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INFINITELY MANY SOLUTIONS FOR SYSTEMS OF MULTI-POINT BOUNDARY VALUE PROBLEMS USING VARIATIONAL METHODS

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Abstract. In this paper, we obtain the existence of infinitely many classical solutions to the multi-point boundary value system

$$\begin{cases} -(\phi_{p_i}(u_i'))' = \lambda F_{u_i}(x, u_1, \ldots, u_n), & t \in (0,1), \\ u_i(0) = \displaystyle\sum_{j=1}^m a_j u_i(x_j), \quad u_i(1) = \sum_{j=1}^m b_j u_i(x_j), & \end{cases} \qquad i = 1, \ldots, n.$$

Our analysis is based on critical point theory.

1. Introduction

In this paper, we discuss the existence of infinitely many classical solutions to the multi-point boundary value system

$$\begin{cases} -(\phi_{p_i}(u_i'))' = \lambda F_{u_i}(x, u_1, \ldots, u_n), & t \in (0,1), \\ u_i(0) = \displaystyle\sum_{j=1}^m a_j u_i(x_j), \quad u_i(1) = \sum_{j=1}^m b_j u_i(x_j), & \end{cases} \qquad i = 1, \ldots, n, \tag{1.1}$$

where $p_i > 1$, m, $n \geq 1$ are integers, $\phi_{p_i}(t) = |t|^{p_i - 2}t$ for $i = 1, \ldots, n$, λ is a positive parameter, a_j, $b_j \in \mathbb{R}$ for $j = 1, \ldots, m$, and $0 < x_1 < \ldots < x_m < 1$.

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Here, $F\colon [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a function such that the mapping $(t_1, \dots, t_n) \to F(x, t_1, \dots, t_n)$ is in C^1 in $\mathbb{R}^n$ for all $x \in [0,1]$, F_{t_i} is continuous in $[0,1] \times \mathbb{R}^n$ for $i = 1, \dots, n$, where F_{t_i} denotes the partial derivative of $F(x, t_1, \dots, t_n)$ with respect to t_i.

In the sequel, we let X be the Cartesian product of n spaces

$$X_i = \bigg\{ y \in W^{1,p_i}([0,1]) : y(0) = \sum_{j=1}^{m} a_j y(x_j), \quad y(1) = \sum_{j=1}^{m} b_j y(x_j) \bigg\},$$

for $i = 1, \dots, n$, i.e. $X = X_1 \times \dots \times X_n$, endowed with the norm

$$||u|| = ||(u_1, \dots, u_n)|| = \sum_{i=1}^{n} ||u_i||_{p_i} \quad \text{for } u = (u_1, \dots, u_n) \in X,$$

where

$$||u_i||_{p_i} = \bigg(\int_0^1 |u_i'(x)|^{p_i} \, dx \bigg)^{1/p_i}, \quad i = 1, \dots, n.$$

Then X is a reflexive real Banach space.

By a *classical solution* of (1.1), we mean a function $u = (u_1, \dots, u_n)$ such that, for $i = 1, \dots, n$, $u_i \in C^1[0,1]$, $\phi_{p_i}(u_i') \in C^1[0,1]$, and $u_i(t)$ satisfies (1.1). We say that a function $u = (u_1, \dots, u_n) \in X$ is a *weak solution* of (1.1) if

$$\int_0^1 \sum_{i=1}^{n} \phi_{p_i}(u_i'(x)) v_i'(x) \, dx - \lambda \int_0^1 \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \dots, u_n(x)) v_i(x) \, dx = 0$$

for any $v = (v_1, \dots, v_n) \in W_0^{1,p_1}([0,1]) \times W_0^{1,p_2}([0,1]) \times \dots \times W_0^{1,p_n}([0,1])$. From Lemma 2.2 below, we see that a weak solution of (1.1) is also a classical solution.

Multi-point boundary value problems arise in many applications and have been studied by many researchers in recent years. We refer the reader to [5]–[16], [18], [19] for some recent work on this topic. Our goal in this paper is to obtain some sufficient conditions to guarantee that the system (1.1) has infinitely many classical solutions. Our analysis is mainly based on a recent critical points theorem that appeared in [1], [17] and is contained in Lemma 2.1 below. This lemma and its variations have been frequently used to obtain multiplicity results for nonlinear problems of a variational nature; see, for example, [1]–[4], [17] and the references therein. Our proofs are partly motivated by these papers.

