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IMPULSIVE HYPERBOLIC DIFFERENTIAL INCLUSIONS WITH VARIABLE TIMES

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Abstract. In this paper the nonlinear alternative of Leray–Schauder type is used to investigate the existence of solutions for second order impulsive hyperbolic differential inclusions with variable times.

1. Introduction

In this paper, we shall be concerned with the existence of solutions for the following second order impulsive hyperbolic differential inclusions with variable times:

$$
\begin{aligned}
&(1.1) \qquad \frac{\partial^2 u(t,x)}{\partial t \partial x} \in F(t,x,u(t,x)), && \text{a.e. } (t,x) \in J_a \times J_b, \\
& && t \neq \tau_k(u(t,x)),\ k=1,\dots,m, \\
&(1.2) \qquad u(t^+,x) = I_k(u(t,x)), && t = \tau_k(u(t,x)),\ k=1,\dots,m, \\
&(1.3) \qquad u(t,0) = \psi(t), && t \in J_a,\ u(0,x) = \phi(x),\ x \in J_b,
\end{aligned}
$$

where $F\colon J_a \times J_b \times \mathbb{R}^n \to P(\mathbb{R}^n)$ is a multivalued map with compact values, $J := J_a \times J_b := [0,a] \times [0,b]$, $I_k \in C^1(\mathbb{R}^n,\mathbb{R}^n)$, $\phi \in C(J_a,\mathbb{R}^n)$, $u(t^+,y) = \lim_{(h,x)\to(0^+,y)} u(t+h,x)$ and $u(t^-,y) = \lim_{(h,x)\to(0^-,y)} u(t-h,x)$ represent the

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right and left limits of $u(t,x)$ at (t,x), respectively and $\mathbb{R}^n$ a Euclidean space with norm $|\cdot|$.

Impulsive differential and partial differential equations with fixed moments have become more important in recent years in some mathematical models of real phenomena, especially in control, biological or medical domains, see the mongraphs of Lakshmikantham et al ([12]), Samoilenko and Perestyuk ([16]), and the papers of Bainov et al ([2]), Kirane and Rogovchenko ([11]), Liu ([14]) and Liu and Zhang ([15]). However the theory of impulsive partial differential equations with variable time is relatively less developed due to the difficulties created by the state-dependent impulses.

Very recently, by means of a Martelli's fixed point theorem for condensing multivalued maps, a particular case ($I_k = 0$, $k = 1, \ldots, m$) of the problem (1.1)–(1.3) was studied by Benchohra in [3]. Let us mention that that with the aid of the Leray–Schauder nonlinear alternative ([6]), the problem (1.1)–(1.3) was considered by the authors (see [4]) in the case where the instant of impulses are fixed. Hence the present result is an extension of the problem to variable moments. Our proof is based also on the nonlinear alternative. It can also be considered as a contribution to the title literature.

2. Preliminaries

We will briefly recall some basic definitions and facts from multivalued analysis that we will use in the sequel.

$C(J_a \times J_b, \mathbb{R}^n)$ is the Banach space of all continuous functions from $J_a \times J_b$ into $\mathbb{R}^n$ with the norm

$$\|u\|_\infty = \sup\{|u(t,s)| : (t,s) \in J_a \times J_b\}.$$

A measurable function $z\colon J_a \times J_b \to \mathbb{R}^n$ is integrable if and only if z is Lebesgue integrable.

$L^1(J_a \times J_b, \mathbb{R}^n)$ denotes the Banach space of functions $z\colon J_a \times J_b \to \mathbb{R}^n$ which are Lebesgue integrable normed by

$$\|z\|_{L^1} = \int_0^a \int_0^b |z(t,s)|\,dt\,ds.$$

Let $(X, \|\cdot\|)$ be a normed space and

$$\begin{aligned}
P_{\mathrm{cl}}(X) &= \{Y \in P(X) : Y \text{ closed}\},\\
P_{\mathrm{b}}(X) &= \{Y \in P(X) : Y \text{ bounded}\},\\
P_{\mathrm{cp}}(X) &= \{Y \in P(X) : Y \text{ compact}\},\\
P_{\mathrm{cp,c}}(X) &= \{Y \in P(X) : Y \text{ compact, convex}\}.
\end{aligned}$$

A multivalued map $G\colon X \to P(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$.

