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MULTIPLE PERIODIC SOLUTIONS OF ASYMPTOTICALLY LINEAR HAMILTONIAN SYSTEMS VIA CONLEY INDEX THEORY

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Abstract. In this paper we study the existence of periodic solutions of asymptotically linear Hamiltonian systems which may not satisfy the Palais-Smale condition. By using the Conley index theory and the Galerkin approximation methods, we establish the existence of at least two nontrivial periodic solutions for the corresponding systems.

1. Introduction

In this paper we study the following Hamiltonian system

$$\dot{z} = JH'(t,z) \tag{1.1}$$

where $H'(t,z)$ denotes the gradient of $H(t,z)$ with respect to the z variable, $J = \left(\begin{smallmatrix} 0 & -I_N \\ I_N & 0 \end{smallmatrix}\right)$ is the standard $2N \times 2N$ symplectic matrix, and N is a positive integer. Denote by (x,y) and $|x|$ the usual inner product and norm in $\mathbb{R}^{2N}$, respectively. We assume the system (1.1) is asymptotically linear both at the origin and at infinity, i.e.

$$|H'(t,z) - B_0(t)z| = o(|z|), \quad \text{as } |z| \to 0, \tag{1.2}$$

$$|H'(t,z) - B_\infty(t)z| = o(|z|), \quad \text{as } |z| \to \infty, \tag{1.3}$$

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where $B_0(t)$ and $B_\infty(t)$ are $2N \times 2N$ symmetric matrices, continuous and 1-periodic in t. Obviously, 0 is a trivial solution. We are interested in the nontrivial 1-periodic solutions.

The existence of periodic solutions of asymptotically linear Hamiltonian systems was first studied by H. Amann and E. Zehnder ([3], [4]). They considered the case that $B_0(t)$ and $B_\infty(t)$ are constant matrices and $B_\infty(t)$ is nondegenerate. Later, for nonconstant matrices $B_0(t)$ and $B_\infty(t)$, C. Conley and E. Zehnder in [10] studied the problem with nondegenerate $B_0(t)$ and $B_\infty(t)$. After then, many works have been done about this problem (see [1], [5], [6], [8], [11]–[15], [18]–[21], [25]–[27]. $B_0(t)$ and $B_\infty(t)$ are allowed to be degenerate and nonconstant, and the Landesman–Lazer type condition and the strong resonance condition are often used (see [8], [14]). Since the corresponding functional is strongly indefinite, many variational methods have been developed to handle it ([2], [7], [17], [22], [23]).

The goal of this paper is to establish the existence of multiple periodic solutions of the system (1.1). We combine Conley index theory with the Galerkin approximation procedure to show that the system (1.1) possesses at least two nontrivial 1-periodic solutions if the "twist" between the origin and the infinity is large enough. From now on, denote

$$\begin{aligned} G_\infty(t,z) &= H(t,z) - \frac{1}{2}(B_\infty(t)z,z), \\ G_0(t,z) &= H(t,z) - \frac{1}{2}(B_0(t)z,z). \end{aligned}$$

We assume the following conditions for H.

(H1) $H \in C^2([0,1] \times \mathbb{R}^{2N}, \mathbb{R})$ is a 1-periodic function in t, and satisfies

$$|H''(t,z)| \le a_1|z|^s + a_2, \quad \text{for all } (t,z) \in \mathbb{R} \times \mathbb{R}^{2N}, \text{ where } s \in (1,\infty), a_1, a_2 > 0.$$

(H2$^\pm$) There exist $2 < \alpha_0 < 2\beta_0$ and $c_1, c_2, L_0 > 0$ such that

$$\begin{aligned} \pm(G_0'(t,z),z) &\ge c_1|z|^{\alpha_0} \quad \text{for all } |z| \le L_0, \\ |G_0'(t,z)| &\le c_2|z|^{\beta_0} \quad \text{for all } |z| \le L_0. \end{aligned} \tag{1.4}$$

(H3$^\pm$) There exist $c_3, c_4, c_5 > 0$, $L_\infty > 0$ and $\delta > 0$ such that

$$\begin{aligned} \pm(G_\infty'(t,z),z) &\ge \frac{c_3}{|z|^\delta} \quad |G_\infty'(t,z)| \le c_5 \quad \text{for all } |z| \ge L_\infty, \\ |G_\infty'(t,z)||z| &\le c_4|(G_\infty'(t,z),z)| \qquad \text{for all } |z| \ge L_\infty. \end{aligned}$$

According to [10], [19], [21], for a given continuous 1-periodic and symmetric matrix function $B(t)$, one can assign a pair of integers $(i,n) \in \mathbb{Z} \times \{0,\ldots,2N\}$ to it, which is called the Maslov-type index of $B(t)$. Let (i_0,n_0) and (i_∞,n_∞) be the Maslov-type indices of $B_0(t)$ and $B_\infty(t)$, respectively. Our first result reads as:

Theorem 1.1. *Suppose that H satisfies* (H1). *Then the system* (1.1) *possesses a nontrivial* 1*-periodic solution if one of the following four cases occurs:*

(a) $(\mathrm{H2}^+)$ *and* $(\mathrm{H3}^+)$ *hold,* $i_\infty + n_\infty \neq i_0 + n_0$,
(b) $(\mathrm{H2}^+)$ *and* $(\mathrm{H3}^-)$ *hold,* $i_\infty \neq i_0 + n_0$,
(c) $(\mathrm{H2}^-)$ *and* $(\mathrm{H3}^+)$ *hold,* $i_\infty + n_\infty \neq i_0$,
(d) $(\mathrm{H2}^-)$ *and* $(\mathrm{H3}^-)$ *hold,* $i_\infty \neq i_0$.

Moreover, the system (1.1) *possesses at least two nontrivial* 1*-periodic solutions if one of the following four cases occurs:*

(e) $(\mathrm{H2}^+)$ *and* $(\mathrm{H3}^+)$ *hold,* $|i_\infty + n_\infty - i_0 - n_0| > 2N+1$,
(f) $(\mathrm{H2}^+)$ *and* $(\mathrm{H3}^-)$ *hold,* $|i_\infty - i_0 - n_0| > 2N+1$,
(g) $(\mathrm{H2}^-)$ *and* $(\mathrm{H3}^+)$ *hold,* $|i_\infty + n_\infty - i_0| > 2N+1$,
(h) $(\mathrm{H2}^-)$ *and* $(\mathrm{H3}^-)$ *hold,* $|i_\infty - i_0| > 2N+1$.

Remark 1.2. (a) It is easy to show that $(\mathrm{H2}^\pm)$ and $(\mathrm{H3}^\pm)$ imply (1.2) and (1.3), respectively. Under $(\mathrm{H3}^\pm)$, the Palais–Smale condition may not hold and the strong resonance method ([8], [14]) may not work here, too. See Example 3.6 for more details.

(b) Conditions (b) and (c) of Theorem 1.1 include a special case that $B_0(t) = B_\infty(t)$, i.e. the system (1.1) may be resonance at 0 and at ∞ with the same asymptotical matrix. As far as I know, this case has been studied only in [14], [15], [25], where the Palais–Smale condition is always required.

(c) In order to get the second nontrivial solution, one usually assumes that the first obtained one is nondegenerate (see [18]). Here in (e)–(h), we do not require any condition on the first obtained solution. Conditions (e)–(h) of Theorem 1.1 are a kind of generalization of the corresponding results in [19], [20], where $B_\infty(t)$ is assumed to be nondegenerate.

(d) To prove Theorem 1.1, we first apply the Galerkin approximation procedure to consider functions $\{f_m\}$ defined on finite dimensional spaces $\{E_m\}$. Then we construct the isolating blocks $D_{\infty m}$ at ∞ and D_m at 0 in a way that $\{D_{\infty m}\}$ are uniformly bounded. This allows us to avoid the Palais–Smale condition. The different Conley indices of $D_{\infty m}$ and D_m give us the critical point z_m of f_m, which converges to the first nontrivial solution. The Morse type inequality of Conley index theory gives us the second nontrivial solution.

(e) Special attention is paid on the control of the small eigenvalues of $P_n(A-B)P_n$. (See Theorem 2.3 and Remark 2.4.) This is a very important part in building the uniformly bounded isolating blocks.

Now we consider the case with unbounded $|G'_\infty(t,z)|$. Assume

(H4$^\pm$) There exist $1 \le \alpha_\infty < 2$, $0 < \beta_\infty < \alpha_\infty/2$ and $c_6, c_7, L_\infty > 0$ such that

$$\begin{aligned} \pm(G'_\infty(t,z), z) &\ge c_6|z|^{\alpha_\infty} \quad \text{for all } |z| \ge L_\infty, \\ |G'_\infty(t,z)| &\le c_7|z|^{\beta_\infty} \quad \text{for all } |z| \ge L_\infty. \end{aligned} \tag{1.5}$$

Theorem 1.3. *Suppose that H satisfies* (H1). *Then the same conclusions as those in Theorem* 1.1 *hold if we replace* (H3$^\pm$) *by* (H4$^\pm$).

Remark 1.4. (a) It is easy to see that (H4$^\pm$) implies (1.3). The Palais–Smale condition does hold under (H4$^\pm$). But we do not need it in our approach.

(b) [25, Theorem 1.3] is a special case of our Theorem 1.3(a)–(d), where $B_\infty(t)$ and $B_0(t)$ are required to be finitely degenerate and the conditions about $G_\infty(t,z)$ and $G_0(t,z)$ are special cases of (H4$^\pm$).

(c) In [15], under different conditions about $G_\infty(t,z)$ and $G_0(t,z)$, they got a result similar to Theorem 1.3(a)–(d) by computing the critical groups $C_*(f,0)$ and $C_*(f,\infty)$.

This paper is organized as follows. In Section 2, we introduce the Galerkin approximation scheme and Conley index theory. In Section 3, we construct the isolating blocks and prove our results.