We assume throughout, and without further mention, that the following condition holds:

(H) $\sum\limits_{j=1}^{m} a_j \neq 1$ and $\sum\limits_{j=1}^{m} b_j \neq 1$.

In Section 2 we will present our main results and their proofs. One example is also included to illustrate the applicability of our results. We end this section with the following theorem which is a direct consequence of Theorem 2.3 below.

THEOREM 1.1. *Assume that there exists a function $F:\mathbb{R}^n \to \mathbb{R}$ such that $F(x_1,\dots,x_n) \ge 0$ in $\mathbb{R}^n$, $F(x_1,\dots,x_n)$ is continuously differentiable in x_i, and $\partial F/\partial x_i = f_i$ for $i = 1,\dots,n$. Suppose further that there exists $l \in \{1,\dots,n\}$ such that*

$$\liminf_{\xi\to\infty} \frac{\sup_{(t_1,\dots,t_n)\in K(\xi^{p_l})} F(t_1,\dots,t_n)}{\xi^{p_l}} = 0$$

and

$$\limsup_{\xi\to\infty} \frac{F(0,\dots,\xi,\dots,0)}{\xi^{p_l}} = \infty,$$

where, in $F(0,\dots,\xi,\dots,0)$, ξ is the l-th argument and

$$K(\xi^{p_l}) = \bigg\{(t_1,\dots,t_n) \in \mathbb{R}^n : \sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i} \le \xi^{p_l}\bigg\}.$$

Then the system

$$\begin{cases} -(\phi_{p_i}(u_i'))' = f_i(u_1,\dots,u_n), \quad t \in (0,1), \\ u_i(0) = \displaystyle\sum_{j=1}^m a_j u_i(x_j), \quad u_i(1) = \sum_{j=1}^m b_j u_i(x_j), \end{cases} \qquad i = 1,\dots,n,$$

has an unbounded sequence of classical solutions.

2. Main results

Our main tool is the following critical points theorem obtained in [1, Theorem 2.1]. This result is a refinement of the variational principle of Ricceri [17, Theorem 2.5].

LEMMA 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi: X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\Big(\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)\Big) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r\to\infty} \varphi(r) \quad \text{and} \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then:

(a) *For every $r > \inf_X \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional $I_\lambda := \Phi - \lambda\Psi$ to $\Phi^{-1}(-\infty,r)$ admits a global minimum, which is a critical point* (*local minimum*) *of I_λ in X.*

(b) *If $\gamma < \infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either*

  (b_1) *I_λ possesses a global minimum, or* (b$_2$) *there is a sequence* $\{u_n\}$ *of critical points* (*local minima*) *of* I_λ *such that*

$$\lim_{n\to\infty} \Phi(u_n) = \infty.$$

(c) *If* $\delta < \infty$, *then for each* $\lambda \in (0, 1/\delta)$, *the following alternative holds: either*

(c$_1$) *there is a global minimum of* Φ *which is a local minimum of* I_λ, *or*

(c$_2$) *there is a sequence* $\{u_n\}$ *of pairwise distinct critical points* (*local minima*) *of* I_λ *that converges weakly to a global minimum of* Φ.

The following lemma is taken from [10, Lemma 2.5].

Lemma 2.2. *A weak solution to* (1.1) *coincides with a classical solution to* (1.1).

In what follows we let X^* denote the dual space to X, and for X_i defined as in Section 1, we set

$$c = \max\bigg\{ \sup_{u_i \in X_i \setminus \{0\}} \frac{\max_{x\in[0,1]} |u_i(x)|^{p_i}}{||u_i||_{p_i}^{p_i}} : \text{for } 1 \le i \le n \bigg\}. \tag{2.1}$$