G is bounded on bounded sets if $G(\mathcal{B}) = \bigcup_{x\in\mathcal{B}} G(x)$ is bounded in X for all $\mathcal{B} \in P_b(X)$ (i.e. $\sup_{x\in\mathcal{B}}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

G is called *upper semi-continuous* (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set $\mathcal{U}$ of X containing $G(x_0)$, there exists an open neighbourhood $\mathcal{V}$ of x_0 such that $G(\mathcal{V}) \subseteq \mathcal{U}$.

G is said to be *completely continuous* if $G(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in P_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $\operatorname{Fix} G$.

A multivalued map $N\colon J_a \times J_b \times \mathbb{R}^n \to P_{\rm cl}(\mathbb{R}^n)$ is said to be *measurable*, if for every $w \in \mathbb{R}^n$, the function $t \mapsto d(w, N(t,x,u)) = \inf\{\|w - v\| : v \in N(t,x,u)\}$ is measurable where d is the distance induced from the normed space $\mathbb{R}^n$. For more details on multivalued maps see the books of Aubin and Cellina ([1]), Deimling ([5]), Górniewicz ([8]) and Hu and Papageorgiou ([10]).

Definition 2.1. The multivalued map $F\colon J_a \times J_b \times \mathbb{R}^n \to P(\mathbb{R}^n)$ is said to be an L^1-*Carathéodory* if

(a) $(t,x) \mapsto F(t,x,u)$ is measurable for each $u \in \mathbb{R}^n$,
(b) $u \mapsto F(t,x,u)$ is upper semicontinuous for almost all $(t,x) \in J_a \times J_b$,
(c) for each $r > 0$, there exists $\varphi_r \in L^1(J_a \times J_b, \mathbb{R}_+)$ such that

$$\|F(t,x,u)\| = \sup\{|v| : v \in F(t,x,u)\} \le \varphi_r(t,x)$$

for all $|u| \le r$ and for a.e. $(t,x) \in J_a \times J_b$.

For each $u \in C(J_a \times J_b, \mathbb{R}^n)$, define the set of selections of F by

$$S_{F,u} = \{v \in L^1(J_a \times J_b, \mathbb{R}^n) : v(t,s) \in F(t,x,u(t,x)) \text{ a.e. } t \in J_a,\ x \in J_b\}.$$

Lemma 2.2 ([13]). *Let X be a Banach space. Let $F\colon J_a \times J_b \times X \to P_{\rm cp,c}(X)$ be an L^1-Carathéodory multivalued map with $S_F \neq \emptyset$ and let Ψ be a linear continuous mapping from $L^1(J_a \times J_b, X)$ to $C(J \times J_b, X)$, then the operator*

$$\Psi \circ S_F\colon C(J_a \times J_b, X) \to P_{cp,c}(C(J_a \times J_b, X)),$$
$$u \mapsto (\Psi \circ S_F)(u) := \Psi(S_{F,u})$$

is a closed graph operator in $C(J_a \times J_b, X) \times C(J_a \times J_b, X)$.

Lemma 2.3 ([6]). *Let X be a Banach space with $C \subset X$ a convex. Assume U is a relatively open subset of C with $0 \in U$ and $G: X \to P_{\mathrm{cp,c}}(X)$ be an upper semi-continuous and compact map. Then either*

(a) *there is a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda G(u)$ or*
(b) *G has a fixed point in $\overline{U}$.*

Remark 2.4. By $\overline{U}$ and ∂U we denote the closure of U and the boundary of U, respectively.