2. Conley index and Galerkin approximation

First of all, we recall some results about the Conley index. Let $\eta\colon (\mathbb{R}^n \times \mathbb{R}) \to \mathbb{R}^n$ be the flow on $\mathbb{R}^n$. Let $D \subset \mathbb{R}^n$ be a closed set and $x \in \partial D$ be a boundary point. Then x is called a strict egress (strict ingress, bounce-off, respectively) point of D, if there are $c, d > 0$ such that for $0 < t \le c$: $\eta(x,t) \notin D$ ($\eta(x,t) \in \operatorname{int}(D)$, $\eta(x,t) \notin D$, respectively) and for $0 < -t \le d$: $\eta(x,t) \in \operatorname{int}(D)$ ($\eta(x,t) \notin D$, $\eta(x,t) \notin D$, respectively). We use D^e (D^i, D^b, respectively) to denote the set of strict egress (strict ingress, bounce-off) points of the closed set D. Let $D^- = D^e \cup D^b$.

A closed set $D \subset \mathbb{R}^n$ is called an isolating block if $\partial D = D^e \cup D^i \cup D^b$ and $D^- = D^e \cup D^b$ is closed.

Let $D \subset \mathbb{R}^n$ be a bounded isolating block under the flow η. We define

$$I(\eta, D) = \sum_{k \ge 0} r^k(D, D^-)t^k, \tag{2.1}$$

where $r^k(D,D^-) = \operatorname{rank}(H_k(D,D^-))$ is the rank of the k-th homology group $H_k(D,D^-)$.

Let $h\colon \mathbb{R}^n \to \mathbb{R} \in C^2$. η is the gradient flow generated by

$$\frac{dx(t)}{dt} = -h'(x(t)).$$

Let $D_\infty, D_0 \subset \mathbb{R}^n$ be two bounded isolating blocks under the flow η such that $D_0 \subset \operatorname{int}(D_\infty)$. Using the results in [9], [10], [24], one can prove the following theorem.

THEOREM 2.1.

(a) *If $\theta \in D_0$ is the only critical point of h in D_∞,*

$$I(\eta, D_\infty) = I(\eta, D_0).$$

(b) *Suppose θ is the only critical point of h in D_0 and all critical points of h in $D_\infty \setminus D_0$, say $\{x_1, \dots, x_m\}$, are nondegenerate with the Morse indices $\{i_1, \dots, i_m\}$ respectively. Then*

$$\sum_{j=1}^{m} t^{i_j} + I(\eta, D_0) = I(\eta, D_\infty) + (1+t)Q(t),$$

where $Q(t)$ is a polynomial with nonnegative integer coefficients.

PROOF. (a) The conclusion comes from the fact that the Conley homotopy index is independent of the choice of index pairs (see [24]).

(b) Obviously, there is an admissible Morse decomposition of D_∞ with Morse sets $\{\theta, x_1, \dots, x_m\}$ (see Salamon [24]). The conclusion comes directly from the Morse type inequality for $\{\theta, x_1, \dots, x_m\}$ (see [9], [10], [20]). We omit the details. □

Now we focus on the Galerkin approximation. We would rather work in an abstract framework. Let E be a seperable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Assume

(A) A is a bounded selfadjoint operator with a finite dimensional kernel, and its zero eigenvalue is isolated in the spectrum of A.

Note that the restriction $A|_{\mathrm{Im}(A)}$ is invertible.

The following definition of a Galerkin approximation procedure is due to [8].

DEFINITION 2.2. Let $\Gamma = \{P_m : m = 1, 2, \dots\}$ be a sequence of orthogonal projections. We call Γ an *approximation scheme with respect to A*, if the following properties hold:

(a) $E_m = P_m E$ is finite dimensional, for all $n \geq 1$,
(b) $P_m \to I$ strongly as $n \to \infty$,
(c) $[P_m, A] = P_m A - A P_m \to 0$ in the operator norm.

For a self adjoint bounded operator T, denote $T^\# = (T_{\mathrm{Im}(T)})^{-1}$, and denote by $M^+(T)$, $M^-(T)$ and $M^0(T)$ the positive definite, negative definite and null subspaces of T, respectively. For $d > 0$, we also use $M_d^+(T)$, $M_d^-(T)$ and $M_d^0(T)$

to denote the eigenspaces corresponding to the eigenvalues λ belonging to $[d,\infty)$, $(-\infty,-d]$ and $(-d,d)$, respectively.

For a linear symmetric compact operator B, it is easy to show that

$$\dim\ker(A-B)<\infty,\quad \text{and}$$
$$\tau_m = \|(I-P_m)B\| + \|B(I-P_m)\| \to 0 \quad \text{as } m\to\infty, \tag{2.2}$$

where $\Gamma=\{P_m : m=1,2,\dots\}$ is an approximation scheme with respect to A. Let $P_B\colon E\to\ker(A-B)$ be the orthogonal projection. Obviously, P_B is compact. Then by (2.2) and Definition 2.2(c),

$$\varepsilon_m = \|P_mA-AP_m\|+\tau_m+\|(I-P_m)P_B\|(1+\|A-B\|)\to 0 \quad \text{as } m\to\infty. \tag{2.3}$$

Theorem 2.3. *Let B be a linear symmetric compact operator. For any fixed constant $0<d\le\|(A-B)^{\#}\|^{-1}/4$, there exists $m^*>0$ such that for $m\ge m^*$ we have*

(a) $\dim M^0_{2\varepsilon_m}(P_m(A-B)P_m)=\dim\ker(A-B)$,

(b) $E_m = M^+_d(P_m(A-B)P_m)\oplus M^-_d(P_m(A-B)P_m)\oplus M^0_{2\varepsilon_m}(P_m(A-B)P_m)$,

where ε_m is given by (2.3), *and $2\varepsilon_m<\min(1,d)$.*

Proof. *Step 1.* Set $E^0=P_BE=\ker(A-B)$. Then there exists $m_0>0$ such that for $m\ge m_0$,

$$\dim P_mE^0=\dim E^0. \tag{2.4}$$

For otherwise, there exist $\{m_k\}$ such that $\dim P_{m_k}E^0<\dim E^0$. This implies that there exist $\{x_k\}\subseteq E^0$ such that

$$P_{m_k}x_k=0,\quad \|x_k\|=1. \tag{2.5}$$

Since $\dim E^0<\infty$, passing to a subsequence if necessary, $x_k\to x^*$ as $k\to\infty$. By (2.5) we have

$$1=\|x_*\|=\lim_{k\to\infty}\|P_{m_k}x^*-P_{m_k}x_k\|\le\lim_{k\to\infty}\|x^*-x_k\|=0,$$

a contradiction. Therefore (2.4) holds. Moreover, for any $x\in P_mE^0$, there is a unique $\widetilde{x}\in E^0$ such that $x=P_m\widetilde{x}$. By (2.3), for m large enough,

$$x=P_m\widetilde{x}=\widetilde{x}-(I-P_m)P_B\widetilde{x}\quad\text{and}\quad \|x\|\ge(1-\varepsilon_m)\|\widetilde{x}\|.$$

Therefore we have

$$\begin{aligned} P_m(A-B)P_mx &= P_m(A-B)P_m\widetilde{x}=P_m(A-B)(P_m-I)P_B\widetilde{x},\\ \|P_m(A-B)P_mx\| &\le \varepsilon_m\|\widetilde{x}\|\le\frac{\varepsilon_m}{1-\varepsilon_m}\|x\|. \end{aligned}$$

By (2.3), there exists $m_1 \geq m_0$ such that for $m \geq m_1$

$$\|P_m(A-B)P_m x\| \leq \frac{4}{3}\varepsilon_m \|x\| \quad \text{for all } x \in P_m E^0. \tag{2.6}$$

Step 2. For $m \geq m_1$, let Y_m be the orthogonal complement of $P_m E^0$ in E_m, i.e. $E_m = Y_m \oplus P_m E^0$. Then there exists $m_2 \geq m_1$ such that for $m \geq m_2$,

$$\|P_m(A-B)P_m y\| \geq 2d\|y\| \quad \text{for all } y \in Y_m. \tag{2.7}$$

In fact, for all $y \in Y_m$ and for all $x \in E^0$, we have

$$0 = \langle y, P_m x\rangle = \langle P_m y, x\rangle = \langle y, x\rangle.$$

By Step 1, we know that $y \perp E^0$, i.e. $y \in \mathrm{Im}(A-B)$. Moreover,

$$\begin{aligned} P_m(A-B)P_m y &= (A-B)y + (P_m - I)(A-B)P_m y \\ &= (A-B)y + (P_m - I)AP_m y - (P_m - I)By. \end{aligned}$$

By (2.2) and Definition 2.2(c),

$$\|(P_m - I)AP_m\| + \|(P_m - I)B\| \to 0 \quad \text{as } m \to \infty.$$

This means that there exists $m_2 \geq m_1$ such that (2.7) holds.

Step 3. There exists $m^* \geq m_2$ such that for $m \geq m^*$, we have $2\varepsilon_m < d$ and

$$\dim M^0_{2\varepsilon_m}(P_m(A-B)P_m) = \dim P_m E^0. \tag{2.8}$$

In fact, if $\dim M^0_{2\varepsilon_m}(P_m(A-B)P_m) > \dim P_m E^0$, there must exist $y \neq 0$ and $y \in M^0_{2\varepsilon_m}(P_m(A-B)P_m) \cap Y_m$. This implies that

$$\begin{aligned} \|P_m(A-B)P_m y\| &\leq 2\varepsilon_m \|y\|, \\ \|P_m(A-B)P_m y\| &\geq 2d\|y\| \qquad \text{(by (2.7))}. \end{aligned}$$

We get a contradiction. If $\dim M^0_{2\varepsilon_m}(P_m(A-B)P_m) < \dim P_m E^0$, there must exist $y \neq 0$ and $y \in P_m E^0 \cap (M^+_{2\varepsilon_m}(P_m(A-B)P_m) \oplus M^-_{2\varepsilon_m}(P_m(A-B)P_m))$. This implies that

$$\begin{aligned} \|P_m(A-B)P_m y\| &\leq \frac{4}{3}\varepsilon_m \|y\| \quad \text{(by (2.6))}, \\ \|P_m(A-B)P_m y\| &\geq 2\varepsilon_m \|y\|, \end{aligned}$$

and we get a contradiction again. Thus (2.8) holds. By (2.4) we have (a).

