta, \Theta_i(\beta) r + \widetilde{\Theta}_i(\beta), r\Theta\} \tag{2.11}$$

where

$$\begin{aligned}\Lambda_i(\beta) &= C_0 e^{\beta(\widehat{s}_i - s_i)} L_{h_i}(s_i)\widetilde{\Psi}_i(s_i),\\ \Theta_i(\beta) &= e^{-\beta(t_i-\widehat{s}_i)}\|L_{h_i}\|_{C((t_i,s_i];\mathbb{R})}\sup_{s\in(t_i,s_i]}\widetilde{\Psi}_i(s),\\ \widetilde{\Lambda}_i(\beta) &= C_0 e^{\gamma(t_{i+1}-s_i)-\beta s_i}\|h_i(s_i,0)\|,\\ \widetilde{\Theta}_i(\beta) &= e^{-\beta t_i}\|h_i(\,\cdot\,,0)\|_{C((t_i,s_i];X)}.\end{aligned}$$

Moreover, from the proof of Theorem 2.3 it is easy to note that

$$\|\Gamma^1 u - \Gamma^1 v\|_{\mathcal{P}_\beta\mathcal{C}(X)} \le \max_{i=1,\dots,N}\{\Lambda_i(\beta),\Theta_i(\beta)\}\|u-v\|_{\mathcal{P}_\beta\mathcal{C}(X)}. \tag{2.12}$$

We divide the remainder of the proof into several steps.

Step 1. $\Gamma^2 B_r(0,\mathcal{P}_\beta\mathcal{C}(X)) \subset B_r(0,\mathcal{P}_\beta\mathcal{C}(X))$ and Γ^2 is a condensing map on $B_r(0,\mathcal{P}_\beta\mathcal{C}(X))$ for all $r \ge r_0$,

Let $r \ge r_0$. The fact that $\Gamma^2 B_r(0,\mathcal{P}\mathcal{C}(X)) \subset B_r(0,\mathcal{P}\mathcal{C}(X))$ follows directly from the estimate (2.10). The proof that Γ^2 is condensing on $B_r(0,\mathcal{P}_\beta\mathcal{C}(X))$ follows from the proof of Steps 3–5 in the proof of [8, Theorem 2.2]. We omit the details.

Step 2. If $r \ge r_0$ and condition (**a**) is valid, then there exists $\beta \ge \gamma$ large enough such that the problem (1.1)-(1.3) has a mild solution $u \in B_r(0,\mathcal{P}_\beta\mathcal{C}(X))$.

From condition (a) and the definition of the numbers $\Lambda_i(\beta)$, $\widetilde{\Lambda}_i(\beta)$, $\Theta_i(\beta)$, $\widetilde{\Theta}_i(\beta)$ it is easy to see that $\max_{i=1,\dots,N}\{\Lambda_i(\beta)+\widetilde{\Lambda}_i(\beta)+\Theta_i(\beta)+\widetilde{\Theta}_i(\beta)\} \to 0$ as $\beta\to\infty$. Thus, there exists $\beta \ge \gamma$ large enough such that

$$\max_{i=1,\dots,N}\{\Lambda_i(\beta)r+\widetilde{\Lambda}_i(\beta),\Theta_i(\beta)r+\widetilde{\Theta}_i(\beta)\} \le (1-\Theta)r, \tag{2.13}$$

$$\max_{i=1,\dots,N}\{\Lambda_i(\beta),\Theta_i(\beta)\} < 1. \tag{2.14}$$

From the inequalities (2.11)–(2.14) we have that

$$\Gamma B_r(0,\mathcal{P}_\beta\mathcal{C}(X)) \subset B_r(0,\mathcal{P}_\beta\mathcal{C}(X))$$

and Γ^1 is a contraction on $B_r(0,\mathcal{P}_\beta\mathcal{C}(X))$.

From the above and Step 1 it follows that Γ is condensing on $B_r(0,\mathcal{P}_\beta\mathcal{C}(X))$, which implies that there exists a mild solution of the problem (1.1)–(1.3).

Step 3. If condition (b) is valid, then there exists $r > r_0$ large enough such that problem (1.1)–(1.3) has a mild solution $u \in B_r(0,\mathcal{P}_\gamma\mathcal{C}(X))$.

From condition (b) we select $\Phi \in (0,1-\Theta)$, $\widetilde{\Phi} \in (0,1)$ and $r > r_0$ large enough such that $\Phi+\widetilde{\Phi} < 1-\Theta$ and

$$\max_{i=1,\dots,N}\{\Lambda_i(\gamma),\Theta_i(\gamma)\} < \Phi, \tag{2.15}$$

$$\max_{i=1,\dots,N}\{\widetilde{\Lambda}_i(\gamma),\widetilde{\Theta}_i(\gamma)\} < \widetilde{\Phi}r. \tag{2.16}$$

From the above, for $u \in B_r(0, \mathcal{P}_\beta\mathcal{C}(X))$ we have that

$$\begin{aligned}\|\Gamma u\|_{\mathcal{P}_\beta\mathcal{C}(X)} &\leq \max_{i=1,\dots,N}\{\Lambda_i(\beta)r + \widetilde{\Lambda}_i(\beta) + r\Theta, \Theta_i(\beta)r + \widetilde{\Theta}_i(\beta), r\Theta\}\\ &\leq \max_{i=1,\dots,N}\{\Phi r + \widetilde{\Phi}r + \Theta r, \Phi r + \widetilde{\Phi}r, r\Theta\} \leq (1-\Theta)r + \Theta r = r,\end{aligned}$$

which implies that $\Gamma B_r(0, \mathcal{P}_\beta\mathcal{C}(X)) \subset B_r(0, \mathcal{P}_\beta\mathcal{C}(X))$. Moreover, from (2.15) and (2.12) we have that Γ^1 is a contraction on $B_r(0, \mathcal{P}_\gamma\mathcal{C}(X)))$ which allows us to conclude via Step 1 that Γ is condensing on $B_r(0, \mathcal{P}_\gamma\mathcal{C}(X)))$. This proves that Γ has a fixed point in $B_r(0, \mathcal{P}_\beta\mathcal{C}(X))$ and there exists a mild solution of (1.1)–(1.3).

Step 4. If condition (c) is satisfied, then there exists $r > r_0$ large enough such that problem (1.1)–(1.3) has a mild solution $u \in B_r(0, \mathcal{P}_\gamma\mathcal{C}(X))$.

The proof of this step follows the argument uses in the proof of Step 3. We omit the details. The proof of this theorem is complete. □

2.1. Regularity of mild solutions. In this section we study the existence of mild solution with values in fractional spaces. In the remainder of this section we assume $0 \in \rho(A)$, $\alpha \in (0,1)$ and $(T(t))_{t\geq 0}$ is analytic. Next, we use the notation X_α for the domain of the fractional power $(-A)^\alpha$ of $-A$ endowed with the α-norm given by $\|x\|_\alpha = \|(-A)^\alpha x\|$. We note that X_α is a Banach space and there is $C_\alpha > 0$ such that $\|(-A)^\alpha T(t)\|_{\mathcal{L}(X)} \leq C_\alpha e^{\gamma t} t^{-\alpha}$ for $t > 0$.