3. Main result

In this section we are concerned with the existence of solutions for problem (1.1)–(1.3). In order to define the solution of (1.1)–(1.3) we shall consider the following space

$$\begin{aligned}\Omega = \{u: J_a \times J_b \to \mathbb{R}^n : &\text{ there exist } 0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = a \\ &\text{such that } t_k = \tau_k(u(t_k, \cdot\,)) \text{ and } u_k \in C(\Gamma_k, \mathbb{R}^n), k = 0, \ldots, m \\ &\text{and there exist } u(t_k^-, \cdot\,), \text{ and } u(t_k^+, \cdot\,),\ k = 1, \ldots, m \\ &\text{with } u(t_k^-, \cdot\,) = u(t_k, \cdot\,)\}\end{aligned}$$

which is a Banach space with the norm

$$\|u\|_\Omega = \max\{\|u_k\|, k = 0, \ldots, m\},$$

where u_k is the restriction of u to $\Gamma_k = (t_k, t_{k+1}) \times J_b$, $k = 0, \ldots, m$. So let us start by defining what we mean by a solution of problem (1.1)–(1.3).

Definition 3.1. A function $u \in \Omega \cap \bigcup_{k=1}^m A^1(\Gamma_k, \mathbb{R}^n)$ is said to be a *solution* of (1.1)–(1.3) if there exist $v \in L^1(J_a \times J_b)$ such that $v(t,x) \in F(t,x,u(t,x))$ satisfied a.e. on $J_a \times J_b$, $\partial^2 u(t,x)/\partial t \partial x = v(t,x)$ a.e. on $J_a \times J_b$, and the conditions (1.2)–(1.3).

Let us introduce the following hypotheses:

(H1) There exist constants c_k such that $|I_k(u)| \leq c_k$, $k = 1, \ldots, m$ for each $u \in \mathbb{R}^n$.
(H2) There exist functions $p, q \in L^1(J_a \times J_b, \mathbb{R}_+)$ such that

$$\|F(t,x,u)\| \leq p(t,x) + q(t,x)|u|$$

for a.e. $(t,x) \in J_a \times J_b$ and each $u \in \mathbb{R}^n$.
(H3) The functions $\tau_k \in C^1(\mathbb{R}^n, \mathbb{R})$ for $k = 1, \ldots, m$. Moreover,

$$0 < \tau_1(x) < \ldots < \tau_m(x) < a \quad \text{for all } x \in \mathbb{R}^n.$$

(H4) For all $u \in C(J_a \times J_b, \mathbb{R}^n)$ and all $v \in S_{F,u}$ we have

$$\left\langle \tau_k'(x), \int_{\overline{t}}^t v(s,x)\,ds \right\rangle \neq 1$$

for all $(t, \overline{t}, x) \in J_a \times J_a \times \mathbb{R}^n$ and $k = 0, \dots, m$, where $\langle \cdot\,, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$.

(H5) For all $x \in \mathbb{R}^n$

$$\tau_k(I_k(x)) \leq \tau_k(x) < \tau_{k+1}(I_k(x)) \quad \text{for } k = 1, \dots, m.$$

THEOREM 3.2. *Assume that the hypotheses* (H1)–(H5) *are satisfied. Then the* IVP (1.1)–(1.3) *has at least one solution.*

PROOF. The proof will be given in several steps.

Step 1. Consider the following problem

$$\frac{\partial^2 u(t,x)}{\partial t \partial x} \in F(t,x,u(t,x)), \quad \text{a.e. } (t,x) \in J_a \times J_b \tag{3.1}$$

$$u(t,0) = \psi(t), \quad t \in J_a, \quad u(0,x) = \phi(x), \quad x \in J_b. \tag{3.2}$$

A solution to problem (3.1)–(3.2) is a fixed point of the operator

$$N\colon C(J_a \times J_b, \mathbb{R}^n) \to P(C(J_a \times J_b, \mathbb{R}^n))$$

defined by:

$$N(u) = \left\{ h \in C(J_a \times J_b, \mathbb{R}^n) : h(t,x) = z_0(t,x) + \int_0^t \int_0^x v(s,y)\,ds\,dy,\ v \in S_{F,u} \right\},$$

where $z_0(t,x) := \psi(t) + \phi(x) - \psi(0)$. The proof will be given in several claims.

