Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 31, 2008, 341–348

APPLICATION OF THE HOMOTOPY PERTURBATION METHOD TO COUPLED SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS WITH TIME FRACTIONAL DERIVATIVES

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Abstract. The homotopy perturbation method (HPM) is applied to solve nonlinear partial differential equations of fractional orders. The corresponding solutions for integer orders of the fractional derivatives are found to be special cases of the fractional differential equations. It is predicted that HPM can be found widely applicable in engineering.

1. Introduction

In recent years, it has turned out that many phenomena in engineering, physics, chemistry and other sciences can be described very successfully by models using fractional calculus, i.e. the theory of derivatives and integrals of fractional (non-integer) order. For instance, the nonlinear oscillation of earthquakes can be modeled by fractional derivatives [and the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising in the assumption of continuum traffic flow. Most fractional differential equations do not have exact analytic solutions, so approximate and numerical techniques have to be used. The variational iteration method [2]–[7], [12], homotopy perturbation

2000 *Mathematics Subject Classification*. 35B20, 65C20.

Key words and phrases. Homotopy perturbation method (HPM), fractional partial differential coupled systems equations, Caputo derivative.

method [2]–[4], [7], [9]–[11], [13], [19]–[23], [25]–[30] and Adomian's decomposition method [1], [14], [15] are relatively new approaches to providing analytic approximations to linear and nonlinear problems. Recently, Odibat and Momani [17] have implemented the variational iteration method to solve nonlinear ordinary differential equations of fractional order. In this study, He's homotopy perturbation method is implemented to derive analytical approximate solutions to linear partial differential equations of fractional order.

2. Basic definitions

Definition 2.1. A real function $f(x)$, $x > 0$ is said to be in the space C_α, $\alpha \in \Re$ if there exists a real number p $(> \alpha)$ such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C[0, \infty)$. Clearly $C_\alpha \subset C_\beta$ if $\beta \leq \alpha$.

Definition 2.2. A function $f(x)$, $x > 0$ is said to be in the space C_α^m, $m \in N \cup \{0\}$, if $f^{(m)} \in C_\alpha$.

Definition 2.3. The left sided Riemann–Liouville fractional integral of order $\mu > 0$, [21], [24] of a function $f \in C_\alpha$, $\alpha \geq -1$ is defined as:

$$\begin{aligned} I^\mu f(x) &= \frac{1}{\Gamma(\mu)} \int_0^x \frac{f(\tau)}{(t-\tau)^{1-\mu}}\, d\tau, \quad \mu > 0,\ x > 0, \\ I^0 f(x) &= f(x). \end{aligned}$$

Definition 2.4. The fractional derivative of $f(x)$ in the Caputo sense is defined as:

$$D_\alpha^* f(x) = J^{m-\alpha}\ D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt,$$

for $m - 1 < \alpha < m$, $m \in N$, $x > 0$, $f \in C_{-1}^m$

Note that (see [21], [24])

$$\begin{aligned} I^\mu t^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\mu+1)} t^{\gamma+\mu}, \qquad \mu > 0,\ \gamma > -1,\ t > 0, \\ I^\mu D_*^\mu f(t) &= f(t) - \sum_{k=0}^{m-1} f^{(k)}(0+) \frac{t^k}{k!}, \quad m-1 < \mu \leq m,\ m \in \mathbb{N}. \end{aligned}$$

Definition 2.5. For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order$\alpha > 0$ is defined as:

$$D_{*t}^\alpha\ u(x,t) = \frac{\partial^\alpha\ u(x,t)}{\partial t^\alpha} = \begin{cases} J^{m-\alpha}\left(\dfrac{\partial^m u(x,t)}{\partial t^m}\right) & m-1 < \alpha < m,\ m \in \mathbb{N}, \\ \dfrac{\partial^m u(x,t)}{\partial t^m} & \alpha = m. \end{cases}$$

3. Basic idea of He's homotopy perturbation method

To illustrate the basic ideas of this method, we consider the following equation (see [11]):

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{3.1}$$

with the boundary condition:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma,$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and G is the boundary of the domain Ω.

A can be divided into two parts which are L and N, where L is linear and N is nonlinear. Equation (3.1) can therefore be rewritten as follows:

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega,$$

Homotopy perturbation structure is shown as follows:

$$H(v,p) = (1-p) \cdot [L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \tag{3.2}$$

where $v \colon \Omega \times [0,1] \to \mathbb{R}$.