Next, for $\beta > 0$ we use the notation $\mathcal{P}_\beta\mathcal{C}(X, X_\alpha)$ for the space formed by all the functions $u \in \mathcal{PC}(X)$ such that $u_{|_{(t_i,s_i]}} \in C((t_i, s_i]; X)$, $u_{|_{(s_i,t_{i+1}]}} \in C((s_i, t_{i+1}]; X_\alpha)$ for all $i = 1, \dots, N$ and

$$[|\ u\ |]_{\beta,\alpha,i} = \sup_{t\in(s_i,t_{i+1}]} (t-s_i)^\alpha e^{-\beta t}\|u(t)\|_\alpha < \infty$$

for every $i = 0, \dots, N$, endowed with the norm

$$\|u\|_{\beta,\alpha} = \max_{i=1,\dots,N}\left\{\sup_{s\in(t_i,s_i]} e^{-\beta s}\|u(s)\|_{C((t_i,s_i];X)}, [|\ u\ |]_{\beta,\alpha,i}, [|\ u\ |]_{\beta,\alpha,0}\right\}.$$

It is easy to see that $(\mathcal{P}_\beta\mathcal{C}(X, X_\alpha), \|\cdot\|_{\beta,\alpha})$ is a Banach space.

To avoid confusion with the problem studied in the first section, we believe it is convenient to introduce some new notations and assumptions. Next, we study the existence of a mild solution for the problem

$$u'(t) = Au(t) + f_\alpha(t, u(t)), \quad t \in (s_i, t_{i+1}],\ i = 0, \dots, N, \tag{2.17}$$

$$u(t) = h_{i,\alpha}(t, u_{|_{I_i(t)}}), \qquad t \in (t_i, s_i],\ i = 1, \dots, N, \tag{2.18}$$

$$u(0) = x_0. \tag{2.19}$$

In this section, $(\mathcal{C}_{i,\alpha}(t), \|\cdot\|_{\mathcal{C}_{i,\alpha}(t)})$ are Banach spaces formed by functions defined from $I_i(t)$ into X_α and we assume that the functions $h_{i,\alpha}(\,\cdot\,)$ and $f_\alpha(\,\cdot\,)$

are defined from $\mathcal{C}_{i,\alpha}(t)$ into X and from $[0,a]\times X_\alpha$ into X, respectively. To prove the results of this section, we include the following conditions:

($\mathrm{H}_{\alpha,1}$) For $t\in(t_i,s_i]$, the function $h_{i,\alpha}(t,\cdot)$ belongs to $C(\mathcal{C}_{i,\alpha}(t);X)$ and there is a function $L_{h_{i,\alpha}}\in C((t_i,s_i];\mathbb{R}^+)$ such that

$$\|h_{i,\alpha}(t,u)-h_{i,\alpha}(t,v)\|\le L_{h_{i,\alpha}}(t)\|u-v\|_{\mathcal{C}_{i,\alpha}(t)},$$

for all $u,v\in\mathcal{C}_{i,\alpha}(t)$, $t\in(t_i,s_i]$ and each $i=1,\dots,N$.

($\mathrm{H}_{\alpha,2}$) The function $f_\alpha(\cdot)$ belongs to $C([0,a]\times X_\alpha;X)$ and there is $L_{f_\alpha}\in L^p([0,a];\mathbb{R}^+)$ (with $p>1$) such that $\|f_\alpha(t,x)-f_\alpha(t,y)\|\le L_{f_\alpha}(t)\|x-y\|_\alpha$ for all $x,y\in X_\alpha$ and each $t\in[0,a]$.

($\mathrm{H}_{\alpha,4}$) For $i\in\{1,\dots,N\}$ and $u\in\mathcal{PC}(X,X_\alpha)$, the function $t\to h_{i,\alpha}(t,u_{|_{I_i(t)}})$ belongs to $C((t_i,s_i];X)$ and $\lim_{t\downarrow t_i}h_{i,\alpha}(t,u_{|I_i(t)})$ exists.

($\mathrm{H}_{\alpha,5}$) For $t\in(t_i,s_i]$ and $i\in\{1,\dots,N\}$, the operator

$$\Psi_{i,\alpha}(t)\colon \mathcal{PC}(X,X_\alpha)_{|_{I_i(t)}}=\{u_{|_{I_i(t)}}:u\in\mathcal{PC}(X,X_\alpha)\}\to\mathcal{C}_{i,\alpha}(t),$$

given by $\Psi_{i,\alpha}(t)u=u_{|_{I_i(t)}}$ is a bounded linear operator and the operator family $\{\Psi_{i,\alpha}(t):t\in(t_i,s_i],\ i=1,\dots,N\}$ is bounded. Next, we use the notation $\widetilde{\Psi}_{i,\alpha}(s)=\|\Psi_i(s)\|_{\mathcal{L}(\mathcal{PC}(X,X_\alpha)_{|_{I_i(s)}},\mathcal{C}_i(s))}$.

To simplify, in the remainder of this work, for $q>1$ we use the notation q' for the conjugate of q given by $q'=q/(q-1)$ and δ_i is the number defined by $\delta_i=t_{i+1}-s_i$.

Lemma 2.6. *Assume $\xi\in L^p([s_i,t_{i+1}];\mathbb{R})$ for some $i\in\{i,\dots,N\}$ and $p>1/(1-\alpha)$. Then the function $s\to\xi(s)/((t-s)^\alpha(s-s_i)^\alpha)$ is integrable on $[s_i,t]$ for all $t\in[s_i,t_{i+1}]$ and*

$$\lim_{\beta\to\infty}\sup_{t\in[s_i,t_{i+1}]}(t-s_i)^\alpha\int_{s_i}^t\frac{\xi(s)e^{(\gamma-\beta)(t-s)}}{(t-s)^\alpha(s-s_i)^\alpha}\,ds=0. \tag{2.20}$$

Proof. Let $t\in[s_i,t_{i+1}]$ and $\delta=(t-s_i)/2$. The integrability of the function $s\to\xi(s)/((t-s)^\alpha(s-s_i)^\alpha)$ follows from the fact that $p>1/(1-\alpha)$ and from the inequality,

$$\begin{aligned}\int_{s_i}^t\frac{\xi(s)}{(t-s)^\alpha(s-s_i)^\alpha}\,ds&\le\bigg(\int_{s_i}^{s_i+\delta}\frac{\xi(s)}{\delta^\alpha(s-s_i)^\alpha}\,ds+\int_{s_i+\delta}^t\frac{\xi(s)}{(t-s)^\alpha\delta^\alpha}\,ds\bigg)\\&\le\|\xi\|_{L^p([s_i,t_{i+1}];\mathbb{R})}\delta^{-\alpha}\bigg(\frac{\delta^{1/p'-\alpha}}{[1-\alpha p']^{1/p'}}+\frac{\delta^{1/p'-\alpha}}{[1-\alpha p']^{1/p'}}\bigg).\end{aligned}$$