Since $p_i > 1$ for $i = 1, \ldots, n$, the embedding $X = X_1 \times \ldots \times X_n \hookrightarrow C^0([0,1]) \times \ldots \times C^0([0,1])$ is compact, and so $c < \infty$. Moreover, under condition (H), from [6, Lemma 3.1], we have

$$\sup_{v \in X_i \setminus \{0\}} \frac{\max_{x\in[0,1]} |v(x)|}{||v||_{p_i}} \le \frac{1}{2} \left(1 + \frac{\sum_{j=1}^m |a_j|}{\left|1 - \sum_{j=1}^m a_j\right|} + \frac{\sum_{j=1}^m |b_j|}{\left|1 - \sum_{j=1}^m b_j\right|} \right)$$

for $i = 1, \ldots, n$. For $i = 1, \ldots, n$, let

$$\sigma_i = \left[2^{p_i - 1} \left(x_1^{1-p_i} \left| 1 - \sum_{j=1}^m a_j \right|^{p_i} + (1 - x_m)^{1-p_i} \left| 1 - \sum_{j=1}^m b_j \right|^{p_i} \right) \right]^{1/p_i}, \tag{2.2}$$

and for any $\gamma > 0$, define the set $K(\gamma)$ by

$$K(\gamma) = \bigg\{ (t_1, \ldots, t_n) \in \mathbb{R}^n : \sum_{i=1}^n \frac{|t_i|^{p_i}}{p_i} \le \gamma \bigg\}. \tag{2.3}$$

The set $K(\gamma)$ will be used in some of our hypotheses with appropriate choices of γ.

We formulate our main result as follows.

THEOREM 2.3. *Assume that*

(A1) $F(x,t_1,\dots,t_n)\geq 0$ *for each* $x\in[0,x_1/2]\cup[(1+x_m)/2,1]$ *and* $(t_1,\dots,t_n)$ *in* $\mathbb{R}^n$;

(A2) *there exists* $l\in\{1,\dots,n\}$ *such that*

$$0\leq \liminf_{\xi\to\infty}\frac{\displaystyle\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\xi^{p_l})} F(x,t_1,\dots,t_n)\,dx}{\xi^{p_l}} < \frac{p_l}{c\sigma_l^{p_l}}\limsup_{\xi\to\infty}\frac{\displaystyle\int_{x_1/2}^{(1+x_m)/2} F(x,0,\dots,\xi,\dots,0)\,dx}{\xi^{p_l}},$$

where, in $F(x,0,\dots,\xi,\dots,0)$, ξ *is the* $(l+1)$*-th argument. Then, for each* $\lambda\in\Lambda$, *the system* (1.1) *has an unbounded sequence of classical solutions, where*

$$\Lambda=\left(\frac{\sigma_l^{p_l}}{p_l\limsup\limits_{\xi\to\infty}\left(\dfrac{1}{\xi^{p_l}}\displaystyle\int_{x_1/2}^{(1+x_m)/2}F(x,0,\dots,\xi,\dots,0)\,dx\right)},\ \frac{1}{c\liminf\limits_{\xi\to\infty}\left(\dfrac{1}{\xi^{p_l}}\displaystyle\int_0^1\sup_{(t_1,\dots,t_n)\in K(\xi^{p_l})}F(x,t_1,\dots,t_n)\,dx\right)}\right). \tag{2.4}$$

PROOF. Our aim is to apply Lemma 2.1(b) to our problem. To this end, for each $u=(u_1,\dots,u_n)\in X$, we introduce the functionals Φ, $\Psi\colon X\to\mathbb{R}$ as follows:

$$\Phi(u)=\sum_{i=1}^n\frac{||u_i||_{p_i}^{p_i}}{p_i} \tag{2.5}$$

and

$$\Psi(u)=\int_0^1 F(x,u_1(x),\dots,u_n(x))\,dx. \tag{2.6}$$

It is well known that Φ and Ψ are well defined and continuously differentiable functionals whose derivatives at the point $u=(u_1,\dots,u_n)\in X$ are the functionals $\Phi'(u)$, $\Psi'(u)\in X^*$ given by

$$\Phi'(u)(v)=\int_0^1\sum_{i=1}^n|u_i'(x)|^{p_i-2}u_i'(x)v_i'(x)\,dx$$

and

$$\Psi'(u)(v)=\int_0^1\sum_{i=1}^n F_{u_i}(x,u_1(x),\dots,u_n(x))v_i(x)\,dx$$

for every $v = (v_1, \dots , v_n) \in X$. Moreover, Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous.