In (3.2), $p \in [0,1]$ is an embedding parameter and is the first approximation that satisfies the boundary conditions. We can assume that the solution of (3.1) can be written as a power series in p, as following:

$$v = v_0 + p\ v_1 + p^2 v_2 + \dots\,, \tag{3.3}$$

and the best approximation is:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \dots$$

Recently, Momani applied homotopy perturbation method to fractional differential equations. To illustrate the basic ideas of the new modification, we consider the following nonlinear differential equation of fractional order (see [16] and [18]):

$$D_*^{\alpha} u(t) + L(u(t)) + N(u(t)) = f(t), \quad t > 0,\ m-1 < \alpha < m,$$

where L is a linear operator which might include other fractional derivatives of order less than a, N is a nonlinear operator which also might include other fractional derivatives of order less than a, f is a known analytic function and D_*^{α} is the Caputo fractional derivative of order a, subject to the initial conditions

$$u^k(0) = c_k, \quad k = 0, \dots, m-1.$$

In view of the homotopy technique, we can construct the following homotopy:

$$u^{(m)} - f(t) = p\ [u^{(m)} - L(u(t)) - N(u(t)) - D_*^{\alpha} u], \quad p \in [0,1].$$

4. Application and numerical result

We consider the following nonlinear system [5]:

$$
\begin{aligned}
D_t^\alpha u + v_x w_y - v_y w_x &= -u,\\
D_t^\alpha v + w_x u_y + u_v w_x &= v,\\
D_t^\alpha w + u_x v_y + u_y v_x &= w,
\end{aligned}
\tag{4.1}
$$

$(0 < \alpha < 1)$ with the initial conditions as:

$$u(x,0) = e^{x+y}, \quad v(x,0) = e^{x-y}, \quad w(x,0) = e^{-x+y},$$

The exact solutions, when $\alpha = 1$ are:

$$u(x,t) = e^{x+y-t}, \quad v(x,t) = e^{x-y+t}, \quad w(x,t) = e^{-x+y+t},$$

In order to solve (4.1) using HPM, we construct the following homotopy for these equations:

$$
\begin{aligned}
u_t &= p[u_t - v_x w_y + v_y w_x - u - u_t^\alpha],\\
v_t &= p[u_t - w_x u_y - u_v w_x + v - v_t^\alpha],\\
w_t &= p[w_t - u_x v_y - u_y v_x + w - wt^\alpha],
\end{aligned}
\tag{4.2}
$$

Substituting v from (3.3) into (4.2) and rearranging based on powers of p-terms, we can obtain:

$$p^0\colon \frac{\partial u_0}{\partial t} = 0, \qquad p^0\colon \frac{\partial v_0}{\partial t} = 0, \qquad p^0\colon \frac{\partial w_0}{\partial t} = 0,$$

$$
\begin{aligned}
p^1\colon \frac{\partial u_1(x,y,t)}{\partial t} &= -\frac{\partial v_0(x,y,t)}{\partial x}\cdot\frac{\partial w_0(x,y,t)}{\partial y} + \frac{\partial v_0(x,y,t)}{\partial y}\frac{\partial w_0(x,y,t)}{\partial x}\\
&\quad - u_0(x,y,t) + \frac{\partial u_0(x,y,t)}{\partial t} - \frac{\partial^\alpha u_0(x,t)}{\partial t^\alpha},\\
p^1\colon \frac{\partial v_1(x,y,t)}{\partial t} &= -\frac{\partial w_0(x,y,t)}{\partial x}\cdot\frac{\partial u_0(x,y,t)}{\partial y} + \frac{\partial w_0(x,y,t)}{\partial y}\cdot\frac{\partial u_0(x,y,t)}{\partial x}\\
&\quad - v_0(x,y,t) + \frac{\partial v_0(x,y,t)}{\partial t} - \frac{\partial^\alpha v_0(x,t)}{\partial t^\alpha},\\
p^1\colon \frac{\partial w_1(x,y,t)}{\partial t} &= -\frac{\partial v_0(x,y,t)}{\partial x}\cdot\frac{\partial u_0(x,y,t)}{\partial y} + \frac{\partial v_0(x,y,t)}{\partial y}\cdot\frac{\partial u_0(x,y,t)}{\partial x}\\
&\quad - w_0(x,y,t) + \frac{\partial w_0(x,y,t)}{\partial t} - \frac{\partial^\alpha w_0(x,t)}{\partial t^\alpha},\\
p^2\colon \frac{\partial u_2(x,y,t)}{\partial t} &= -\frac{\partial v_0(x,y,t)}{\partial x}\cdot\frac{\partial w_0(x,y,t)}{\partial y}\\
&\quad + \frac{\partial v_0(x,y,t)}{\partial y}\cdot\frac{\partial w_1(x,y,t)}{\partial x}\cdot\frac{\partial v_1(x,y,t)}{\partial y}\cdot\frac{\partial w_0(x,y,t)}{\partial x}\\
&\quad - u_1(x,y,t) + \frac{\partial u_1(x,y,t)}{\partial t} - \frac{\partial^\alpha u_1(x,t)}{\partial t^\alpha},
\end{aligned}
$$