To prove the second assertion, by noting that $p>1/(1-\alpha)$ we select $1<r$ such that $1/(\alpha p')>r$. Then

$$\begin{aligned}(t-s_i)^\alpha\int_{s_i}^t\frac{\xi(s)e^{(\gamma-\beta)(t-s)}}{(t-s)^\alpha(s-s_i)^\alpha}\,ds&=\int_{s_i}^t\xi(s)e^{(\gamma-\beta)(t-s)}\bigg(\frac{1}{(t-s)}+\frac{1}{(s-s_i)}\bigg)^\alpha ds\\&\le2^\alpha\int_{s_i}^t\xi(s)e^{(\gamma-\beta)(t-s)}\bigg(\frac{1}{(t-s)^\alpha}+\frac{1}{(s-s_i)^\alpha}\bigg)ds\end{aligned}$$

$$
\begin{aligned}
&\le 2^{\alpha}\|\xi\|_{L^p([s_i,t_i])}\bigg(\bigg(\int_{s_i}^{t}\frac{e^{p'(\gamma-\beta)(t-s)}}{(t-s)^{\alpha p'}}\,ds\bigg)^{1/p'}+\bigg(\int_{s_i}^{t}\frac{e^{p'(\gamma-\beta)(t-s)}}{(s-s_i)^{\alpha p'}}\,ds\bigg)^{1/p'}\bigg)\\
&\le 2^{\alpha}\|\xi\|_{L^p([s_i,t_i])}\bigg(\int_{s_i}^{t}e^{p'r'(\gamma-\beta)(t-s)}\,ds\bigg)^{1/(r'p')}\bigg(\int_{s_i}^{t}\frac{ds}{(t-s)^{\alpha p'r}}\bigg)^{1/(p'r)}\\
&\quad+2^{\alpha}\|\xi\|_{L^p([s_i,t_i])}\bigg(\int_{s_i}^{t}e^{p'r'(\gamma-\beta)(t-s)}\,ds\bigg)^{1/(r'p')}\bigg(\int_{s_i}^{t}\frac{ds}{(s-s_i)^{\alpha p'r}}\bigg)^{1/p'r}\\
&\le 2^{\alpha}\|\xi\|_{L^p([s_i,t_i])}\bigg(\frac{1}{p'r'(\beta-\gamma)}\bigg)^{1/(r'p')}2\bigg(\frac{\delta_i^{1-\alpha p'r}}{1-\alpha p'r}\bigg)^{1/(\alpha p'r)},
\end{aligned}
$$

and hence

$$
\begin{aligned}
\text{(2.21)}\qquad &\sup_{t\in[s_i,t_{i+1}]}(t-s_i)^{\alpha}\int_{s_i}^{t}\frac{\xi(s)e^{(\gamma-\beta)(t-s)}}{(t-s)^{\alpha}(s-s_i)^{\alpha}}\,ds\\
&\le \Xi_{\beta,\alpha,i}(\xi):=2^{\alpha+1}\|\xi\|_{L^p([s_i,t_i])}\bigg(\frac{1}{p'r'(\beta-\gamma)}\bigg)^{1/(r'p')}\bigg(\frac{\delta_i^{1-\alpha p'r}}{1-\alpha p'r}\bigg)^{1/(\alpha p'r)},
\end{aligned}
$$

which completes the proof since $\Xi_{\beta,\alpha,i}(\xi)\to 0$ as $\beta\to\infty$. □

Proceeding as in the proof of Lemma 2.6 we can prove the next result.

LEMMA 2.7. *If $\xi\in L^p([s_i,t_{i+1}];\mathbb{R})$ for some $i\in\{i,\dots,N\}$ and $p>1/(1-\alpha)$, then*

$$
\begin{aligned}
\text{(2.22)}\qquad &\sup_{t\in[s_i,t_{i+1}]}(t-s_i)^{\alpha}\int_{s_i}^{t}\frac{\xi(s)}{(t-s)^{\alpha}(s-s_i)^{\alpha}}\,ds\\
&\qquad\qquad\le 2^{\alpha+1}\delta_i^{1/(\alpha p'-1}[1-\alpha p']^{-1/(\alpha p')}\|\xi\|_{L^p([s_i,t_i])}.
\end{aligned}
$$

In Theorem 2.8 we establish the existence of a mild solution for (2.17)–(2.19). In this result, $\Xi_{\beta,\alpha,i}(L_{f_\alpha})$ is defined as in (2.21) and $\widetilde{s_{i,j}}$, $\widehat{s_{i,j}}$, $s^*_{i,j}$ with $1\le j\le i$, are the numbers defined by

$$
\begin{aligned}
\widetilde{s_{i,j}}&=\sup\bigcup_{t\in(t_i,s_i]}I_i(t)\cap[s_{j-1},t_j],\\
\widehat{s_{i,j}}&=\sup\bigcup_{t\in(t_i,s_i]}I_i(t)\cap(t_j,s_j],\\
s^*_{i,j}&=\inf\bigcup_{t\in(s_i,t_{i+1}]}I_i(t)\cap[s_{j-1},t_j].
\end{aligned}
$$

THEOREM 2.8. *Assume the conditions* $(\mathrm{H}_{\alpha,1})$, $(\mathrm{H}_{\alpha,2})$, $(\mathrm{H}_{\alpha,4})$ *and* $(\mathrm{H}_{\alpha,5})$ *are satisfied, $t_i>\max\{\widetilde{s_{i,j}},\widehat{s_{i,j}}\}$ and $s^*_{i,j}>s_{j-1}$ for all $1\le j\le i$. Then there exists a mild solution $u\in\mathcal{P}_\beta\mathcal{C}(X,X_\alpha)$ of the impulsive problem* (2.17)–(2.19).

PROOF. Let $\Gamma\colon\mathcal{P}_\beta\mathcal{C}(X,X_\alpha)\to\mathcal{P}_\beta\mathcal{C}(X,X_\alpha)$ be defined as in the proof of Theorem 2.3. We will prove that there exists $\beta\ge\gamma$ large enough such that Γ is a contraction on $\mathcal{P}_\beta\mathcal{C}(X,X_\alpha)$.