Let $\{\xi_k\}$ be a sequence of positive numbers such that $\xi_k \to \infty$ as $k \to \infty$ and

$$\lim_{k\to\infty} \frac{\displaystyle\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\xi_k^{p_l})} F(x, t_1, \dots , t_n)\, dx}{\xi_k^{p_l}} = \liminf_{\xi\to\infty} \frac{\displaystyle\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\xi^{p_l})} F(x, t_1, \dots , t_n)\, dx}{\xi^{p_l}}. \tag{2.7}$$

In view of (2.1), for each $(u_1, \dots , u_n) \in X$, we have

$$\sup_{x\in[0,1]} |u_i(x)|^{p_i} \le c||u_i||_{p_i}^{p_i} \quad \text{for } i = 1, \dots , n,$$

and so

$$\sup_{x\in[0,1]} \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} \le c \sum_{i=1}^{n} \frac{||u_i||_{p_i}^{p_i}}{p_i}.$$

Let $r_k = c^{-1}\xi_k^{p_l}$ for $k \in \mathbb{N}$. Then, for $v = (v_1, \dots , v_n) \in X$ with $\sum_{i=1}^{n} \frac{||v_i||_{p_i}^{p_i}}{p_i} < r_k$, we have

$$\sup_{x\in[0,1]} \sum_{i=1}^{n} \frac{|v_i(x)|^{p_i}}{p_i} \le \xi_k^{p_l}.$$

Note that $0 \in \Phi^{-1}(-\infty, r_k)$ and $\Psi(0) \ge 0$ by (A1). Then,

$$\begin{aligned}\varphi(r_k) &= \inf_{u\in\Phi^{-1}(-\infty,r_k)} \frac{\Big(\sup_{v\in\Phi^{-1}(-\infty,r_k)} \Psi(v)\Big) - \Psi(u)}{r_k - \Phi(u)} \\ &\le \frac{\sup_{v\in\Phi^{-1}(-\infty,r_k)} \Psi(v)}{r_k} \le \frac{c\displaystyle\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\xi_k^{p_l})} F(x, t_1, \dots , t_n)\, dx}{\xi_k^{p_l}}.\end{aligned}$$

Then, from (2.7) and (A2), we see that

$$\begin{aligned}\gamma &:= \liminf_{k\to\infty} \varphi(r_k) \\ &\le c \liminf_{\xi\to\infty} \frac{\displaystyle\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\xi^{p_l})} F(x, t_1, \dots , t_n)\, dx}{\xi^{p_l}} < \infty.\end{aligned} \tag{2.8}$$

By (A2), (2.4), and (2.8), one has $\Lambda \subseteq (0, 1/\gamma)$. Let $\lambda \in \Lambda$ be fixed. By Lemma 2.1(b), it follows that one of the following alternatives holds:

(b$_1$) either $I_\lambda = \Phi - \lambda\Psi$ has a global minimum, or (b$_2$) there exists a sequence $\{(u_{1k},\dots,u_{nk})\}$ of critical points of I_λ such that

$$\lim_{k\to\infty}\|(u_{1k},\dots,u_{nk})\|=\infty.$$

In what follows, we show that alternative (b$_1$) does not hold. Since $\lambda\in\Lambda$, by (2.4), we have

$$\frac{1}{\lambda}<\frac{p_l}{\sigma_l^{p_l}}\limsup_{\xi\to\infty}\frac{\displaystyle\int_{x_1/2}^{(1+x_m)/2}F(x,0,\dots,\xi,\dots,0)\,dx}{\xi^{p_l}}.$$

Then, there exists a sequence $\{d_k\}$ of positive numbers and a constant τ such that $d_k\to\infty$ as $k\to\infty$ and

$$\frac{1}{\lambda}<\tau<\frac{p_l}{\sigma_l^{p_l}}\frac{\displaystyle\int_{x_1/2}^{(1+x_m)/2}F(x,0,\dots,d_k,\dots,0)\,dx}{d_k^{p_l}}\tag{2.9}$$

for $k\in\mathbb{N}$ large enough.