Step 4. For $m \geq m^*$, we have

$$\dim M^+_d(P_m(A-B)P_m) \oplus M^-_d(P_m(A-B)P_m) = \dim Y_m. \tag{2.9}$$

In fact, if $\dim M_d^+(P_m(A-B)P_m)\oplus M_d^-(P_m(A-B)P_m) > \dim Y_m$, there must exist $y\neq 0$ and $y\in (M_d^+(P_m(A-B)P_m)\oplus M_d^-(P_m(A-B)P_m))\cap P_mE^0$. This implies

$$\begin{aligned}\|P_m(A-B)P_my\| &\geq d\|y\|,\\ \|P_m(A-B)P_my\| &\leq \frac{4}{3}\varepsilon_m\|y\| \quad \text{(by (2.6))}.\end{aligned}$$

We get a contradicton. If $\dim(M_d^+(P_m(A-B)P_m)\oplus M_d^-(P_m(A-B)P_m)) < \dim Y_m$, there must exist $y\in Y_m\cap M_d^0(P_m(A-B)P_m$ and $y\neq 0$. This implies

$$\begin{aligned}\|P_m(A-B)P_my\| &\geq 2d\|y\| \quad \text{(by (2.7))},\\ \|P_m(A-B)P_mY\| &\leq d\|y\|.\end{aligned}$$

We get a contradiction again. Therefore (2.9) holds. By (2.8), (2.9) and the fact $E_m = Y_m\oplus P_mE^0$, we have (b). □

Remark 2.4. (a) Since $A-B$ may not commute with P_m, how to compute the Morse index of $P_m(A-B)P_m$ becomes a very difficult part in applications. Theorem 2.3 shows a way to describe the behavior of the operator $P_m(A-B)P_m$.

(b) All eigenvalues of $P_m(A-B)P_m$ split into two parts for m large enough. One part falls into $(-\infty,-d]\cup[d,\infty)$ and they will stay there as $m\to\infty$. Another part falls into $(-2\varepsilon_m, 2\varepsilon_m)$ and they will go to 0 as $m\to\infty$.

(c) There is no eigenvalues of $P_m(A-B)P_m$ in $(-d,-2\varepsilon_m]\cup[2\varepsilon_m,d)$ and $\varepsilon_m\to 0$ as $m\to\infty$.

Remark 2.5. The idea in Theorem 2.3 and Remark 2.4 is very close to the idea of the L-index of a compact selfadjoint operator given by M. Izydorek in [16]. The author wants to thank the referee for pointing out this.

3. Periodic solutions of Hamiltonian systems

Let $S^1=\mathbb{R}/\mathbb{Z}$, $E=W^{1/2,2}(S^1,\mathbb{R}^{2N})$. Then E is a Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, and E consists of those $z(t)$ in $L^2(S^1,\mathbb{R}^{2N})$ whose Fourier series

$$z(t)=a_0+\sum_{n=1}^{\infty}(a_n\cos(2\pi nt)+b_n\sin(2\pi nt))$$

satisfies

$$\|z\|^2=|a_0|^2+\frac{1}{2}\sum_{n=1}^{\infty}n(|a_n|^2+|b_n|^2)<\infty,$$

where $a_j, b_j\in\mathbb{R}^{2N}$. For a given continuous 1-periodic and symmetric matrix function $B(t)$, we define

$$\langle Ax,y\rangle=\int_0^1(-J\dot{x},y)\,dt,\quad \langle Bx,y\rangle=\int_0^1(B(t)x,y)\,dt \tag{3.1}$$

on E. Then A satisfies (A) in Section 2 with $\ker A = \mathbb{R}^{2N}$, and B is a linear symmetric compact operator ([21]). For $B(t)$, by [10], [19], [21], we can define its Maslov-type index as a pair of integers $(i(B), n(B)) \in \mathbb{Z} \times \{0, \ldots, 2N\}$. Using the Floquet theory, we have

$$n(B) = \dim \ker(A - B).$$

Let $B_0(t)$ and $B_\infty(t)$ be the matrix functions in (1.2) and (1.3) with the Maslov-type index (i_0, n_0) and (i_∞, n_∞), respectively. Let B_0 and B_∞ be operators, defined by (3.1), corresponding to $B_0(t)$ and $B_\infty(t)$. Then we have

$$n_0 = \dim \ker(A - B_0), \quad n_\infty = \dim \ker(A - B_\infty).$$

Let $\ldots \leq \lambda_2' \leq \lambda_1' < 0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be the eigenvalues of $A - B_0$, and Let $\{e_j'\}$ and $\{e_j\}$ be the eigenvectors of $A - B_0$ corresponding to $\{\lambda_j'\}$ and $\{\lambda_j\}$, respectively. For $m \geq 0$, set

$$\begin{aligned} E_0 &= \ker(A - B_0), \\ E_m &= E_0 \oplus \operatorname{span}\{e_1, \ldots, e_m\} \oplus \operatorname{span}\{e_1', \ldots, e_m'\} \end{aligned}$$

and let P_m be the orthogonal projection from E to E_m. Then $\Gamma_0 = \{P_m : m = 1, 2, \ldots\}$ is an approximation scheme with respect to A. Moreover,

$$(A - B_0)P_m = P_m(A - B_0) \quad \text{for all } m \geq 0.$$

The following result was proved in [14].

Theorem 3.1 ([14]). *For any continuous* 1*-periodic and symmetric matrix function $B(t)$ with the Maslov-type index (i_∞, n_∞), there exists a $m^* > 0$ such that for $m \geq m^*$ we have*

$$\begin{aligned} \dim M_d^+(P_m(A - B)P_m) &= m + i_0 - i_\infty + n_0 - n_\infty, \\ \dim M_d^-(P_m(A - B)P_m) &= m - i_0 + i_\infty, \\ \dim M_d^0(P_m(A - B)P_m) &= n_\infty, \end{aligned} \tag{3.2}$$

where $d = \|(A - B)^\#\|^{-1}/4$, and B is the operator, defined by (3.1)*, corresponding to $B(t)$.*

For any $z \in E$, we define

$$\begin{gathered} g_0(z) = \int_0^1 G_0(t, z)\, dt, \quad g_\infty(z) = \int_0^1 G_\infty(t, z)\, dt, \\ f(z) = \frac{1}{2}\langle (A - B_\infty)z, z\rangle - g_\infty(z) = \frac{1}{2}\langle (A - B_0)z, z\rangle - g_0(z). \end{gathered}$$

Then (H1) implies that $f(z) \in C^2(E, R)$. Looking for 1-periodic solutions of (1.1) is equivalent to looking for the critical points of f (see [23]).

For m≥ 1, let f_m be the restriction of f to the subspace E_m. Then

$$f'_m(z) = (A - B_0)z - P_m g'_0(z) \quad \text{for all } z \in E_m. \tag{3.3}$$

LEMMA 3.2. *Assume* (H1), (1.3) *and* (H2$^+$) (*or* (H2$^-$)). *Then there exist* $m_1 > 0$ *and* $r_0 > 0$ *independent of* m *such that for* $m \geq m_1$, 0 *is the only critical point of* f_m *inside* $Q = \{z \in E_m : \|z\| \leq r_0\}$.

PROOF. Suppose the conclusion is not true. Then for any $k \geq 1$, there exists $z_k \in E_{m_k}$ such that

$$f'_{m_k}(z_k) = 0, \quad z_k \neq 0, \quad \text{and} \quad z_k \to 0 \quad \text{as } k \to \infty. \tag{3.4}$$

Without lossing generality, suppose $\|z_k\| \leq 1$ for $k \geq 1$. By the special structure of $\Gamma_0 = \{P_m : m = 1, 2, \dots\}$,

$$E_m = \ker(A - B_0) \oplus (\operatorname{Im}(A - B_0) \cap E_m).$$

Write $z_k = x_k + y_k \in \ker(A - B_0) \oplus (\operatorname{Im}(A - B_0) \cap E_{m_k})$. Then $x_k \to 0$, $y_k \to 0$ as $k \to \infty$, and by (3.3)

$$(A - B_0)y_k = P_m g'_0(z_k). \tag{3.5}$$

By (1.3) and (1.4), we have $a_1 > 0$ such that

$$|G'_0(t,z)| \leq a_1 \|z\|^{\beta_0} \quad \text{for all } (t,z) \in [0,1] \times \mathbb{R}^{2N}.$$

This implies that there exists $a_2 > 0$ such that

$$\|g'_0(z)\| \leq a_2 \|z\|^{\beta_0} \quad \text{for all } z \in E. \tag{3.6}$$

By (3.5) and (3.6), we have $a_3 > 0$ such that

$$\|y_k\| \leq a_3(\|x_k\| + \|y_k\|)^{\beta_0}.$$

This implies that for k large enough

$$\|y_k\| \leq \|x_k\|, \quad \|y_k\| \leq a_3 2^{\beta_0} \|x_k\|^{\beta_0}. \tag{3.7}$$

By (3.5)–(3.7),

$$|\langle g'_0(z_k), z_k \rangle| = |\langle g'_0(z_k), y_k \rangle| \leq a_2 \|z_k\|^{\beta_0} \|y_k\| \leq a_2 a_3 2^{2\beta_0} \|x_k\|^{2\beta_0}. \tag{3.8}$$