To begin, for $u \in \mathcal{P}_\beta\mathcal{C}(X, X_\alpha)$ and $t \in (t_i, s_i]$ it is convenient to estimate the expression $\|u_{|_{I_i(t)}}\|_{\mathcal{PC}(X,X_\alpha)_{|_{I_i(t)}}}$. By noting that $I_i(t) \cap [s_i, t_{i+1}] = \emptyset$, for $u \in \mathcal{P}_\beta\mathcal{C}(X, X_\alpha)$ and $t \in (t_i, s_i]$ we note that

$$\begin{aligned}
\|u_{|_{I_i(t)}}\|_{\mathcal{PC}(X,X_\alpha)_{|_{I_i(t)}}} &\leq \max_{1\leq j\leq i}\bigg\{\sup_{s\in I_i(t)\cap[s_{j-1},t_j]}\|u(s)\|_\alpha, \sup_{s\in I_i(t)\cap(t_j,s_j]}\|u(s)\|\bigg\}\\
&\leq \max_{1\leq j\leq i}\bigg\{\sup_{s\in I_i(t)\cap[s_{j-1},t_j]}\frac{e^{\beta s}\|u\|_{\mathcal{P}_\beta\mathcal{C}(X,X_\alpha)}}{(s-s_{j-1})^\alpha}, \sup_{s\in I_i(t)\cap(t_j,s_j]}e^{\beta s}\|u\|_{\mathcal{P}_\beta\mathcal{C}(X,X_\alpha)}\bigg\},
\end{aligned}$$

so that

$$\|u_{|_{I_i(t)}}\|_{\mathcal{PC}(X,X_\alpha)_{|_{I_i(t)}}} \leq \max_{1\leq j\leq i}\bigg\{\frac{e^{\beta\widetilde{s_{i,j}}}}{(s^*_{i,j}-s_{j-1})^\alpha}, e^{\beta\widehat{s_{i,j}}}\bigg\}\|u\|_{\mathcal{P}_\beta\mathcal{C}(X,X_\alpha)}. \tag{2.23}$$

From the above inequality, for $t \in [s_i, t_{i+1}]$ we get

$$\begin{aligned}
\|(-A)^\alpha\Gamma u(t)\| &\leq \frac{C_\alpha e^{\gamma(t-s_i)}}{(t-s_i)^\alpha}\|L_{h_{i,\alpha}}\|_{C((t_i,s_i];\mathbb{R})}\widetilde{\Psi}_{i,\alpha}(s_i)\|u_{|_{I_i(s_i)}}\|_{\mathcal{PC}(X_\alpha)_{|_{I_i(s_i)}}}\\
&\quad+\frac{C_\alpha e^{\gamma(t-s_i)}}{(t-s_i)^\alpha}\|h_{i,\alpha}(s_i,0)\|\\
&\quad+C_\alpha\int_{s_i}^t\frac{L_{f_\alpha}(s)e^{\gamma(t-s)}}{(t-s)^\alpha}\|u(s)\|_\alpha\,ds + C_\alpha\int_{s_i}^t\frac{e^{\gamma(t-s)}}{(t-s)^\alpha}\|f_\alpha(s,0)\|\,ds\\
&\leq \frac{C_\alpha e^{\gamma(t-s_i)}}{(t-s_i)^\alpha}\|L_{h_{i,\alpha}}\|_{C((t_i,s_i];\mathbb{R})}\widetilde{\Psi}_{i,\alpha}(s_i)\\
&\quad\times\max_{1\leq j\leq i}\bigg\{\frac{e^{\beta\widetilde{s_{i,j}}}}{(s^*_{i,j}-s_{j-1})^\alpha}, e^{\beta\widehat{s_{i,j}}}\bigg\}\|u\|_{\mathcal{P}_\beta\mathcal{C}(X,X_\alpha)}\\
&\quad+\frac{C_\alpha e^{\gamma(t-s_i)}}{(t-s_i)^\alpha}\|h_{i,\alpha}(s_i,0)\| + C_\alpha\|u\|_{\mathcal{P}_\beta\mathcal{C}(X,X_\alpha)}\int_{s_i}^t\frac{L_{f_\alpha}(s)e^{\gamma(t-s)}e^{\beta s}}{(t-s)^\alpha(s-s_i)^\alpha}\,ds\\
&\quad+C_\alpha\|f_\alpha(\,\cdot\,,0)\|_{C([0,a];X)}e^{\gamma(t_{i+1}-s_i)}\frac{\delta_i^{1-\alpha}}{1-\alpha},
\end{aligned}$$

which implies via (2.21) that

$$\begin{aligned}
\sup_{t\in[s_i,t_{i+1}]}(t-s_i)^\alpha e^{-\beta t}\|(-A)^\alpha\Gamma u(t)\| &\leq C_\alpha\|L_{h_{i,\alpha}}\|_{C((t_i,s_i];\mathbb{R})}\\
&\quad\times\widetilde{\Psi}_{i,\alpha}(s_i)\|u\|_{\mathcal{P}_\beta\mathcal{C}(X,X_\alpha)}e^{\gamma\delta_i-\beta s_i}\max_{1\leq j\leq i}\bigg\{\frac{e^{\beta\widetilde{s_{i,j}}}}{(s^*_{i,j}-s_{j-1})^\alpha}, e^{\beta\widehat{s_{i,j}}}\bigg\}\\
&\quad+C_\alpha\sup_{t\in[s_i,t_{i+1}]}e^{\gamma\delta_i-\beta s_i}\|h_{i,\alpha}(s_i,0)\|\\
&\quad+C_\alpha\|u\|_{\mathcal{P}_\beta\mathcal{C}(X,X_\alpha)}\Xi_{\beta,\alpha,i}(L_{f_\alpha})\\
&\quad+e^{\gamma\delta_i-\beta s_i}C_\alpha\|f_\alpha(\,\cdot\,,0)\|_{C([0,a];X)}\frac{\delta_i}{1-\alpha}.
\end{aligned}\tag{2.24}$$

Proceeding as above, we prove that

$$\begin{aligned}
(2.25)\qquad & \sup_{t\in[0,t_1]} t^{\alpha}e^{-\beta t}\|(-A)^{\alpha}\Gamma u(t)\| \\
&\le C_{\alpha}\|u\|_{\mathcal{P}_{\beta}\mathcal{C}(X,X_{\alpha})}\Xi_{\beta,\alpha,0}(L_{f_{\alpha}})(f_{\alpha}) + e^{\gamma t_1}C_{\alpha}\|f_{\alpha}(\,\cdot\,,0)\|_{C([0,a];X)}\frac{t_1}{1-\alpha}.
\end{aligned}$$

On the other hand, by using the estimate (2.23), for $t\in[t_i,s_i]$ we see that

$$\begin{aligned}
\|\Gamma u(t)\| &\le L_{h_{i,\alpha}}(t)\widetilde{\Psi}_{i,\alpha}(t)\|u_{|_{I_i(t)}}\|_{\mathcal{P}\mathcal{C}(X,X_{\alpha})_{|_{I_i(t)}}} + \|h_{i,\alpha}(s_i,0)\| \\
&\le \|L_{h_{i,\alpha}}\|_{C((t_i,s_i];\mathbb{R})}\widetilde{\Psi}_{i,\alpha}(t)\|u\|_{\mathcal{P}_{\beta}\mathcal{C}(X,X_{\alpha})}\max_{1\le j\le i}\left\{\frac{e^{\beta\widetilde{s_{i,j}}}}{(s^{*}_{i,j}-s_{j-1})^{\alpha}}, e^{\beta\widehat{s_{i,j}}}\right\} \\
&\quad + \|h_{i,\alpha}(s_i,0)\|,
\end{aligned}$$

and hence,

$$\begin{aligned}
\sup_{t\in(t_i,s_i]} e^{-\beta t}\|\Gamma u(t)\| &\le \|L_{h_{i,\alpha}}\|_{C((t_i,s_i];\mathbb{R})} \\
&\times \sup_{t\in(t_i,s_i]}\widetilde{\Psi}_{i,\alpha}(t)\|u\|_{\mathcal{P}_{\beta}\mathcal{C}(X,X_{\alpha})}\max_{1\le j\le i}\left\{\frac{e^{\beta(\widetilde{s_{i,j}}-t_i)}}{(s^{*}_{i,j}-s_{j-1})^{\alpha}}, e^{\beta(\widehat{s_{i,j}}-t_i)}\right\} \\
&\qquad + \|h_{i,\alpha}(s_i,0)\|.
\end{aligned}$$