Let $\{w_k\}$ be a sequence in X defined by $w_k(x)=(0,\dots,w_{lk}(x),\dots,0)$, where w_{lk} is the l-th argument of w_k and is defined by

$$w_{lk}(x)=\begin{cases} d_k\left(\displaystyle\sum_{j=1}^m a_j+\frac{2}{x_1}\left(1-\sum_{j=1}^m a_j\right)x\right) & \text{if } x\in\left[0,\dfrac{x_1}{2}\right),\\[2ex] d_k & \text{if } x\in\left[\dfrac{x_1}{2},\dfrac{1+x_m}{2}\right],\\[2ex] d_k\left(\dfrac{1}{1-x_m}\left(2-\displaystyle\sum_{j=1}^m b_j-x_m\sum_{j=1}^m b_j\right)-\dfrac{2}{1-x_m}\left(1-\sum_{j=1}^m b_j\right)x\right) & \\ & \text{if } x\in\left(\dfrac{1+x_m}{2},1\right].\end{cases}\tag{2.10}$$

For any fixed $k\in\mathbb{N}$, it is easy to see that $w_k=(0,\dots,w_{lk},\dots,0)\in X$ and $||w_{lk}||_{p_l}^{p_l}=(\sigma_l d_k)^{p_l}$. Thus, (2.5) yields

$$\Phi(w_k)=\frac{(\sigma_l d_k)^{p_l}}{p_l}.\tag{2.11}$$

From (A1) and (2.6),

$$\Psi(w_k)\geq\int_{x_1/2}^{(1+x_m)/2}F(x,0\dots,d_k,\dots,0)\,dx.\tag{2.12}$$

By (2.9), (2.11) and (2.12), we see that, for every $k\in\mathbb{N}$ large enough,

$$\begin{aligned}\Phi(w_k)-\lambda\Psi(w_k)&\leq\frac{(\sigma_l d_k)^{p_l}}{p_l}-\lambda\int_{x_1/2}^{(1+x_m)/2}F(x,0,\dots,d_k,\dots,0)\,dx\\ &<\frac{(\sigma_l d_k)^{p_l}}{p_l}(1-\lambda\tau).\end{aligned}\tag{2.13}$$

From the fact that $\lambda\tau > 1$ (by (2.9) and $d_k \to \infty$, it follows that $\lim_{k\to\infty}(\Phi(w_k) - \lambda\Psi(w_k)) = -\infty$. Then, the functional $\Phi - \lambda\Psi$ is unbounded from below. This shows that alternative (b_1) does not hold. Therefore, there exists a sequence $\{(u_{1k}, \dots, u_{nk})\}$ of critical points of I_λ such that $\lim_{k\to\infty} \|(u_{1k}, \dots, u_{nk})\| = \infty$. Finally, taking into account the fact that the weak solutions of the system (1.1) are exactly critical points of I_λ and applying Lemma 2.2, we have completed the proof of the theorem. $\square$

The following result is a special case of Theorem 2.3 with $F(x, t_1, \dots, t_n) \equiv F(t_1, \dots, t_n)$.

Corollary 2.4. *Assume that*

(B1) $F(t_1, \dots, t_n) \geq 0$ *for each* $(t_1, \dots, t_n) \in \mathbb{R}^n$;
(B2) *there exists* $l \in \{1, \dots, n\}$ *such that*

$$\liminf_{\xi\to\infty} \frac{\sup_{(t_1,\dots,t_n)\in K(\xi^{p_l})} F(t_1, \dots, t_n)}{\xi^{p_l}} < \frac{p_l(1 + x_m - x_1)}{2c\sigma_l^{p_l}} \limsup_{\xi\to\infty} \frac{F(0, \dots, \xi, \dots, 0)}{\xi^{p_l}}.$$

Then, for each

$$\lambda \in \left(\frac{2\sigma_l^{p_l}}{p_l(1 + x_m - x_1) \limsup_{\xi\to\infty} \dfrac{F(0, \dots, \xi, \dots, 0)}{\xi^{p_l}}}, \frac{1}{c \liminf_{\xi\to\infty} \dfrac{\sup_{(t_1,\dots,t_n)\in K(\xi^{p_l})} F(t_1, \dots, t_n)}{\xi^{p_l}}} \right),$$

the system

$$\begin{cases} -(\phi_{p_i}(u_i'))' = \lambda F_{u_i}(u_1, \dots, u_n), \quad t \in (0,1), \\ u_i(0) = \displaystyle\sum_{j=1}^m a_j u_i(x_j), \quad u_i(1) = \sum_{j=1}^m b_j u_i(x_j), \end{cases} \qquad i = 1, \dots, n,$$

has an unbounded sequence of classical solutions.