On the other hand, for $L_0 > 0$ given in (H2$^\pm$), denote

$$\Omega = \{t \in [0,1] : |z_k(t)| \leq L_0\}, \quad \Omega^\perp = [0,1] \setminus \Omega.$$

Then for α_0 given in (H2$^\pm$),

$$a_4 \|z_k\|^{2\alpha_0} \geq \int_0^1 |z_k|^{2\alpha_0}\, dt \geq \int_{\Omega^\perp} |z_k(t)|^{2\alpha_0}\, dt \geq L_0^{2\alpha_0} \operatorname{meas}(\Omega^\perp).$$

This implies

$$\text{meas}(\Omega^{\perp}) \leq a_5 \|z_k\|^{2\alpha_0}, \tag{3.9}$$

where $a_5 > 0$. By (H2$^{\pm}$), we have

$$\begin{aligned} |\langle g_0'(z_k), z_k\rangle| &= \left| \int_0^1 \pm (G_0'(t,z_k), z_k)\,dt \right| \\ &\geq \left| \int_{\Omega} \pm (G_0'(t,z_k), z_k)\,dt \right| - \left| \int_{\Omega^{\perp}} |G_0'(t,z_k)||z_k|\,dt \right| \\ &\geq \int_{\Omega} c_1 |z_k(t)|^{\alpha_0}\,dt - \int_{\Omega^{\perp}} |G_0'(t,z_k)||y_k|\,dt \\ &\quad - \int_{\Omega^{\perp}} |G_0'(t,z_k)||x_k|\,dt. \end{aligned} \tag{3.10}$$

Using the same argument as (3.6) and (3.8), we have

$$\begin{aligned} \int_{\Omega^{\perp}} |G_0'(t,z_k)||y_k|\,dt &\leq \int_0^1 a_1 |z_k|^{\beta_0} |y_k|\,dt \\ &\leq a_1 \left(\int_0^1 |z_k|^{2\beta_0}\,dt \right)^{1/2} \left(\int_0^1 |y_k|^2\,dt \right)^{1/2} \\ &\leq a_6 \|z_k\|^{\beta_0} \|y_k\| \leq a_7 \|x_k\|^{2\beta_0}. \end{aligned} \tag{3.11}$$

Notice that there exist $\lambda_1, \lambda_2 > 0$ such that for any $x \in \ker(A - B_0)$,

$$\lambda_1 \|x\| \leq |x(t)| \leq \lambda_2 \|x\| \quad \text{for all } t \in [0,1]. \tag{3.12}$$

By (1.3) and the fact that $\alpha_0 > 2$, we have

$$|G_0'(t,z)| \leq a_8 + a_9 |z|^{\alpha_0}, \quad \text{for all } (t,z) \in [0,1] \times \mathbb{R}^{2N}.$$

Combining this with (3.7), (3.9) and (3.12) yields

$$\begin{aligned} \int_{\Omega^{\perp}} |G_0'(t,z_k)||x_k|\,dt &\leq \int_{\Omega^{\perp}} a_8 |x_k|\,dt + \int_{\Omega^{\perp}} a_9 |z_k|^{\alpha_0} |x_k|\,dt \\ &\leq a_8 \lambda_2 \|x_k\| \text{meas}(\Omega^{\perp}) + \int_0^1 a_9 |z_k|^{\alpha_0} |x_k|\,dt \\ &\leq a_8 \lambda_2 a_5 \|x_k\|^{2\alpha_0+1} + a_{10} \|x_k\|^{\alpha_0+1} \leq a_{11} \|x_k\|^{\alpha_0+1}. \end{aligned} \tag{3.13}$$

By (3.7) and (3.12), we have

$$\begin{aligned} \int_{\Omega} (z_k, x_k)\,dt &= \int_{\Omega} |x_k|^2\,dt + \int_{\Omega} (y_k, x_k)\,dt \\ &\geq \lambda_1^2 \|x_k\|^2 \text{meas}(\Omega) - \int_0^1 |y_k||x_k|\,dt \\ &\geq \lambda_1^2 \|x_k\|^2 \text{meas}(\Omega) - a_{12} \|x_k\|^{\beta_0+1}. \end{aligned}$$

Since $\beta_0+1>2$ and $\|z_k\|\to 0$ as $k\to\infty$, by (3.9) there exists $k^*>0$ such that for $k\geq k^*$

$$\operatorname{meas}(\Omega^\perp)\leq\frac{1}{2},\quad \|x_k\|^{\beta_0-1}\leq\frac{\lambda_1^2}{4a_{12}}.$$

This implies that

$$\int_\Omega (z_k,x_k)\,dt\geq\frac{\lambda_1^2}{4}\|x_k\|^2. \tag{3.14}$$

Since $\alpha_0>1$, we have

$$\begin{aligned}\int_\Omega (z_k,x_k)\,dt&\leq\bigg(\int_\Omega |z_k|^{\alpha_0}\,dt\bigg)^{1/\alpha_0}\bigg(\int_\Omega |x_k|^{\alpha_0/(\alpha_0-1)}\,dt\bigg)^{(\alpha_0-1)/\alpha_0}\\&\leq a_{13}\bigg(\int_\Omega |z_k|^{\alpha_0}\,dt\bigg)^{1/\alpha_0}\|x_k\|.\end{aligned}$$

Combing this with (3.14), we have

$$\int_\Omega |z_k|^{\alpha_0}\,dt\geq a_{14}\|x_k\|^{\alpha_0}. \tag{3.15}$$

By (3.8), (3.10), (3.11), (3.13) and (3.15), we have

$$c_1a_{14}\|x_k\|^{\alpha_0}\leq a_2a_32^{2\beta^0}\|x_k\|^{2\beta_0}+a_7\|x_k\|^{2\beta_0}+a_{11}\|x_k\|^{\alpha_0+1}. \tag{3.16}$$

Since all the constants $c_1,a_1,\ldots,a_{14}$ are independent of k, $\alpha_0<2\beta_0$ and $x_k\to 0$ as $k\to\infty$, we get a contradiction from (3.16). Therefore the conclusion of Lemma 3.2 is true. □

For $m\geq 1$, let η_m be the gradient flow generated by

$$\frac{dz}{dt}=-(A-B_0)z+P_mg_0'(z)\quad\text{on } E_m. \tag{3.17}$$

Lemma 3.3. *Assume* (H1), (1.3) *and* (H2$^\pm$). *Then there exists $m_2>0$ such that for $m\geq m_2$, there exists an isolating block D_m of η_m satisfying the following properties*

(a) *0 is the only critical point of f_m inside D_m,*
(b) *$I(\eta,D_m)=t^m$ if* (H2$^-$) *holds,*
(c) *$I(\eta,D_m)=t^{m+n_0}$ if* (H2$^+$) *holds.*

Proof. By Lemma 3.2, for $m\geq m_1$, f_m has only one critical point 0 inside Q, where $Q=\{z\in E_m:\|z\|\leq r_0\}$. Set

$$\begin{aligned}V_m^\pm&=\{y^\pm\in P_mM^\pm(A-B_0):\|y^\pm\|\leq r_\pm\},\\ W&=\{x\in M^0(A-B_0):\|x\|\leq r_w\}.\end{aligned}$$

We want to show that there are $r_+, r_-, r_w > 0$, which do not depend on m, such that $D_m = V_m^+ \times V_m^- \times W \subset Q$ is an isolating block of the gradient flow η_m generated by (3.17). Denote

$$\begin{gathered}\rho_\pm = \inf_{\|y^\pm\|=1} |\langle y^\pm, (A-B_0)y^\pm\rangle|, \quad y^\pm \in M^\pm(A-B_0), \\ \rho = \min(\rho_+, \rho_-) > 0.\end{gathered} \tag{3.18}$$

For $z = z^+ + z^- + z^0 \in \partial V_m^+ \times V_m^- \times W$, by (3.6) and (3.17),

$$\begin{aligned}\left\langle \frac{dz}{dt}, z^+\right\rangle\bigg|_{t=0} &= -\langle (A-B_0)z^+, z^+\rangle + \langle g_0'(z), z^+\rangle \\ &\le -\rho\|z^+\|^2 + a_2\|z\|^{\beta_0}\|z^+\| \\ &= -\rho\|z^+\|\left(\|z^+\| - \left(\frac{a_2}{\rho}\right)\|z\|^{\beta_0}\right) \\ &\le -\rho r_+\left(r_+ - \left(\frac{a_2}{\rho}\right)(r_+ + r_- + r_w)^{\beta_0}\right) \\ &\le -\rho r_+\left(r_+ - \left(\frac{a_2}{\rho}\right)3^{\beta_0} r_w^{\beta_0}\right) \le -\rho r_+\left(\frac{1}{2}r_+\right) < 0,\end{aligned} \tag{3.19}$$

provided

$$r_+ = r_- \le r_w, \quad r_+ = 2(a_2/\rho)3^{\beta_0} r_w^{\beta_0}. \tag{3.20}$$

Similarly, for $z = z^+ + z^- + z^0 \in V_m^+ \times \partial V_m^- \times W$

$$\begin{aligned}\left\langle \frac{dz}{dt}, z^-\right\rangle\bigg|_{t=0} &= -\langle (A-B_0)z^-, z^-\rangle + \langle g_0'(z), z^-\rangle \\ &\ge \rho\|z^-\|^2 - a_2\|z\|^{\beta_0}\|z^-\| \ge \rho r_-(r_-/2) > 0,\end{aligned} \tag{3.21}$$

provided (3.20) holds.

For $z = z^+ + z^- + z^0 \in V_m^+ \times V_m^- \times \partial W$, denote $\Omega = \{t \in [0,1] : |z(t)| \le L_0\}$, $\Omega^\perp = [0,1] \setminus \Omega$.