From the above estimates is obvious that $\|\Gamma u\|_{\alpha,\beta}$ is finite, which proves that Γ is a well defined $\mathcal{P}_{\beta}\mathcal{C}(X,X_{\alpha})$-valued function. Moreover, from the above we also infer that

$$\begin{aligned}
(2.26)\qquad & \|\Gamma u-\Gamma v\|_{\alpha,\beta} \\
&\le \max_{i=1,\dots,N}\{\Theta_{i,\alpha}(\beta)+C_{\alpha}\Xi_{\beta,\alpha,i}(L_{f_{\alpha}}), C_{\alpha}\Xi_{\beta,\alpha,0}(L_{f_{\alpha}}), \Phi_{i,\alpha}(\beta)\}\|u-v\|_{\mathcal{P}_{\beta}\mathcal{C}(X,X_{\alpha})},
\end{aligned}$$

where

$$\Theta_{i,\alpha}(\beta) = C_{\alpha}\|L_{h_{i,\alpha}}\|_{C((t_i,s_i];\mathbb{R})}\widetilde{\Psi}_{i,\alpha}(s_i)\max_{1\le j\le i}\left\{\frac{e^{\gamma\delta_i-\beta(s_i-\widetilde{s_{i,j}})}}{(s^{*}_{i,j}-s_{j-1})^{\alpha}}, e^{\gamma\delta_i-\beta(s_i-\widehat{s_{i,j}})}\right\},$$

$$\Phi_{i,\alpha}(\beta) = \|L_{h_{i,\alpha}}\|_{C((t_i,s_i];\mathbb{R})}\sup_{t\in(t_i,s_i]}\widetilde{\Psi}_{i,\alpha}(t)\max_{1\le j\le i}\left\{\frac{e^{\beta(\widetilde{s_{i,j}}-t_i)}}{(s^{*}_{i,j}-s_{j-1})^{\alpha}}, e^{\beta(\widehat{s_{i,j}}-t_i)}\right\}.$$

From the assumptions and Lemma 2.6 we have that

$$\Theta_{i,\alpha}(\beta)+\Xi_{\beta,\alpha,i}(L_{f_{\alpha}})+\Xi_{\beta,\alpha,0}(L_{f_{\alpha}})+\Phi_{i,\alpha}(\beta)\to 0,\quad \text{as } \beta\to\infty,$$

which implies via (2.26) that there exists $\beta>\gamma$ large enough such that Γ is a contraction on $\mathcal{P}_{\beta}\mathcal{C}(X,X_{\alpha})$. Thus, there exists a unique mild solution $u\in\mathcal{P}_{\beta}\mathcal{C}(X,X_{\alpha})$ of problem (2.17)–(2.19). $\square$

From the proof of Theorem 2.8, it is easy to infer the following result.

Proposition 2.9. *Let conditions* $(\mathrm{H}_{\alpha,1})$, $(\mathrm{H}_{\alpha,2})$, $(\mathrm{H}_{\alpha,4})$–$(\mathrm{H}_{\alpha,5})$ *be holded,* $C_{\alpha}\Xi_{\gamma,\alpha,0}(L_{f_{\alpha}})<1$ *and*

$$C_{\alpha}\|L_{h_{i,\alpha}}\|_{C((t_i,s_i];\mathbb{R})}\widetilde{\Psi}_{i,\alpha}(s_i)\max_{1\le j\le i}\left\{\frac{e^{\gamma\delta_i-\gamma(s_i-\widetilde{s_{i,j}})}}{(s^{*}_{i,j}-s_{j-1})^{\alpha}}, e^{\gamma\delta_i-\gamma(s_i-\widehat{s_{i,j}})}\right\} < 1,$$

$$\|L_{h_{i,\alpha}}\|_{C((t_i,s_i];\mathbb{R})} \sup_{t\in(t_i,s_i]} \widetilde{\Psi}_{i,\alpha}(t) \max_{1\le j\le i} \left\{ \frac{e^{\gamma(\widetilde{s_{i,j}}-t_i)}}{(s^*_{i,j}-s_{j-1})^\alpha}, e^{\gamma(\widehat{s_{i,j}}-t_i)} \right\} < 1,$$

for all $i \in \{1,\ldots,N\}$. Then there exists a unique mild solution $u \in \mathcal{P}_\gamma\mathcal{C}(X,X_\alpha)$ of problem (2.17)–(2.19).

2.2. Classical solutions. In this section we study the existence of a classical solution for the impulsive problem (1.1)–(1.3). For convenience, we remark on some well known concepts on abstract systems of the form

$$u'(t) = Au(t) + \xi(t), \quad t \in [0,a], \tag{2.27}$$

$$u(0) = x \in X, \tag{2.28}$$

where $\xi \in L^1([0,a],X)$. A mild solution of the problem (2.27)–(2.28) on $[0,b]$, $0 < b \le a$, is the function defined by $u(t) = T(t)x + \int_0^t T(t-s)\xi(s)\,ds$. In addition, a function $u \in C([0,b],X)$, $0 < b \le a$, is said to be a classical solution of (2.27)–(2.28) on $[0,b]$ if $u \in C^1([0,b],X) \cap C([0,b],\mathcal{D})$, $u(0) = x$ and $u(\,\cdot\,)$ is a solution of (2.27) on $[0,b]$.

Our first result follows directly from [14, Theorem 4.3.1] and [16, Theorem 6.1.5]. Concerning the proof of this result we only note that the space X_α is continuously embedded in the space $D_A(\alpha,\infty)$, the space in the statement of [14, Theorem 4.3.1]. In the remainder of this section, we always assume that $u(\,\cdot\,)$ is a mild solution of the problem (1.1)–(1.3).