CLAIM 1. *$N(u)$ is convex for each $u \in \Omega$.*

Indeed, if h_1, h_2 belong to $N(u)$, then there exist $v_1, v_2 \in S_{F,u}$ such that for each $(t,x) \in J_a \times J_b$ we have

$$h_i(t,x) = z_0(t,x) + \int_0^t \int_0^x v_i(s,y)\,ds\,dy, \quad i = 1,2.$$

Let $0 \leq d \leq 1$. Then for each $(t,x) \in J_a \times J_b$ we have

$$(dh_1 + (1-d)h_2)(t) = z_0(t,x) + \int_0^t \int_0^x [dv_1(s,y) + (1-d)v_2(s,y)]\,ds\,dy.$$

Since $S_{F,u}$ is convex (because F has convex values) then

$$dh_1 + (1-d)h_2 \in N(u).$$

CLAIM 2. *N maps bounded sets into bounded sets in $C(J_a \times J_b, \mathbb{R}^n)$.*

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $u \in \mathcal{B}_q = \{u \in C(J_a \times J_b, \mathbb{R}^n) : \|u\|_\infty \leq q\}$ one has $\|N(u)\|_\infty \leq \ell$.

Let $h \in N(u)$ then there exist $v \in S_{F,u}$ such that

$$h(t,x) = z_0(t,x) + \int_0^t \int_0^x v(s,y)\,ds\,dy.$$

Since F is an L-Carathéodory we have for each $(t,x) \in J_a \times J_b$

$$|h(t,x)| \le |z_0(t,x)| + \int_0^a \int_0^b |\varphi_q(t,x)|\,ds \le \|z_0\|_\infty + \|\varphi_q\|_{L^1} := \ell.$$

Claim 3. *N maps bounded sets into equicontinuous sets of $C(J_a \times J_b, \mathbb{R}^n)$.*

Let $(\overline{t}_1, x_1), (\overline{t}_2, x_2) \in J_a \times J_b$, $\overline{t}_1 < \overline{t}_2$, $x_1 < x_2$ and $\mathcal{B}_q$ be a bounded set of $C(J_a \times J_b, \mathbb{R}^n)$, as in Claim 2. Then

$$\begin{aligned} |h(\overline{t}_2, x_2)) - h(\overline{t}_1, x_1)| \le{}& |z_0(\overline{t}_2, x_2)) - z_0(\overline{t}_1, x_1)| \\ &+ \int_0^{\overline{t}_2} \int_{x_1}^{x_2} \varphi_q(t,s)\,dt\,ds + \int_{\overline{t}_1}^{\overline{t}_2} \int_0^{x_1} \varphi_q(t,s)\,dt\,ds. \end{aligned}$$

The right-hand side tends to zero as $\overline{t}_2 - \overline{t}_1 \to 0$, $x_2 - x_1 \to 0$.

As a consequence of Claims 2 and 3 with the Arzela–Ascoli Theorem we can conclude that $N: C(J_a \times J_b, \mathbb{R}^n) \to C(J_a \times J_b, \mathbb{R}^n)$ is completely continuous.

Claim 4. *N has a closed graph.*

Let $u_n \to u_*$, $h_n \in N(u_n)$ and $h_n \to h_*$. We shall prove that $h_* \in N(u_*)$. $h_n \in N(u_n)$ means that there exists $v_n \in S_{F,u_n}$ such that for each $t \in J$

$$h_n(t,x) = z_0(t,x) + \int_0^t \int_0^x v_n(s,x)\,ds\,dx.$$

We must prove that there exists $v_* \in S_{F,u_*}$ such that for each $(t,x) \in J_a \times J_b$

$$h_*(t,x) = z_0(t,x) + \int_0^t \int_0^x v_*(s,x)\,ds\,dx.$$

Clearly, since ϕ is continuous we have that

$$\|(h_n - z_0(t,x)) - (h_* - z_0(t,x))\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Consider the linear continuous operator

$$\begin{aligned} &\Psi: L^1(J_a \times J_b, \mathbb{R}^n) \to C(J_a \times J_b, \mathbb{R}^n), \\ &v \mapsto \Psi(v)(t,x) = \int_0^t \int_0^x v(s,\tau)\,ds\,d\tau. \end{aligned}$$

From Lemma 2.2, it follows that $\Psi \circ S_F$ is a closed graph operator. Moreover, we have that

$$(h_n(t,x) - z_0(t,x)) \in \Psi(S_{F,u_n}).$$

Since $u_n \to u_*$, it follows from Lemma 2.2 that

$$h_*(t,x) = z_0(t,x) + \int_0^t \int_0^x v_*(s,y)\,ds\,dy$$

for some $v_* \in S_{F,u_*}$.