Similar to the proof of (3.9), we have

$$\operatorname{meas}(\Omega^\perp) \le a_5\|z\|^{2\alpha_0}. \tag{3.22}$$

Case 1. (H2$^+$) holds. By (3.20) and the same arguments as those in the proof (3.10), (3.11), (3.13) and (3.15), we have

$$\begin{aligned}\left\langle \frac{dz}{dt}, z^0\right\rangle\bigg|_{t=0} &= \int_0^1 (G_0'(t,z), z^0)\,dt \\ &= \int_\Omega (G_0'(t,z), z) - \int_\Omega (G_0'(t,z), z^+ + z^-)\,dt \\ &\quad + \int_{\Omega^\perp} (G_0'(t,z), z^0)\,dt \\ &\ge \int_\Omega c_1|z|^{\alpha_0}\,dt - \int_0^1 |G_0'(t,z)||z^+ + z^-|\,dt\end{aligned} \tag{3.23}$$

$$
\begin{aligned}
&- \int_{\Omega^\perp} |G_0'(t,z)||z^0|\,dt \\
&\geq b_1\|z^0\|^{\alpha_0} - b_2\|z^0\|^{2\beta_0} - b_3\|z^0\|^{\alpha_0+1} \\
&= r_w^{\alpha_0}(b_1 - b_2 r_w^{2\beta_0-\alpha_0} - b_3 r_w) \geq \frac{1}{2} b_1 r_w^{\alpha_0} > 0
\end{aligned}
$$

provided

$$
b_1 \geq 2(b_2 r_w^{2\beta_0-\alpha_0} + b_3 r_w). \tag{3.24}
$$

Notice that all constants $a_2, b_1, b_2, b_3 > 0$ in (3.20) and (3.24) are independent of m. Since $\beta_0 > 1$ and $\alpha_0 < 2\beta_0$, we can choose $r_+, r_-, r_w > 0$ such that (3.20) and (3.24) holds, and $D_m \subset \operatorname{int}(Q)$. By (3.19), (3.21) and (3.23), D_m is an isolating block with $D_m^- = V_m^+ \times \partial V_m^- \times W \cup V_m^+ \times V_m^- \times \partial W$. Therefore

$$
I(\eta_m, D_m) = t^{\dim(P_m M^-(A-B_0) \oplus M^0(A-B_0))} = t^{m+n_0}. \tag{3.25}
$$

Case 2. (H2$^-$) holds. Using the same arguments as (3.23), we have

$$
\begin{aligned}
\left\langle \frac{dz}{dt}, z^0 \right\rangle \bigg|_{t=0} &= \int_0^1 (G_0'(t,z), z^0)\,dt \\
&\leq - \int_\Omega c_1 |z|^{\alpha_0}\,dt + \int_0^1 |G_0'(t,z)||z^+ + z^-|\,dt \\
&\quad + \int_{\Omega^\perp} |G_0'(t,z)||z^0|\,dt \\
&\leq - r_w^{\alpha_0}(b_1 - b_2 r_w^{2\beta_0-\alpha_0} - b_3 r_w) \leq -\frac{1}{2} b_1 r_w^{\alpha_0} < 0
\end{aligned}
$$

provided (3.24) holds. Therefore we can choose $r_+, r_-, r_w > 0$ such that $D_m \subset Q$ is an islating block with $D_m^- = V_m^+ \times \partial V_m^- \times W$. Then we have

$$
I(\eta_m, D_m) = t^{\dim(P_m M^-(A-B_0))} = t^m. \tag{3.26}
$$

$\square$

Lemma 3.4. *Assume* (H1) *and* (H3$^\pm$). *Then there exists $m_3 > 0$ such that, for $m \geq m_3$, there exists an isolating block $D_{\infty m}$ of η_m satisfying the following properties:*

(a) *$D_{\infty m}$ is uniformly bounded by a constant independent of m,*
(b) *$I(\eta_m, D_{\infty m}) = t^{m-i_0+i_\infty}$ if* (H3$^-$) *holds,*
(c) *$I(\eta_m, D_{\infty m}) = t^{m-i_0+i_\infty+n_\infty}$ if* (H3$^+$) *holds.*

Proof. By Theorem 3.1 and Theorem 2.3, there exists $m^* > 0$ such that for $m \geq m^*$ and $d = \|(A-B_\infty)^\#\|^{-1}/4$, the relation (3.2) holds and

$$
E_m = M_d^+(P_m(A-B_\infty)P_m) \oplus M_d^-(P_m(A-B_\infty)P_m) \oplus M_{2\varepsilon_m}^0(P_m(A-B_\infty)P_m),
$$

where ε_m is given by (2.3) for $B = B_\infty$, and $\varepsilon_m \to 0$ as $m \to \infty$. Denote

$$\begin{aligned} U_m^{\pm} &= \{y^{\pm} \in M_d^{\pm}(P_m(A - B_\infty)P_m) : \|y^{\pm}\| \le R_{\pm}\}, \\ W_\infty &= \{x \in M_{2\varepsilon_m}^0(P_m(A - B_\infty)P_m) : \|x\| \le R_w\}. \end{aligned}$$

We want to show that there are $R_+, R_-, R_w > 0$, which do not depend on m, such that $D_{\infty m} = U_m^+ \times U_m^- \times W_\infty$ is an isolating block of the gradient η_m generated by (3.17), which is the same as

$$\frac{dz}{dt} = -P_m(A - B_\infty)P_m z + P_m g_\infty'(z) \quad \text{on } E_m. \tag{3.27}$$

By (H3$^\pm$), there exist $M > 0$ such that

$$|G_\infty'(t, z)| \le M \quad \text{for all } (t, z) \in [0, 1] \times \mathbb{R}^{2N}. \tag{3.28}$$

This implies

$$\|g_\infty'(z)\| \le M \quad \text{for all } z \in E. \tag{3.29}$$

For $\lambda_0 \ge 2$, let

$$R_+ = R_- = \frac{\lambda_0 M}{d} > 0. \tag{3.30}$$

For $z = z^+ + z^- + z^0 \in \partial U_m^+ \times U_m^- \times W_\infty$, by (3.27), (3.29) and (3.30),

$$\begin{aligned} \left\langle \frac{dz}{dt}, z^+ \right\rangle\bigg|_{t=0} &= -\langle P_m(A - B_\infty)P_m z^+, z^+\rangle + \langle g_\infty'(z), z^+\rangle \\ &\le -d\|z^+\|^2 + M\|z^+\| = -dR_+^2 + MR_+ \\ &\le \frac{-\lambda_0^2 M^2 + \lambda_0 M^2}{d} < 0. \end{aligned} \tag{3.31}$$

Similarly, for $z = z^+ + z^- + z^0 \in U_m^+ \times \partial U_m^- \times W_\infty$,

$$\begin{aligned} \left\langle \frac{dz}{dt}, z^- \right\rangle\bigg|_{t=0} &= -\langle P_m(A - B_\infty)P_m z^-, z^-\rangle + \langle g_\infty'(z), z^-\rangle \\ &\ge dR_-^2 - MR_- \ge \frac{\lambda_0^2 M^2 - \lambda_0 M^2}{d} > 0. \end{aligned} \tag{3.32}$$

Notice that there exist $\lambda_3 > 0, \lambda_4 > 0$ such that for any $x \in \ker(A - B_\infty)$

$$\lambda_3\|x\| \le |x(t)| \le \lambda_4\|x\| \quad \text{for all } t \in [0, 1]. \tag{3.33}$$

For any $z^0 \in M_{2\varepsilon_m}^0(P_m(A - B_\infty)P_m)$, according to Step 2 in the proof of Theorem 2.3, we can write $z^0 = y + P_m x \in E_m = Y_m \oplus P_m(\ker(A - B_\infty))$. By (2.6) and (2.7), we have

$$\begin{aligned} \|P_m(A - B_\infty)P_m z^0\| &\le 2\varepsilon_m\|z^0\|, \\ \|P_m(A - B_\infty)P_m z^0\| &\ge \|P_m(A - B_\infty)P_m y\| - \|P_m(A - B_\infty)P_m x\| \\ &\ge 2d\|y\| - \frac{4}{3}\varepsilon_m\|P_m x\| \ge 2d\|y\| - \frac{4}{3}\varepsilon_m\|z^0\|. \end{aligned}$$

This implies that

$$\|y\| \le \left(\frac{5\varepsilon_m}{3d}\right)\|z^0\|, \tag{3.34}$$

$$\|x\| \ge \|P_m x\| \ge \|z^0\| - \|y\| = \left(1 - \frac{5\varepsilon_m}{3d}\right)\|z^0\|,$$

$$\|z^0\| \ge \|P_m x\| \ge \|x\| - \|(I - P_m)P_{B_\infty} x\| \ge (1 - \varepsilon_m)\|x\|. \tag{3.35}$$

Since $\varepsilon_m \to 0$ as $m \to \infty$, there exists $m_1^* \ge m^*$ such that for $m \ge m_1^*$,

$$1 - \varepsilon_m \ge \frac{1}{2}, \quad 1 - \frac{5\varepsilon_m}{3d} \ge \frac{1}{2}. \tag{3.36}$$

For $m \ge m_1^*$ and $z^0 \in \partial W_\infty$, set

$$\Delta_1 = \{t : |z^0(t)| \le \frac{\lambda_3}{4}\|z^0\|\}, \quad \Delta_2 = \{t : |z^0(t)| \ge 4\lambda_4\|z^0\|\},$$
$$\Delta = [0,1] \setminus (\Delta_1 \cup \Delta_2). \tag{3.37}$$