Remark 2.5. Theorem 1.1 is an immediate consequence of Corollary 2.4.

By choosing a special $F(t_1, \dots, t_n)$ in Corollary 2.4, we have the following result.

Corollary 2.6. *Assume that* $g_i\colon \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, n$, *are continuously differentiable functions such that*

(C1) $g_i(t) \geq 0$ *for each* $i = 1, \dots, n$ *and* $t \in \mathbb{R}$;

(C2)
$$\liminf_{\xi \to \infty} \frac{\displaystyle\sup_{(t_1,\dots,t_n)\in K(\xi^{p_n})} \prod_{i=1}^{n} g_i(t_i)}{\xi^{p_n}} < \frac{p_n(1 + x_m - x_1)}{2c\sigma_n^{p_n}} \prod_{i=1}^{n-1} g_i(0) \limsup_{\xi \to \infty} \frac{g_n(\xi)}{\xi^{p_n}}.$$

Then, for each

$$\lambda \in \left(\frac{2\sigma_n^{p_n}}{p_n(1 + x_m - x_1) \displaystyle\prod_{i=1}^{n-1} g_i(0) \limsup_{\xi \to \infty} \frac{g_n(\xi)}{\xi^{p_n}}}, \frac{1}{c \displaystyle\liminf_{\xi \to \infty} \left(\frac{1}{\xi^{p_n}} \sup_{(t_1,\dots,t_n)\in K(\xi^{p_n})} \prod_{i=1}^{n} g_i(t_i) \right)} \right),$$

the system

$$\text{(2.14)} \quad \begin{cases} -(\phi_{p_i}(u_i'))' = \lambda g_i'(u_i) \left(\displaystyle\prod_{j=1, j\neq i}^{n} g_j(u_j) \right), \quad t \in (0,1), \\ u_i(0) = \displaystyle\sum_{j=1}^{m} a_j u_i(x_j), \quad u_i(1) = \displaystyle\sum_{j=1}^{m} b_j u_i(x_j), \end{cases} \quad i = 1, \dots, n,$$

has an unbounded sequence of classical solutions.

Proof. Let

$$F(u_1, \dots, u_n) = \prod_{i=1}^{n} g_i(u_i) \quad \text{for each } (u_1, \dots, u_n) \in \mathbb{R}^n.$$

Then, with $l = n$, the conclusion follows directly from Corollary 2.4. $\square$

Using Lemma 2.1(c) and arguing as in the proof of Theorem 2.3, we can obtain the following result.

Theorem 2.7. *Assume that* (A1) *in Theorem* 2.3 *holds and*

(D1) *there exists* $l \in \{1, \dots, n\}$ *such that*

$$0 \leq \liminf_{\xi \to 0^+} \frac{\displaystyle\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\xi^{p_l})} F(x, t_1, \dots, t_n)\, dx}{\xi^{p_l}}$$

$$< \frac{p_l}{c\sigma_l^{p_l}} \limsup_{\xi\to 0^+} \frac{\displaystyle\int_{x_1/2}^{(1+x_m)/2} F(x,0,\dots,\xi,\dots,0)\,dx}{\xi^{p_l}},$$

where, in $F(x,0,\dots,\xi,\dots,0)$, ξ *is the* $(l+1)$*-th argument. Then, for each* $\lambda\in\Lambda_1$, *the system* (1.1) *has a sequence of classical solutions converging to zero, where*

$$\Lambda_1 = \left(\frac{\sigma_l^{p_l}}{p_l \limsup\limits_{\xi\to 0^+}\left(\dfrac{1}{\xi^{p_l}}\displaystyle\int_{x_1/2}^{(1+x_m)/2} F(x,0,\dots,\xi,\dots,0)\,dx\right)}, \frac{1}{c\liminf\limits_{\xi\to 0^+}\left(\dfrac{1}{\xi^{p_l}}\displaystyle\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\xi^{p_l})} F(x,t_1,\dots,t_n)\,dx\right)}\right). \tag{2.15}$$