Claim 5. *A priori bounds on solutions.*

Let $u \in \Omega$ be a possible solution to (3.1)–(3.2). Then there exists $v \in S_{F,u}$ such that for each $(t,x) \in J$

$$u(t,x) = z_0(t,x) + \int_0^t \int_0^x v(s,y)\,ds\,dy.$$

This implies by (H2)–(H4) that for each $(t,x) \in J_a \times J_b$ we have

$$\begin{aligned}|u(t,x)| &\leq \|z_0\|_\infty + \int_0^t \int_0^x [|p(s,\tau)| + |q(s,\tau)||u(s,\tau)|]\,ds\,d\tau \\ &\leq \|z_0\|_\infty + \int_0^t \int_0^x |q(s,\tau)||u(s,\tau)|\,ds\,d\tau + \|p\|_{L^1}.\end{aligned}$$

Invoking Gronwall's inequality (see for instance [9]) we get that

$$|u(t,x)| \leq [\|z_0\|_\infty + \|p\|_{L^1}] exp(\|q\|_{L^1}) := M.$$

Then $\|u\|_\Omega < M$. Set

$$U_1 = \{u \in C(J_a \times J_b, \mathbb{R}^n) : \|u\|_\infty < M + 1\}.$$

$N \colon \overline{U}_1 \to P(C(J_a \times J_b, \mathbb{R}^n))$ is completely continuous. From the choice of U_1 there is no $u \in \partial U_1$ such that $u \in \lambda N(u)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray Schauder type (see [6]) we deduce that N has a fixed point u in $\overline{U_1}$ which is a solution of (3.1)–(3.2). Denote this solution by u_1.

Define the function $r_{k,1}(t,x) = \tau_k(u_1(t,x)) - t$ for $t \geq 0$. (H3) implies that $r_{k,1}(0,0) \neq 0$ for $k = 1, \dots, m$. If $r_{k,1}(t,x) \neq 0$ on $J_a \times J_b$ for $k = 1, \dots, m$, i.e. $t \neq \tau_k(u_1(t,x))$ on $J_a \times J_b$ and for $k = 1, \dots, m$, then u_1 is a solution of the problem (1.1)–(1.3).

It remains to consider the case when $r_{1,1}(t,x) = 0$ for some $(t,x) \in J_a \times J_b$. Now since $r_{1,1}(0,0) \neq 0$ and $r_{1,1}$ is continuous, there exists $t_1 > 0, x_1 > 0$ such that

$$r_{1,1}(t_1,x_1) = 0 \quad \text{and} \quad r_{1,1}(t,x) \neq 0 \quad \text{for all } (t,x) \in [0,t_1) \times [0,x_1].$$

Thus by (H4) we have

$$r_{1,1}(t_1,x_1) = 0 \quad \text{and} \quad r_{1,1}(t,x) \neq 0 \quad \text{for all } (t,x) \in [0,t_1) \times [0,x_1] \cup (x_1,b].$$

Suppose that there exist $(\bar{t},\overline{x}) \in [0,t_1) \times [0,x_1) \cup (x_1,b]$ such that $r_{1,1}(\bar{t},\overline{x}) = 0$. The function $r_{1,1}$ attains a maximum at some point $(s,\overline{s}) \in [0,t_1] \times J_b$. Since

$$\frac{\partial^2 u(t,x)}{\partial t \partial x} \in F(t,x,u_1(t,x)), \quad \text{a.e. } (t,x) \in J_a \times J_b$$

then there exist $v(\,\cdot\,,\,\cdot\,) \in L^1(J_a \times J_b)$ with $v(t,x) \in F(t,x,u_1(t,x))$, a.e. $(t,x) \in J_a \times J_b$ such that

$$\frac{\partial^2 u(t,x)}{\partial t \partial x} = v(t,x) \quad \text{a.e. } t \in J_a \times J_b.$$