By (3.33)–(3.36), we have

$$\int_{\Delta_1} |z^0|\,dt \le \frac{\lambda_3}{4}\|z^0\|\mathrm{meas}(\Delta_1),$$
$$\begin{aligned}\int_{\Delta_1} |z^0|\,dt &= \int_{\Delta_1} |y + P_m x|\,dt = \int_{\Delta_1} |x + y - (I - P_m)x|\,dt \\ &\ge \int_{\Delta_1} |x(t)|\,dt - \|y\| - \|(I - P_m)P_{B_\infty} x\| \\ &\ge \lambda_3\|x\|\mathrm{meas}(\Delta_1) - \frac{5\varepsilon_m}{3d}\|z^0\| - 2\varepsilon_m\|z^0\| \\ &\ge \frac{\lambda_3}{2}\|z^0\|\mathrm{meas}(\Delta_1) - \left(\frac{5}{3d} + 2\right)\varepsilon_m\|z^0\|.\end{aligned}$$

Therefore

$$\mathrm{meas}(\Delta_1) \le \frac{4}{\lambda_3}\left(\frac{5}{3d} + 2\right)\varepsilon_m = b_4\varepsilon_m. \tag{3.38}$$

By (3.33)–(3.36), we also have

$$\int_{\Delta_2} |z^0|\,dt \ge 4\lambda_4\|z^0\|\mathrm{meas}(\Delta_2),$$
$$\begin{aligned}\int_{\Delta_2} |z^0|\,dt &\le \int_{\Delta_2} |x + y - (I - P_m)x|\,dt \\ &\le \int_{\Delta_2} |x(t)|\,dt + \|y\| + \|(I - P_m)x\| \\ &\le \lambda_4\|x\|\mathrm{meas}(\Delta_2) + \|y\| + \|(I - P_m)P_{B_\infty}\|\|x\| \\ &\le 2\lambda_4\|z^0\|\mathrm{meas}(\Delta_2) + \left(\frac{5\varepsilon_m}{3d} + 2\varepsilon_m\right)\|z^0\|.\end{aligned}$$

This implies

$$\text{meas}(\Delta_2) \le \frac{1}{2\lambda_4}\left(\frac{5}{3d} + 2\right)\varepsilon_m = b_5\varepsilon_m. \tag{3.39}$$

For $z^+ + z^- \in U_m^+ \times U_m^-$, $\beta > 1$ and $k > 0$, let

$$\Omega_k = \{t \in [0,1] : |z^+ + z^-| \le k\}, \quad \Omega_k^\perp = [0,1] \setminus \Omega_k. \tag{3.40}$$

Then we have

$$\begin{aligned}\int_0^1 |z^+ + z^-|^\beta\,dt &\ge \int_{\Omega_k^\perp} |z^+ + z^-|^\beta\,dt \ge k^\beta \text{meas}(\Omega_k^\perp),\\ \int_0^1 |z^+ + z^-|^\beta\,dt &\le c_\beta \|z^+ + z^-\|^\beta \le c_\beta (R_+ + R_-)^\beta,\end{aligned}$$

where c_β is the embedding constant for $E \subset L^\beta(S^1, \mathbb{R}^{2N})$. This implies

$$\text{meas}(\Omega_k^\perp) \le \frac{c_\beta^\beta |R_+ + R_-|^\beta}{k^\beta}. \tag{3.41}$$

Case 1. (H3$^+$) holds, i.e. for $|z| \ge L_\infty$,

$$(G_\infty'(t,z), z) \ge \frac{c_3}{|z|^\delta}, \quad |G_\infty'(t,z)||z| \le c_4 |(G_\infty'(t,z), z)|.$$

Choose

$$R_w = \frac{4(4c_4 + 1)}{\lambda_3} k \tag{3.42}$$

with $k > 0$ being determined later.

For $z = z^+ + z^- + z^0 \in U_m^+ \times U_m^- \times \partial W_\infty$, Let Δ and Ω_k be given by (3.37) and (3.40). For any $t \in \Delta \cap \Omega_k$ and $k \ge L_\infty/(2c_4)$, we have

$$\begin{aligned}|z(t)| &\ge |z^0(t)| - |z^+(t) + z^-(t)| \ge \frac{\lambda_3}{4}\|z^0\| - k \qquad (3.43)\\ &= \frac{\lambda_3}{4} R_w - k = 4c_4 k > 2c_4 k \ge L_\infty,\\ |z(t)| &\le |z^0(t)| + |z^+ + z^-| \le 4\lambda_4 \|z^0\| + k \qquad (3.44)\\ &= \left(\frac{16\lambda_4(4c_4+1)}{\lambda_3} + 1\right)k = b_6 k.\end{aligned}$$

By (3.37)–(3.41), we can choose $k_0 \ge L_\infty/(2c_4)$ and $m_2^* \ge m_1^*$ such that for k$\ge k_0$ and $m \ge m_2^*$,

$$\text{meas}(\Omega_k \cap \Delta) \ge 1/2. \tag{3.45}$$

Now by (H3$^+$), (3.27), (3.28) and (3.37)–(3.45), we have

$$
\begin{aligned}
(3.46)\quad \left\langle \frac{dz}{dt}, z^0 \right\rangle\Big|_{t=0} &= \int_0^1 (G'_\infty(t,z), z^0)\,dt - \langle P_m(A-B_\infty)P_m z^0, z^0\rangle \\
&\geq \int_\Delta (G'_\infty(t,z), z^0)\,dt - \int_{\Delta_1\cup\Delta_2} M|z^0|\,dt - 2\varepsilon_m\|z^0\|^2 \\
&\geq \int_{\Delta\cap\Omega_k} (G'_\infty(t,z), z^0)\,dt - \int_{\Delta\cap\Omega_k^\perp} M|z^0|\,dt \\
&\quad - M\|z^0\|(\mathrm{meas}(\Delta_1\cup\Delta_2))^{1/2} - 2\varepsilon_m\|z^0\|^2 \\
&\geq \int_{\Delta\cap\Omega_k} ((G'_\infty(t,z), z) - k|G'_\infty(t,z)|)\,dt \\
&\quad - M4\lambda_4\|z^0\|\mathrm{meas}(\Omega_k^\perp) \\
&\quad - M(b_4+b_5)^{1/2}\varepsilon_m^{1/2}\|z^0\| - 2\varepsilon_m\|z^0\|^2 \\
&\geq \int_{\Delta\cap\Omega_k} \frac{c_3}{|z|^\delta}\left(1-\frac{kc_4}{|z|}\right)dt - b_7R_w\mathrm{meas}(\Omega_k^\perp) \\
&\quad - b_8R_w\varepsilon_m^{1/2} - 2\varepsilon_m R_w^2 \\
&\geq \frac{c_3}{(b_6k)^\delta}\cdot\frac{1}{2}\mathrm{meas}(\Delta\cap\Omega_k) - b_7R_w\mathrm{meas}(\Omega_k^\perp) \\
&\quad - b_8R_w\varepsilon_m^{1/2} - 2\varepsilon_m R_w^2 \\
&\geq \frac{b_9}{k^\delta} - \frac{b_{10}}{k^{\beta-1}} - b_8R_w\varepsilon_m^{1/2} - 2\varepsilon_m R_w^2 \\
&\geq \frac{b_9}{2k^\delta} - b_8R_w\varepsilon_m^{1/2} - 2\varepsilon_m R_w^2
\end{aligned}
$$

provided

$$
(3.47)\qquad \beta = \delta + 2 \quad\text{and}\quad k \geq 2b_{10}/b_9 + k_0.
$$

In the above arguments, all the constants M, k_0, b_i are independent of m. Therefore R_+, R_- and R_w are independent of m. Since $\varepsilon_m \to 0$ as $m\to\infty$, there exists $m_3 \geq m_2^*$ such that for $m \geq m_3$

$$
\left\langle \frac{dz}{dt}, z^0 \right\rangle\Big|_{t=0} \geq \frac{b_9}{2k^\delta} - b_8R_w\varepsilon_m^{1/2} - 2\varepsilon_m R_w^2 \geq \frac{b_9}{4k^\delta} > 0.
$$

Combining this with (3.31) and (3.32) yields that $D_{\infty m} = U_m^+\times U_m^-\times W_\infty$ is an isolating block with $D_{\infty m}^- = U_m^+\times\partial U_m^-\times W_\infty \cup U_m^+\times U_m^-\times\partial W_\infty$. $D_{\infty m}$ is uniformly bounded by $R_+ + R_- + R_w$, which is independent of m. Moreover, by Theorem 3.1, we have

$$
I(\eta_m, D_{\infty m}) = t^{\dim(M_d^-(P_m(A-B_\infty)P_m)\oplus M^0_{2\varepsilon_m}(P_m(A-B_\infty)P_m))} = t^{m-i_0+i_\infty+n_\infty}.
$$

Case 2. (H3$^-$) holds. By using similar arguments as in the proof of Case 1, we can choose R_w as in (3.42) and show that for $z = z^+ + z^- + z^0 \in U_m^+ \times U_m^- \times \partial W_\infty$,

$$\begin{aligned}\left\langle \frac{dz}{dt}, z^0 \right\rangle\Big|_{t=0} &= \int_0^1 (G'_\infty(t,z), z^0)\,dt - \langle P_m(A - B_\infty)P_m z^0, z^0\rangle \\ &\leq \int_{\Delta\cap\Omega_k} ((G'_\infty(t,z), z) + k|G'_\infty(t,z)|)\,dt \\ &\quad + b_7 R_w \mathrm{meas}(\Omega_k^\perp) + b_8 R_w \varepsilon_m^{1/2} + 2\varepsilon_m R_w^2 \\ &\leq -b_9/(2k^\delta) + b_8 R_w \varepsilon_m^{1/2} + 2\varepsilon_m R_w^2 \leq -b_9/(4k^\delta) < 0\end{aligned}$$

provided $m \geq m_3$. Therefore $D_{\infty m}$ is an isolating block with $D_{\infty m}^- = U_m^+ \times \partial U_m^- \times W_\infty$, and

$$I(\eta_m, D_{\infty m}) = t^{\dim(M_d^-(P_m(A-B)P_m)} = t^{m-i_0+i_\infty}.$$

$\{D_{\infty m}\}$ are uniformly bounded by $R_+ + R_- + R_w$, which is independent of m.$\square$

Remark 3.5. (a) If we increase the constants λ_0 in (3.30) and k in (3.47), we will get bigger R_+, R_- and R_w. This allows us to choose the size of $D_{\infty m}$ as large as we want.