Proof. Let Φ and Ψ be defined as in (2.5) and (2.6), respectively. Let $\{\xi_k\}$ be a sequence of positive numbers such that $\xi_k\to 0^+$ as $k\to\infty$ and

$$\lim_{k\to\infty}\frac{\displaystyle\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\xi_k^{p_l})} F(x,t_1,\dots,t_n)\,dx}{\xi_k^{p_l}} = \liminf_{\xi\to 0^+}\frac{\displaystyle\int_0^1 \sup_{(t_1,\dots,t_n)\in K(\xi^{p_l})} F(x,t_1,\dots,t_n)\,dx}{\xi^{p_l}}.$$

By the fact that $\inf_X \Phi = 0$ and the definition δ, we have $\delta = \liminf\limits_{r\to 0^+}\varphi(r)$. Then, as in showing (2.8) in the proof of Theorem 2.3, we can prove that $\delta<\infty$ and $\Lambda_1\subseteq(0,1/\delta)$. Let $\lambda\in\Lambda_1$ be fixed. By Lemma 2.1(c), we see that one of the following alternatives holds

- (c_1) either there is a global minimum of Φ which is a local minimum of $I_\lambda=\Phi-\lambda\Psi$, or
- (c_2) there exists a sequence $\{(u_{1k},\dots,u_{nk})\}$ of critical points of I_λ which converges weakly to a global minimum of Φ.

In the following, we show that alternative (c_1) does not hold. Since $\lambda\in\Lambda_1$, by (2.15), we see that there exists a sequence $\{d_k\}$ of positive numbers and $\tau>0$ such that $d_k\to 0^+$ as $k\to\infty$ and

$$\frac{1}{\lambda} < \tau < \frac{p_l}{\sigma_l^{p_l}}\frac{\displaystyle\int_{x_1/2}^{(1+x_m)/2} F(x,0,\dots,d_k,\dots,0)\,dx}{d_k^{p_l}} \quad \text{for } k\in\mathbb{N} \text{ large enough.}$$

Let $\{w_k\}$ be a sequence in X defined by $w_k(x)=(0,\dots,w_{lk}(x),\dots,0)$, where $w_{lk}(x)$ is the l-th argument of w_k and is defined by (2.10) with the above d_k.

Note that $\lambda\tau > 1$, Then, as in showing (2.13), we can obtain that

$$\begin{aligned} I_\lambda(w_k) &= \Phi(w_k) - \lambda\Psi(w_k) \\ &\le \frac{(\sigma_l d_k)^{p_l}}{p_l} - \lambda \int_{x_1/2}^{(1+x_m)/2} F(x, 0, \dots, d_k, \dots, 0)\, dx \\ &< \frac{(\sigma_l d_k)^{p_l}}{p_l}(1 - \lambda\tau) < 0 \end{aligned}$$

for every $k \in \mathbb{N}$ large enough. Then, since $\lim_{k\to\infty} I_\lambda(w_k) = I_\lambda(0) = 0$, we see that 0 is not a local minimum of I_λ. This, together with the fact that 0 is the only global minimum of Φ, shows that alternative (c_1) does not hold. Therefore, there exists a sequence $\{(u_{1k}, \dots, u_{nk})\}$ of critical points of I_λ which converges weakly to 0. In view of the fact that the embedding $X = X_1 \times \ldots \times X_n \hookrightarrow C^0([0,1]) \times \ldots \times C^0([0,1])$ is compact, we know that the critical points converge uniformly to zero. Finally, taking into account the fact that the weak solutions of the system (1.1) are exactly critical points of I_λ and applying Lemma 2.2 completes the proof of the theorem. □

Remark 2.8. Applying Theorem 2.7, results similar to Corollaries 2.4, 2.6, and Theorem 1.1 can be obtained. We omit the discussions here.

We conclude this paper with the following example whose construction is motivated by [bf, Example 4.1].