$$\begin{aligned}
p^2\colon \frac{\partial v_2(x,y,t)}{\partial t} &= -\frac{\partial u_1(x,y,t)}{\partial x}\cdot\frac{\partial w_0(x,y,t)}{\partial y}\\
&+\frac{\partial w_1(x,y,t)}{\partial y}\cdot\frac{\partial u_0(x,y,t)}{\partial x}\cdot\frac{\partial u_1(x,y,t)}{\partial y}\cdot\frac{\partial w_0(x,y,t)}{\partial x}\\
&-\frac{\partial w_1(x,y,t)}{\partial x}\cdot\frac{\partial u_0(x,y,t)}{\partial y}-v_1(x,y,t)\\
&+\frac{\partial v_1(x,y,t)}{\partial t}-\frac{\partial^\alpha v_1(x,t)}{\partial t^\alpha},\\
p^2\colon \frac{\partial w_2(x,y,t)}{\partial t} &= -\frac{\partial u_0(x,y,t)}{\partial x}\cdot\frac{\partial v_0(x,y,t)}{\partial y}\\
&+\frac{\partial v_1(x,y,t)}{\partial x}\cdot\frac{\partial u_0(x,y,t)}{\partial y}\cdot\frac{\partial v_0(x,y,t)}{\partial y}\\
&\cdot\frac{\partial u_1(x,y,t)}{\partial x}\cdot\frac{\partial v_0(x,y,t)}{\partial x}\cdot\frac{\partial u_1(x,y,t)}{\partial y}\\
&-w_1(x,y,t)+\frac{\partial w_1(x,y,t)}{\partial t}-\frac{\partial^\alpha w_1(x,t)}{\partial t^\alpha},
\end{aligned}$$

....

and therefore

$$\begin{aligned}
u_0 &= e^{x+y}, & v_0 &= e^{x-y}, & w_0 &= e^{-x+y},\\
u_1 &= -e^{x+y}\ t, & v_1 &= e^{x-y}\ t, & w_1 &= e^{-x+y}\ t,
\end{aligned}$$

$$\begin{aligned}
u_2 &= \frac{t^2}{2}e^{x+y}\ t^2+\frac{e^{x+y}t^{2-\alpha}}{\Gamma 3-\alpha}-e^{x+y}t,\\
v_2 &= \frac{t^2}{2}e^{x-y}\ t^2-\frac{e^{x-y}t^{2-\alpha}}{\Gamma 3-\alpha}+e^{x-y}t,\\
w_2 &= \frac{t^2}{2}e^{-x+y}\ t^2-\frac{e^{-x+y}t^{2-\alpha}}{\Gamma 3-\alpha}+e^{-x+y}t,\\
u_3 &= -\frac{t^3}{6}e^{(x+y)}+e^{(x+y)}t^2-\frac{(t^{(3-\alpha)})^2e^{(x+y)}}{\Gamma(4-2\alpha)t^3}\\
&\quad -\frac{2(t-3+\alpha)t^{(3-\alpha)}e^{(x+y)}}{\Gamma(4-\alpha)t}-e^{x+y}t,\\
v_3 &= \frac{t^3}{6}e^{(x-y)}+e^{(x-y)}\ t^2+\frac{(t^{(3-\alpha)})^2e^{(x-y)}}{\Gamma(4-2\alpha)t^3}\\
&\quad -\frac{2(t-3+\alpha)\ t^{(3-\alpha)}\ e^{(x-y)}}{\Gamma(4-\alpha)t}-e^{x-y}t,\\
w_3 &= \frac{t^3}{6}e^{(-x+y)}+e^{(-x+y)}t^2+\frac{(t^{(3-\alpha)})^2e^{(-x+y)}}{\Gamma(4-2\alpha)t^3}\\
&\quad -\frac{2(t-3+\alpha)t^{(3-\alpha)}e^{(-x+y)}}{\Gamma(4-\alpha)t}-e^{(-x+y)}t,
\end{aligned}$$

....

The solution of this equation, when $p \to 1$ will be as follows:

$$\begin{aligned} u(x,y,z,t) &= u_0 + u_1 + u_2 + u_3, \\ v(x,y,z,t) &= v_0 + v_1 + v_2 + v_3, \\ w(x,y,z,t) &= w_0 + w_1 + w_2 + w_3, \end{aligned}$$

Substituting $\alpha = 1$ in $u(x,y,z,t)$, $v(x,y,z,t)$ and $w(x,y,z,t)$ we ultimately obtain the solutions as below. These are exact solutions confirmed by [8].

$$\begin{aligned} u(x,y,z,t) &= e^{(x+y)}\left(-t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots \right) = e^{(x+y)}e^{-t} = e^{(x+y-t)}, \\ v(x,y,z,t) &= e^{(x-y)}\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) = e^{(x-y)}e^{t} = e^{(x-y+t)}, \\ w(x,y,z,t) &= e^{(-x+y)}\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right) = e^{(-x+y)}e^{t} = e^{(-x+y+t)}, \end{aligned}$$

5. Conclusion

In this paper, the homotopy perturbation method (HPM) was successfully applied to study the partial differential of coupled systems of time-fractional equation. The solution obtained by means of the homotopy perturbation method is an infinite power series with respect to appropriate initial condition, which can be, in turn, expressed in a closed form. The obtained results reinforce the conclusions made by many researchers about the efficiency of HPM. The results show that homotopy perturbation method is a powerful and efficient technique in finding exact and approximate solutions for nonlinear partial differential equations of fractional order.

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Manuscript received October 16, 2007

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