Proposition 2.10. *If any one of the following conditions is satisfied*:

(a) *the function $f(\,\cdot\,,u(\,\cdot\,))$ belongs to $C^\alpha([0,a]X;X)$, $x_0 \in D(A)$, $A_{x_0} + f(0,x_0) \in X_\alpha$, $h_i(s_i,u_{|_{I_i(s_i)}}) \in D(A)$ and $Ah_i(s_i,u_{|_{I_i(s_i)}}) + f(s_i,u(s_i)) \in X_\alpha$ for all $i \in \{1,\ldots,N\}$,*

(b) *the function $f(\,\cdot\,,u(\,\cdot\,))$ belongs to $C^1([0,a];X)$, $x_0 \in D(A)$, $h_i(s_i,u_{|_{I_i(s_i)}}) \in D(A)$ for all $i \in \{1,\ldots,N\}$,*

then $u(\,\cdot\,)$ is a classical solution of the problem (1.1)–(1.3).

The next results are motivated by Proposition 2.10. In Proposition 2.11 (resp. in Proposition 2.12) we establish conditions on $f(\,\cdot\,)$, $h_i(\,\cdot\,)$ and x_0 which implies that the condition (a) (resp. the condition (b)) in Proposition 2.10 is satisfied. In the examples of the last section, we can see that the assumptions used in the next propositions are not restrictive.

Proposition 2.11. *Assume that $f \in C^1([0,a]\times X;X)$, $x_0 \in D(A)$ and the next conditions are satisfied*:

(a) *$h_i(s_i,w) \in D(A)$ if $w \in \mathcal{C}_i(s_i)$ is a $D(A)$-valued continuous function on $I_i(s_i)$.*

(b) *$I_i(s_i) \subset \bigcup_{j<i-1} (s_j,t_{j+1}]$ for all $i \in \{1,\ldots,N\}$.*

Then $u(\,\cdot\,)$ is a classical solution of (1.1)–(1.3).

PROOF. Since $x_0 \in D(A)$, $f \in C^1([0,t_1]\times X; X)$ and $u_{|_{[0,t_1]}}$ is a mild solution of the problem

$$v'(t) = Av(t) + f(t, v(t)), \quad t \in [0, t_1], \quad v(0) = x_0, \tag{2.29}$$

on $[0, t_1]$, from [16, Theorem 6.1.5] it follows that $u_{|_{[0,t_1]}}$ is a classical solution of (2.29) and $u \in C^1([0,t_1]; X) \cap C([0,t_1]; \mathcal{D})$.

Now we prove that $u_{|_{[s_1,t_2]}}$ is a classical solution of the problem

$$v'(t) = Av(t) + f(t, v(t)), \quad t \in [s_1, t_2], \quad v(s_1) = h_1(s_1, u_{|_{I_i(s_1)}}), \tag{2.30}$$

on $[s_1, t_2]$. Since $I_1(s_1) \subset [0, t_1]$ and $u \in C([0,t_1]; \mathcal{D})$, from condition (a) we have that $h_1(s_1, u_{|_{I_1(s_1)}}) \in D(A)$. By noting now that $f \in C^1([s_1,t_2] \times X; X)$ and $u_{|_{[s_1,t_2]}}$ is a mild solution of (2.30), from [16, Theorem 6.1.5] we infer that $u_{|_{[s_1,t_2]}}$ is a classical solution of problem (2.30) and $u \in C^1([s_1,t_2]; X) \cap C([s_1,t_2]; \mathcal{D})$. Continuing as above we prove that $u_{|_{[s_i,t_{i+1}]}} \in C^1([s_i,t_{i+1}]; X) \cap C([s_i,t_{i+1}]; \mathcal{D})$ for all $i \in \{1, \dots, N\}$ and $u_{|_{[s_i,t_{i+1}]}}$ is a classical solution of the problem

$$v'(t) = Av(t) + f(t, v(t)), \quad t \in [s_i, t_{i+1}], \quad v(s_i) = h_1(s_i, u_{|_{s_i}}), \tag{2.31}$$

on $[s_i, t_{i+1}]$. □

PROPOSITION 2.12. *Assume $f \in C([0,a]\times X; X)$, $x_0 \in D(A)$, $A_{x_0} + f(0, x_0)$ in X_α and there exists $L_f > 0$ such that*

$$\|f(t,x) - f(s,y)\| \le L_f(|\, t-s\,|^\alpha + \|x-y\|), \tag{2.32}$$

for all $t, s \in [0,a]$ and $x, y \in X$. Suppose there are $\beta_1 > \beta_2 \ge \alpha$ such that

(a) *for all $i \in \{1, 2, \dots, N\}$, $I_i(s_i) \subset \cup_{j<i}(s_j, t_{j+1}]$ and $h_i(s_i, w) \in X_{1+\alpha}$ if $w \in \mathcal{C}_i(s_i)$ is a $X_{1+\beta_2}$-valued continuous function,*

(b) *the function $f(\,\cdot\,, v(\,\cdot\,)) \in C([c,d]; X_{\beta_1})$ if $v \in C([c,d]; \mathcal{D})$ for some $[c,d] \subset [0,a]$.*

Then $u(\,\cdot\,)$ is a classical solution and

$$u_{|_{[s_i,t_{i+1}]}} \in C^\alpha([s_i,t_{i+1}]; \mathcal{D}) \cap C^{1+\alpha}([s_i,t_{i+1}]; X) \quad \textit{for all } i \in \{1, \dots, N\}.$$

PROOF. Since $f(\,\cdot\,, u(\,\cdot\,))$ is continuous on $[0, t_1]$ and $u(\,\cdot\,)$ is a mild solution of (2.29), from [14, Proposition 4.2.1] we have that $u \in C^\alpha([0,t_1]; X)$ which implies via (2.32) that $f(\,\cdot\,, u(\,\cdot\,)) \in C^\alpha([0,t_1]; X)$. Now, from [14, Theorem 4.3.1] we infer that $u_{|_{[0,t_1]}}$ is a classical solution of the problem (2.29) on $[0, t_1]$ and $u_{|_{[0,t_1]}} \in C^\alpha([0,t_1]; \mathcal{D}) \cap C^{1+\alpha}([0,t_1]; X)$.

We prove now that $u_{|_{[s_1,t_2]}}$ is a classical solution of (2.30) on $[s_1, t_2]$. Arguing as above, from [14, Proposition 4.2.1] and (2.32) we infer that $f(\,\cdot\,, u(\,\cdot\,)) \in C^\alpha([s_1,t_1]; X)$.