(b) Since $A - B_\infty$ does not commute with P_m, we have to control $P_m(A - B_\infty)P_m$ over the possible resonance part $M^0_{2\varepsilon_m}(P_m(A - B_\infty)P_m)$. All the special arguments in the proof of Lemma 3.4 are used to make sure that R_+, R_- and R_w are independent of m.

Proof of Theorem 1.1. *Part 1.* Let $m^* > 0$ be large enough such that for $m \geq m^*$, the conclusions in Lemmas 3.2–3.4 hold. Let η_m be the gradient flow of f_m generated by (3.17). Then we have isolating blocks D_m and $D_{\infty m}$. By Remark 3.5(a), we can adjust the size of $D_{\infty m}$ such that $D_m \subset \mathrm{int}(D_{\infty m})$.

(a) (H2$^+$) and (H3$^+$) hold and $i_0 + n_0 \neq i_\infty + n_\infty$.

In this case, Lemmas 3.3 and 3.4 implies that

$$I(\eta_m, D_m) = t^{m+n_0} \neq I(\eta_m, D_{\infty m}) = t^{m-i_0+i_\infty+n_\infty}.$$

By Theorem 2.1(a), there is a critical point z_m of f_m in $D_{\infty m}\backslash D_m$. By Lemma 3.2 and the fact $z_m \in D_{\infty m}$, we have $r_0 \leq \|z_m\| \leq R_+ + R_- + R_w$, i.e. $\{z_m\}$ are bounded. By standard arguments and passing to a subsequence if necessary, z_m converges to a critical point z^* of f. Moreover,

$$r_0 \leq \|z^*\| \leq R_+ + R_- + R_w.$$

This completes the proof of (a). Cases (b)-(d) follow the same arguments as (a).

Part 2. We only prove (e). Cases (f)–(h) follow the same arguments as (e). Notice that the conditions of (e) implies the conditions of (a). According to the proof of Lemma 3.3, we can have $D_m \subset \mathrm{int}(Q)$, where $Q = \{z \in E_m : \|z\| \leq r_0\}$. By Part 1, for $m \geq m^*$, there is a cirtical point z^* of f with $\|z^*\| \geq r_0$.

Let $B^*(t) = H''(t, z^*(t))$ and B^* be the operator, defined by (3.1), corresponding to $B^*(t)$. Let (i^*, n^*) be the Maslov-type index of $B^*(t)$. It is easy to show that

$$\|f''(z) - (A - B^*)\| \to 0 \quad \text{as } \|z - z^*\| \to 0.$$

Let $d^* = \|(A - B^*)^{\#}\|^{-1}/4$. Then there exists $r > 0$ such that

$$\|f''(z) - (A - B^*)\| < \frac{1}{2}d^*, \quad \text{for all } z \in Q_r(z^*) = \{z \in E : \|z - z^*\|^2 \le 4r\}. \tag{3.48}$$

This implies that

$$\dim M^{\pm}(f''_m(z)) \ge \dim M^{\pm}_{d^*}(P_m(A - B^*)P_m), \tag{3.49}$$

for all $z \in Q_r(z^*) \cap E_m$. For $d^* = \|(A - B^*)^{\#}\|^{-1}/4$, there exists $m_1^* \ge m^*$ such that for $m \ge m_1^*$, the conclusions of Theorems 3.1 and 2.3 hold. Since $\|z^*\| \ge r_0$, we can choose $r > 0$ small enough such that

$$Q_r(z^*) \cap D_m = \phi \quad \text{for } m \ge m_1^*. \tag{3.50}$$

By Remark 3.5(a), we can adjust the size of $D_{\infty m}$ such that

$$Q_r(z^*) \cap E_m \subset \operatorname{int}(D_{\infty m}) \quad \text{for } m \ge m_2^* \ge m_1^*.$$

If there exists another critical point of f in $Q_r(z^*)$, we already have two nontrivial solutions of (1.1) and the proof is complete. Suppose z^* is the only critical point of f in $Q_r(z^*)$. For $m \ge m_2^*$, Set

$$\begin{aligned} C_m(r) &= \{z \in E_m : r < \|z - P_m z^*\|^2 \le 2r\}, \\ V_m(r) &= \{z \in E_m : \|z - P_m z^*\|^2 \le r\}. \end{aligned}$$

Then there exist $m_3^* \ge m_2^*$ such that for $m \ge m_3^*$, we have

$$C_m(r) \subseteq Q_r(z^*) \cap E_m, \quad V_m(r) \subseteq Q_r(z^*) \cap E_m, \tag{3.51}$$

$$\|f'_m(z)\| \ge \rho \quad \text{for all } z \in C_m(r), \tag{3.52}$$

where $\rho > 0$ is a constant independent of m.

For otherwise, there exists $z_{m_k} \in C_{m_k}(r)$ such that $f'_{m_k}(z_{m_k}) \to 0$ as $k \to \infty$. Since $\{z_{m_k}\}$ are bounded, by standard arguments and passing to a subsequence if necessary, z_{m_k} converges to a critical point z^{**} of f, $z^{**} \in Q_r(z^*)$ and $\|z^{**} - z^*\|^2 \ge r$. This is a contradiction to the assumption that z^* is the only critical point of f in $Q_r(z^*)$. Thus (3.52) holds.

Let $a \in E_m$ with $\|a\| < \rho/34$. Define

$$g_m(z) = f_m(z) + \langle a, z - P_m z^*\rangle h(\|z - P_m z^*\|^2) \quad \text{on } E_m,$$

where $h\colon[0,\infty)\to[0,1]$ is a smooth function satisfies

$$|h'(s)|\le 4/r \quad \text{for all } s\in[0,\infty),$$
$$h(s)=0 \quad \text{for } s\ge 2r,\quad h(s)=1 \quad \text{for } s\le 3r/2.$$

Then we have

$$g'_m(z)=f'_m(z)+a,\quad g''_m(z)=f''_m(z) \quad \text{for all } z\in V_m(r). \tag{3.53}$$

$$\|g'_m(z)\|\ge\|f'_m\|-\|a\|-2\|z-P_mz^*\|^2\|a\|4/r\ge\rho/2>0 \quad \text{for all } z\in C_m(r). \tag{3.54}$$

$$g_m(z)=f_m(z) \quad \text{for all } \|z-P_mz^*\|^2\ge 2r. \tag{3.55}$$

By Sard's Lemma, we can choose the vector a in such a way that $g_m(z)$ has only finite number of nondegenerate critical points in $V_m(r)$, say $\{x_1,\dots,x_n\}$. By (3.49), (3.51), (3.53) and Theorem 3.1, we have

$$\dim M^-(g''_m(x_j))=\dim M^-(f''_m(x_j))\in[m-i_0+i^*,m-i_0+i^*+n^*], \tag{3.56}$$

for $j=1,\dots,n$. Now (3.50), (3.51) and (3.55) imply that $D_{\infty m}$ and D_m are also isolating blocks of the gradient flow π_m for g_m generated by

$$\frac{dz}{dt}=-g'_m(z) \quad \text{on } E_m.$$

If $\{0,x_1,\dots,x_n\}$ are all critical points of g_m in $D_{\infty m}$, by Theorem 2.1(b) we have

$$\sum_{j=1}^{n}t^{\dim M^-(g''_m(x_j))}+I(\pi_m,D_m)=I(\pi_m,D_{\infty m})+(1+t)Q(t). \tag{3.57}$$

Notice that

$$\begin{aligned}I(\pi_m,D_m)&=I(\eta_m,D_m)=t^{m+n_0},\\ I(\pi_m,D_{\infty m})&=I(\eta_m,D_{\infty m})=t^{m-i_0+i_\infty+n_\infty}.\end{aligned}$$

By (3.56) and (3.57), we must have

$$\begin{aligned}m-i_0+i_\infty+n_\infty&\in[m-i_0+i^*,m-i_0+i^*+n^*],\\ m+n_0+1&\in[m-i_0+i^*,m-i_0+i^*+n^*],\quad \text{or}\\ m+n_0-1&\in[m-i_0+i^*,m-i_0+i^*+n^*].\end{aligned}$$

This imply that $|i_0+n_0-i_\infty-n_\infty|\le n^*+1\le 2N+1$. This contradicts to the conditions of (e). Therefore g_m must have at least one critical point y_m inside $D_{\infty m}$ other than $\{0,x_1,\dots,x_n\}$. By (3.54) and (3.55), y_m is also a critical point of f_m, and

$$r_0\le\|y_m\|\le R_++R_-+R_w,\quad \|y_m-P_mz^*\|^2\ge 2r.$$

By standard arguments and passing to a subsequence if necessary, y_m converges to a critical point y^* of f. Moreover,

$$\|y^* - z^*\|^2 \geq 2r, \quad \|y^*\| \geq r_0,$$

i.e. y^* is another nontrivial 1-periodic solution of (1.1). $\square$

Proof of Theorem 1.3. By Theorems 3.1 and 2.3, there exists $m^* > 0$ such that for $m \geq m^*$ and $d = \|(A - B_\infty)^\#\|^{-1}/4$, the relation (3.2) holds and

$$E_m = M_d^+(P_m(A-B_\infty)P_m) \oplus M_d^-(P_m(A-B_\infty)P_m) \oplus M_{2\varepsilon_m}^0(P_m(A-B_\infty)P_m),$$

where ε_m is given by (2.3) for $B = B_\infty$, and $\varepsilon_m \to 0$ as $m \to \infty$. Denote

$$\begin{aligned} U_m^\pm &= \{y^\pm \in M_d^\pm(P_m(A-B_\infty)P_m) : \|y^\pm\| \leq R_\pm\}, \\ W_\infty &= \{x \in M_{2\varepsilon_m}^0(P_m(A-B_\infty)P_m) : \|x\| \leq R_w\}. \end{aligned}$$