Example 2.9. Let $n \ge 1$ be an integer. Let

$$f(\xi) = \begin{cases} \dfrac{32(n+1)!^2[(n+1)!^2 - n!^2]}{\pi}\sqrt{\dfrac{1}{16(n+1)!^2} - \left(\xi - \dfrac{n!(n+2)}{2}\right)^2} + 1 & \\ & \text{for } \xi \in \bigcup_{n\in\mathbb{N}} [c_n, d_n], \\ 1 & \text{otherwise}, \end{cases}$$

where

$$c_n = \frac{2n!(n+2)! - 1}{4(n+1)!} \quad \text{and} \quad d_n = \frac{2n!(n+2)! + 1}{4(n+1)!}.$$

Assume that $g_i(\xi) = \cos^2 \xi$ for $i = 1, \dots, n-1$, and g_n is a continuously differentiable function such that $g_n(\xi) \ge 0$ and $g_n'(\xi) = f(\xi)$ for $\xi \in \mathbb{R}$. Then, by simple computations,

$$
\begin{aligned}
g_n(c_n) &= g_n(0) + \int_0^{c_n} f(t)\,dt \\
&= g_n(0) + \int_0^{c_n} 1\,dt + \frac{32(n+1)!^2[(n+1)!^2 - n!^2]}{\pi} \\
&\quad \cdot \int_{t\in \bigcup_{i=1}^{n-1}[c_i,d_i]} \sqrt{\frac{1}{16(n+1)!^2} - \bigg(t - \frac{n!(n+2)}{2}\bigg)^2}\,dt \\
&= g_n(0) + c_n + n!^2 - 1,
\end{aligned} \tag{2.16}
$$

and

$$
\begin{aligned}
g_n(d_n) &= g_n(0) + \int_0^{d_n} f(t)\,dt \\
&= g_n(0) + \int_0^{d_n} 1\,dt + \frac{32(n+1)!^2[(n+1)!^2 - n!^2]}{\pi} \\
&\quad \cdot \int_{t\in \bigcup_{i=1}^{n}[c_i,d_i]} \sqrt{\frac{1}{16(n+1)!^2} - \bigg(t - \frac{n!(n+2)}{2}\bigg)^2}\,dt \\
&= g_n(0) + d_n + (n+1)!^2 - 1.
\end{aligned} \tag{2.17}
$$

From (2.16) and (2.17), it is easy to see that

$$
\lim_{n\to\infty} \frac{g_n(c_n)}{c_n^2} = 0, \quad \text{and} \quad \lim_{n\to\infty} \frac{g_n(d_n)}{d_n^2} = 4.
$$

Note that there is no sequence $\{h_n\}$ such that $h_n \to \infty$ and $\lim_{n\to\infty} g(h_n)/h_n^2 > 4$. Then,

$$
\liminf_{\xi\to\infty} \frac{g_n(\xi)}{\xi^2} = 0 \quad \text{and} \quad \limsup_{\xi\to\infty} \frac{g_n(\xi)}{\xi^2} = 4. \tag{2.18}
$$

We claim that for each $\lambda \in (1/2, \infty)$, the system

$$
\begin{cases}
-u_i'' = \lambda g_i'(u_i)\bigg(\displaystyle\prod_{j=1,\, j\neq i}^{n} g_j(u_j)\bigg), & t \in (0,1), \\[2ex]
u_i(0) = \dfrac{1}{2}u_i\bigg(\dfrac{1}{2}\bigg), \quad u_i(1) = \dfrac{1}{2}u_i\bigg(\dfrac{1}{2}\bigg),
\end{cases}
\qquad i = 1,\ldots,n, \tag{2.19}
$$

has an unbounded sequence of classical solutions.

In fact, with $m = 1$, $a_1 = b_1 = x_1 = 1/2$, and $p_i = 2$ for $i = 1,\ldots,n$, (2.19) is a special case of (2.14). From (2.18), we see that (C1) and (C2) of Corollary 2.6 hold. By (2.2) with $i = n$, we have $\sigma_n = \sqrt{2}$. Note from (2.18) that

$$
\frac{2\sigma_n^{p_n}}{p_n(1 + x_m - x_1)\displaystyle\prod_{i=1}^{n-1} g_i(0) \limsup_{\xi\to\infty} \frac{g_n(\xi)}{\xi^{p_n}}} = \frac{1}{2}
$$

and
$$\frac{1}{c\liminf\limits_{\xi\to\infty}\left(\dfrac{1}{\xi^{p_n}}\sup\limits_{(t_1,\dots,t_n)\in K(\xi^{p_n})}\prod\limits_{i=1}^{n}g_i(t_i)\right)}=\infty.$$
The conclusion then follows directly from Corollary 2.6.

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