On the other hand, since $I_1(s_1) \subset [0,t_1]$ and $u_{|_{[0,t_1]}} \in C([0,t_1];\mathcal{D})$, from condition (b) we have $f(\,\cdot\,,u(\,\cdot\,)) \in C([0,t_1];X_{\beta_1})$. Using this fact and noting that

$$A^{1+\beta_2}u(t) = A^{1+\beta_2}T(t)x_0 + \int_0^t (-A)^{1+\beta_2-\beta_1}T(t-s)(-A)^{\beta_1}f(s,u(s)\,ds,$$

for $t \in (0,t_1]$, we infer that $u_{|_{(0,t_1]}} \in C((0,t_1];X_{1+\beta_2})$, which implies from the condition (a) that $Ah_1(t_1,u_{|_{I_1(t_1)}}) \in X_\alpha$. Moreover, from the above we also obtain that $Ah_1(t_1,u_{|_{I_1(t_1)}}) + f(s_1,u(s_1)) \in X_\alpha + X_{\beta_1} \subset X_\alpha$. Now, from [14, Theorem 4.3.1] and the fact that $u_{|_{[s_1,t_2]}}$ is a mild solution of (2.30) on $[s_1,t_2]$ we can conclude that $u_{|_{[s_1,t_2]}}$ is a classical solution of (2.30) on $[s_1,t_2]$ and $u_{|_{[s_1,t_2]}} \in C^\alpha([s_1,t_2];\mathcal{D}) \cap C^{1+\alpha}([s_1,t_2];X)$. Continuing as above, we can complete the proof. □

3. Examples

In this section we consider some applications of our abstract results. Here, $X = L^2([0,\pi])$ and $A\colon D(A) \subset X \to X$ is the operator given by $Ax = x''$ on $D(A) := \{x \in X : x'' \in X,\ x(0) = x(\pi) = 0\}$. It is well known that A is the infinitesimal generator of a compact semigroup $(T(t))_{t\geq 0}$ on X and that $\|T(t)\| \leq 1$ for all $t \geq 0$.

To begin we consider the impulsive problem

$$\frac{\partial w}{\partial t}(t,\xi) = \frac{\partial^2 w}{\partial \xi^2}(t,\xi) + F(t,w(t,\xi)), \qquad (t,\xi) \in \bigcup_{i=1}^{N}[s_i,t_{i+1}] \times [0,\pi], \tag{3.1}$$

$$w(t,0) = w(t,\pi) = 0, \qquad t \in [0,a], \tag{3.2}$$

$$w(0,\xi) = z(\xi), \qquad \xi \in [0,\pi], \tag{3.3}$$

$$w(t,\xi) = G_i\bigg(t, \int_{t_i}^t \zeta_i(s)w(s,\xi)\,ds\bigg), \quad \xi \in [0,\pi],\ t \in (t_i,s_i], \tag{3.4}$$

where $0 = t_0 = s_0 < t_1 \leq s_1 < \ldots < t_N \leq s_N < t_{N+1} = a$ are fixed real numbers, $z \in X$, $F \in C([0,a]\times\mathbb{R};\mathbb{R})$, $G_i \in C((t_i,s_i]\times\mathbb{R};\mathbb{R})$ and $\zeta_i \in C((t_i,s_i];\mathbb{R})$ for all $i = 1,\ldots,N$.

To represent this problem in the form (1.1)–(1.3), for $t \in (t_i,s_i]$ and $i = 1,\ldots,N$ we introduce the space $\mathcal{C}_i(t) = C((t_i,t];X)$ endowed with the uniform norm denoted by $\|\cdot\|_{\mathcal{C}_i(t)}$. In addition, we consider the maps $f : [0,a]\times X \to X$, $I_i : (t_i,s_i] \to 2^{(t_i,s_i]}$ and $h_i(t,\,\cdot\,)\colon \mathcal{C}_i(t) \to X$ given by $f(t,x)(\xi) = F(t,x(\xi))$, $I_i(t) = (t_i,t]$ and

$$h_i(t,u)(\xi) = G_i\bigg(t, \int_{t_i}^t \zeta_i(s)u(s,\xi)\,ds\bigg).$$

In the next result, which follows from Theorem 2.3, we say that $u \in \mathcal{PC}(X)$ is a mild solution of (3.1)–(3.4) if $u(\,\cdot\,)$ is a mild solution of the associated problem (1.1)–(1.3).

Proposition 3.1. *Assume the functions* $F, G_1, \ldots G_N$, *are globally Lipschitz with Lipschitz constants* $L_F, L_{G_1}, \ldots, L_N$ *respectively, and*

$$\max\Big\{\max_{i=1,\ldots,N}\{L_{G_i}\|\zeta_i\|_{L^2((t_i,s_i])}\delta_i^{1/2} + L_F\delta_i,\ L_F t_1\Big\} < 1. \tag{3.5}$$

Then there exists a unique mild solution of the problem (3.1)–(3.4)*.*

Proof. Is easy to see that the functions $h_i(\,\cdot\,)$, $f(\,\cdot\,)$ satisfies the conditions in Theorem 2.3 with $L_{h_i} = L_G\|\zeta_i\|_{L^2((t_i,s_i])}\delta_i^{1/2}$ and $L_f = L_F$. We also note that $\widetilde{\Psi}_i(t) = 1$ and $\widehat{s}_i = s_i$ for all $t \in (t_i, s_i]$ and each $i = 1, \ldots, N$. From the above and (3.5), we have that condition (c) of Theorem 2.3 is satisfied which implies that there exists a unique mild solution of the problem (3.1)–(3.4). $\square$

We consider now the equations (3.1)–(3.3) submitted to the impulsive conditions

$$w(t,\xi) = G_i\Big(t, \int_0^{\widehat{t_i}} \eta_i(s)w(s,\xi)\,ds, \quad \xi \in [0,\pi],\ t \in (t_i, s_i]. \tag{3.6}$$

In addition to the previous conditions, we assume that $\eta_i \in C([0,\widehat{t_i}];\mathbb{R})$ and $0 < \widehat{t_i} < t_i$ for all $i = 1, \ldots, N$.

Proposition 3.2. *If the functions* $F, G_1, \ldots, G_N$, *are globally Lipschitz, then there exists a unique mild solution of* (3.1)–(3.3) *submitted to the impulsive conditions* (3.6).

Proof. We only note that condition (a) of Theorem 2.3 is satisfied with $\widehat{s}_i = \widehat{t_i}$. $\square$

We consider now the problem (3.1)–(3.3) jointly to the impulsive conditions

$$w(t,\xi) = \sum_{j<i} \int_{s_j}^{t_{j+1}} \eta_j(t,s,w(s,\xi))\,ds, \quad \xi \in [0,\pi],\ t \in (t_i, s_i], \tag{3.7}$$

where $\eta_i \in C^2([0,a] \times [t_i,s_i] \times \mathbb{R};\mathbb{R})$. To simplify, next we assume that the derivatives of the functions $\eta_i(\,\cdot\,)$ are uniformly bounded.