According to the proof of Theorem 1.1, all we need to do is to show that there are $R_+, R_-, R_w > 0$, which do not depend on m, such that $D_{\infty m} = U_m^+ \times U_m^- \times W_\infty$ is an isolating block of the gradient η_m generated by

$$\frac{dz}{dt} = -P_m(A-B_\infty)P_m z + P_m g'_\infty(z) \quad \text{on } E_m.$$

In the following, $\{a_j\}$ are suitable positive constants independent of m. By (H4$^\pm$), we have

$$\|g'_\infty\| \leq a_1\|z\|^{\beta_\infty} \quad \text{for } \|z\| \geq L_\infty. \tag{3.58}$$

For $z = z^+ + z^- + z^0 \in \partial U_m^+ \times U_m^- \times W_\infty$,

$$\begin{aligned} \left\langle \frac{dz}{dt}, z^+ \right\rangle\bigg|_{t=0} &= -\langle P_m(A-B_\infty)P_m z^+, z^+\rangle + \langle g'_m(z), z^+\rangle \\ &\leq -d\|z^+\|^2 + a_1\|z\|^{\beta_\infty}\|z^+\| \leq -dR_+^2 + 3^{\beta_\infty}a_1 R_w^{\beta_\infty}R_+ \\ &= -dR_+(R_+ - a_2R_w^{\beta_\infty}) \leq -dR_+^2/2 < 0, \end{aligned}$$

provided

$$L_\infty \leq R_+ = R_- \leq R_w, \quad R_+ = 2a_2R_w^{\beta_\infty}. \tag{3.59}$$

Similarly, for $z = z^+ + z^- + z^0 \in U_m^+ \times \partial U_m^- \times W_\infty$,

$$\left\langle \frac{dz}{dt}, z^- \right\rangle\bigg|_{t=0} \geq d\|z^-\|^2 - a_1\|z\|^{\beta_\infty}\|z^-\| \geq dR_-^2/2 > 0$$

provided (3.59) holds.

Case 1. (H4$^+$) holds. Then we have

$$\langle G'_\infty(t,z), z\rangle \geq a_3|z|^{\alpha_\infty} - a_4 \quad \text{for all } (t,z) \in [0,1] \times \mathbb{R}^{2N}. \tag{3.60}$$

For $z = z^+ + z^- + z^0 \in U_m^+ \times U_m^- \times \partial W_\infty$, (3.33)–(3.39) still hold. Recall that

$$\Delta = \left\{ t \in [0,1] : \frac{\lambda_3}{4}\|z^0\| \le |z^0(t)| \le 4\lambda_4\|z^0\| \right\}, \quad \Delta^\perp = [0,1] \setminus \Delta.$$

By (3.37)–(3.39), there exists $m_1^* \ge m^*$ such that for $m \ge m_1^*$,

$$\mathrm{meas}(\Delta^\perp) \le (b_4 + b_5)\varepsilon_m, \quad \mathrm{meas}(\Delta) \ge 1/2.$$

Notice that (3.59) implies $\|z^+ + z^-\| \le a_5\|z^0\|^{\beta_\infty}$. We have

$$\begin{aligned}\int_\Delta (z, z^0) dt &\ge \int_\Delta |z^0|^2 dt - \|z^+ + z^-\|\|z^0\| \\ &\ge (\lambda_3/4)\|z^0\|^2 \mathrm{meas}(\Delta) - a_6\|z^0\|^{\beta_\infty + 1} \\ &\ge (\lambda_3/8)\|z^0\|^2 - a_6\|z^0\|^{\beta_\infty+1} \ge (\lambda_3/16)\|z^0\|^2,\end{aligned}$$

provided

$$R_w \ge (16a_6/\lambda_3)R_w^{\beta_\infty}. \tag{3.61}$$

On the other hand, if $\alpha_\infty > 1$,

$$\int_\Delta (z, z^0)\, dt \le a_7 \left(\int_\Delta |z|^{\alpha_\infty}\, dt \right)^{1/\alpha_\infty} \|z^0\|.$$

If $\alpha_\infty = 1$, we have

$$\int_\Delta (z, z^0)\, dt \le \left(\int_\Delta |z|\, dt \right) 4\lambda_4 \|z^0\|.$$

Therefore for $\alpha_\infty \ge 1$,

$$\int_\Delta |z|^{\alpha_\infty}\, dt \ge a_8\|z^0\|^{\alpha_\infty}. \tag{3.62}$$

Now, by (3.58)–(3.62), we have

$$\begin{aligned}\left\langle \frac{dz}{dt}, z^0 \right\rangle\Big|_{t=0} &= \int_0^1 (G'_\infty(t,z), z^0)\, dt - \langle P_m(A - B_\infty)P_m z^0, z^0\rangle \\ &\ge \int_0^1 a_3|z|^{\alpha_\infty}\, dt - a_4 - a_1\|z\|^{\beta_\infty}\|z^+ + z^-\| - 2\varepsilon_m\|z^0\|^2 \\ &\ge \int_\Delta a_3|z|^{\alpha_\infty}\, dt + \int_{\Delta^\perp} a_3|z|^{\alpha_\infty}\, dt - a_4 - a_9R_w^{2\beta_\infty} - 2\varepsilon_m R_w^2 \\ &\ge a_3a_8\|z^0\|^{\alpha_\infty} - a_{10}\|z\|^{\alpha_\infty}(\mathrm{meas}\Delta^\perp)^{1/2} \\ &\quad - a_4 - a_9R_w^{2\beta_\infty} - 2\varepsilon_m R_w^2 \\ &\ge a_3a_8R_w^{\alpha_\infty} - a_9R_w^{2\beta_\infty} - a_4 - a_{11}R_w^{\alpha_\infty}\varepsilon_m^{1/2} - 2\varepsilon_m R_w^2 \\ &\ge (a_3a_8/2)R_w^{\alpha_\infty} - a_{11}R_w^{\alpha_\infty}\varepsilon_m^{1/2} - 2\varepsilon_m R_w^2,\end{aligned} \tag{3.63}$$

provided

$$a_3a_8R_w^{\alpha_\infty} \ge 2(a_9R_w^{2\beta_\infty} + a_4). \tag{3.64}$$

Since $\beta_\infty < 1$ and $\alpha_\infty > 2\beta_\infty$, there exist R_+,R_- and R_w such that (3.59), (3.61) and (3.64) hold. Moreover, R_+,R_- and R_w are independent of m. Since $\varepsilon_m \to 0$ as $m \to \infty$, by (3.63), there exists $m_2^* \geq m_1^*$ such that for $m \geq m_2^*$,

$$\left\langle \frac{dz}{dt}, z^0 \right\rangle \bigg|_{t=0} \geq \left(\frac{a_3 a_8}{4}\right) R_w^{\alpha_\infty} > 0.$$

This implies that $D_{\infty m} = U_m^+ \times U_m^- \times W_\infty$ is an isolating block with

$$D_{\infty m}^- = U_m^+ \times \partial U_m^- \times W_\infty \cup U_m^+ \times U_m^- \times \partial W_\infty.$$

Moreover, by Theorem 3.1, we have

$$I(\eta_m, D_{\infty m}) = t^{\dim(M_d^-(P_m(A-B_\infty)P_m) \oplus M_{2\varepsilon_m}^0(P_m(A-B_\infty)P_m))} = t^{m-i_0+i_\infty+n_\infty}.$$

Case 2. (H4$^-$) holds. Similar to Case 1, we can choose R_+,R_- and R_w independent of m such that $D_{\infty m} = U_m^+ \times U_m^- \times W_\infty$ is an isolating block with

$$\begin{aligned} D_{\infty m}^- &= U_m^+ \times \partial U_m^- \times W_\infty, \\ I(\eta_m, D_{\infty m}) &= t^{\dim(M_d^-(P_m(A-B)P_m)} = t^{m-i_0+i_\infty}. \end{aligned}$$

We omit the details of this part. The proof is complete. □

Example 3.6. Let $H(t,z) \in C^2([0,1] \times \mathbb{R}^{2N}, \mathbb{R})$ be 1-periodic in t such that

$$\begin{aligned} H(t,z) &= -1 + \pi(z,z) + |z|^4 && \text{for } |z| \leq 1, \\ H(t,z) &= \pi - 2\arctan(|z|^2) - \pi(z,z) && \text{for } |z| \geq 100. \end{aligned}$$

Then $B_0(t) = 2\pi I_{2N}, B_\infty(t) = -2\pi I_{2N}$ and

$$G_\infty(t,z) = \pi - 2\arctan(|z|^2), \quad G_0(t,z) = -1 + |z|^4.$$

By direct computation, we have the Maslov-type indices

$$(i_0, n_0) = (N, 2N), \quad (i_\infty, n_\infty) = (-N, 2N).$$

(a) It is easy to see that the Palais–Smale condition fails at level $c = 0$.

(b) One can easily verify that the strong resonance condition holds, i.e.

$$G_\infty(t,z) \to 0, \quad G_\infty'(t,z) \to 0 \quad \text{as } |z| \to \infty.$$

Notice that $\int_0^1 H(t,0)\,dt < 0$ and

$$i_\infty + n_\infty = N \in [N, 3N] = [i_0, i_0 + n_0].$$

If we apply the strong resonance results [8], [14] here, we can not get any conclusion.

(c) By direct computation, $G_0(t,z)$ satisfies (H2$^+$) and $G_\infty(t,z)$ satisfies (H3$^-$).

$$i_0 + n_0 - i_\infty = 4N > 2N + 1.$$

Theorem 1.1(f) implies that the system (1.1) possesses at least two nontrivial 1-periodic solutions. It seems that this example can not be solved by previous results in the references.

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