Proposition 3.3. *Assume that* $F(\,\cdot\,)$ *is globally Lipschitz with Lipschitz constant* L_F, $L_F t_1 < 1$ *and*

$$\max_{i\le N}\Big\{\sum_{j<i}\Big\|\frac{\partial\eta_j}{\partial x}\Big\|_{L^\infty([s_i,t_{i+1}]\times[s_j,t_{j+1}]\times\mathbb{R})}(t_{j+1}-s_j)^{1/2} + L_F(s_i - t_i)\Big\} < 1. \tag{3.8}$$

Then there exists a unique mild $u(\,\cdot\,)$ *solution of the problem* (3.1)–(3.3) *submitted to the impulsive condition* (3.7)*.*

Proof. In this case, we define f as in the first example and for $t \in (t_i, s_i]$ we take $\bigcup_{j<i} C([s_j, t_{j+1}] : X)$ endowed with the norm

$$\|u\|_{\mathcal{C}_i(t)} = \sum_{j<i} \|\cdot\|_{C([s_j,t_{j+1}]:X)},$$

$t = t$ and $I_i(s) = \bigcup_{j<i} [s_j, t_{j+1}]$ for $s \in (t_i, s_i]$. In addition, we define maps $h_i(t, \cdot)\colon \mathcal{C}(t) \to X$ by

$$h_i(t,u)(\xi) = \sum_{j<i} \int_{s_j}^{t_{j+1}} \eta_j(t, s, u(s,\xi))\, ds.$$

Since the derivatives of of the function $\eta_i(\cdot)$ are uniformly bounded, we have that each $h_i(\cdot)$ is globally Lipstchiz with Lipschitz constant

$$L_{h_i} \leq \sum_{j<i} \left\| \frac{\partial \eta_i}{\partial x} \right\|_{L^\infty([s_i,t_{i+1}]\times[s_j,t_{j+1}]\times\mathbb{R})} (t_{j+1} - s_j).$$

From the above and (3.8) we have that condition (c) in Theorem 2.3 is satisfied. Thus, there exists a unique mild solution of the problem (3.1)–(3.3) with impulsive conditions (3.7). $\square$

Concerning the existence of classical solutions, we have the next corollary which follows directly from Proposition 2.11. We omit the proof.

Corollary 3.4. *Assume the conditions in Proposition* 3.3 *are fulfilled and let $u(\cdot)$ be a mild solution of* (3.1)–(3.3) *with impulsive condition* (3.7). *Suppose that $F(\cdot)$ is continuously differentiable, there is $L_F > 0$ such that*

$$\left| \frac{\partial F}{\partial x}(t,z) - \frac{\partial F}{\partial x}(t,w) \right| \leq L_F |z - w|$$

for all $z, w \in \mathbb{R}$ and every $t \in [0,a]$, $\eta_i(t,0) = 0$ for every $t \in (t_i, s_i]$ and all $i = 1, \ldots, N$ and $x_0 \in D(A)$. Then $u(\cdot)$ is a classical solution.

References

[1] N. Abada, M. Benchohra, Mouffak and H. Hammouche, *Existence and controllability results for nondensely defined impulsive semilinear functional differential inclusions*, J. Differential Equations **246** (2009), no. 10, 3834–3863.

[2] M. Benchohra, J. Henderson and S. Ntouyas, *Impulsive differential equations and inclusions*, Contemporary Mathemathics and its Applications, 2, Hindawi Publishing Corporation, New York, 2006.

[3] J. Chu, J.J. Nieto, *Impulsive periodic solutions of first-order singular differential equations*, Bull. Lond. Math. Soc. **40** (2008), no. 1, 143–150.

[4] Z. Fan and G. Li, *Existence results for semilinear differential equations with nonlocal and impulsive conditions*, J. Funct. Anal. **258** (2010), no. 5, 1709–1727.

[5] D. Franco, E. Liz, J.J. Nietoand Y.V. Rogovchenko, *A contribution to the study of functional differential equations with impulses*, Math. Nachr. **218** (2000), 49–60.

[6] M. Frigon and D. O'Regan, *First order impulsive initial and periodic problems with variable moments*, J. Math. Anal. Appl. **233** (1999), no. 2, 730–739.

[7] ———, *Existence results for first-order impulsive differential equations*, J. Math. Anal. Appl. **193** (1995), no. 1, 96–113.

[8] E. Hernández and D. O'Regan, *On a new class of abstract impulsive differential equations*, Proc. Amer. Math. Soc. **141** (2013), no. 5, 1641–1649.

[9] E. Hernández, H. Henríquez and M. Rabello, *Existence of solutions for a class of impulsive partial neutral functional differential equations*, J. Math. Anal. Appl. **331** (2007), 1135–1158.

[10] C. Kou, S. Zhang and S. Wu, *Stability analysis in terms of two measures for impulsive differential equations*, J. London Math. Soc. (2) **66** (2002), no. 1, 142–152.

[11] V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of Impulsive Differential Equations*, Series in Modern Applied Mathematics, vol. 6, World Scientific Publishing Co.; Inc., Teaneck, NJ, 1989.

[12] Y. Liao and J. Wang, *A note on stability of impulsive differential equations*, Boundary Value Problems 2014, 2014:67. http://www.boundaryvalueproblems.com/content/2014/1/67

[13] J.H. Liu, *Nonlinear impulsive evolution equations*,Dynam. Contin. Discrete Impuls. Systems **6** (1999), no. 1, 77–85.

[14] A. Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, PNLDE Vol. 16, Birkhäauser Verlag, Basel, 1995.

[15] J.J. Nieto and D. O'Regan, *Variational approach to impulsive differential equations*, Nonlinear Anal. Real World Appl. **10** (2009), no. 2, 680–690.

[16] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer–Verlag, New York–Berlin, 1983.

[17] M. Pierri, D. O'Regan and V. Rolnik,*Existence of solutions for semi-linear abstract differential equations with not instantaneous impulses*, Appl. Math. Comput. **219** (2013), no. 12, 6743–6749.

[18] Y.V. Rogovchenko, *Impulsive evolution systems: main results and new trends*, Dynam. Contin. Discrete Impuls. Systems **3** (1997), no. 1, 57–88.

[19] Y.V. Rogovchenko, *Nonlinear impulse evolution systems and applications to population models*, J. Math. Anal. Appl. **207** (1997), no. 2, 300–315.

[20] A.M. Samoilenko and N.A. Perestyuk, *Impulsive differential equations*, With a preface by Yu.A. Mitropol'skiĭ and a supplement by S.I. Trofimchuk. Transl. from the Russian by Y. Chapovsky. World Scientific Series on Nonlinear Science, Ser A. Monographs and Treatises, vol. 14, World Scientific Publishing Co., Inc., River Edge, NJ, 1995.

[21] J. Wang and X. Li, *Periodic* BVP *for integer/fractional order nonlinear differential equations with non-instantaneous impulses*, J. Appl. Math. Comput., DOI: 10.1007/s12190-013-0751-4.

[22] J. Wang and Z. Lin, *A class of impulsive nonautonomous differential equations and Ulam–Hyers–Rassias stability*, Math. Methods Appl. Sci., DOI: 10.1002/mma.3113.

[23] J. Wang and Y. Zhang, *Existence and stability of solutions to nonlinear impulsive differential equations in β-normed spaces*, Electron. J, Differential Equations **2014** (2014), No. 83, 1–10.

[24] ———, *A class of nonlinear differential equations with fractional integrable impulses*, Commun. Nonlinear Sci. Numer. Simul. **19** (2014), 3001–3010.

[25] J.S. Yu and X.H. Tang, *Global attractivity in a delay population model under impulsive perturbations*, Bull. London Math. Soc. **34** (2002), no. 3, 319–328.

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