$\partial u_1(t,x)/\partial t$ and $\partial u_1(t,x)/\partial x$ exist. Then

$$\frac{\partial r_{1,1}(s,\overline{s})}{\partial t} = \tau_1'(u_1(s,\overline{s}))\frac{\partial u_1(s,\overline{s})}{\partial t} - 1 = 0.$$

Since

$$\frac{\partial u_1(t,x)}{\partial t} = \int_0^t v(s,x,u_1(s,x))\,ds,$$

then

$$\tau_1'(u_1(s,\overline{s}))\int_0^s v(\tau,\overline{s})\,d\tau - 1 = 0.$$

Therefore

$$\left\langle \tau_1'(u_1(s,\overline{s})), \int_0^s v(\tau,\overline{s})\,d\tau \right\rangle = 1,$$

which contradicts (H4). From (H3) we have

$$r_{k,1}(t,x) \neq 0 \quad \text{for all } t \in [0,t_1) \times J_b \text{ and } k = 1,\ldots,m.$$

Step 2. Consider now the following problem

$$\frac{\partial^2 u(t,x)}{\partial t \partial x} \in F(t,x,u(t,x)), \quad \text{a.e. } t \in [t_1,a] \times J_b, \tag{3.3}$$

$$u(t_1^+,x) = I_1(u_1(t_1,x)). \tag{3.4}$$

Transform the problem (3.3)–(3.4) into a fixed point problem. Consider the operator $N_1\colon C([t_1,a] \times J_b,\mathbb{R}^n) \to C([t_1,a] \times J_b,\mathbb{R}^n)$ defined by

$$N_1(u) = \left\{ h \in C([t_1,a] \times J_b,\mathbb{R}^n) : \\ h(t,x) = I_1(u_1(t_1,x)) + \int_0^x v(s,y)\,ds\,dy,\ v \in S_{F,u} \right\}.$$

As in Step 1 we can show that N_1 is completely continuous, and each possible solution of (3.3)–(3.4) is a priori bounded by constant M_2. Set

$$U_2 := \{u \in C([t_1,a] \times J_b,\mathbb{R}^n) : \|u\|_\infty < M_2 + 1\}.$$

From the choice of U_2 there is no $u \in \partial U_2$ such that $u = \lambda N_1(u)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray–Schauder type (see [6]) we deduce that N_1 has a fixed point u in U_2 which is a solution of (3.3)–(3.4). Denote this solution by u_2. Define

$$r_{k,2}(t,x) = \tau_k(u_2(t,x)) - t \quad \text{for } (t,x) \in [t_1, a] \times J_b.$$

If $r_{k,2}(t,x) \neq 0$ on $(t_1, a] \times J_b$ and for all $k = 1, \dots, m$ then

$$u(t,x) = \begin{cases} u_1(t,x) & \text{if } (t,x) \in [0, t_1) \times J_b, \\ u_2(t,x) & \text{if } (t,x) \in [t_1, a] \times J_b, \end{cases}$$

is a solution of the problem (1.1)–(1.3). It remains to consider the case when $r_{2,2}(t,x) = 0$, for some $(t,x) \in (t_1, a] \times J_b$. By (H5) we have

$$\begin{aligned} r_{2,2}(t_1^+, x_1) &= \tau_2(u_2(t_1^+, x_1)) - t_1 = \tau_2(I_1(u_1(t_1, x_1)) - t_1 \\ &> \tau_1(u_1(t_1, x_1)) - t_1 = r_{1,1}(t_1, x_1) = 0. \end{aligned}$$

Since $r_{2,2}$ is continuous and by (H3) there exists $t_2 > t_1, x_2 > x_1$ such that

$$r_{2,2}(u_2(t_2, x_2) = 0 \quad \text{and} \quad r_{2,2}(t,x) \neq 0 \quad \text{for all } (t,x) \in (t_1, t_2) \times J_b.$$

It is clear by (H3) that

$$r_{k,2}(t,x) \neq 0 \quad \text{for all } (t,x) \in (t_1, t_2) \times J_b,\ k = 2, \dots, m.$$

Suppose now that there is $(s, \overline{s}) \in (t_1, t_2] \times [0, x_2) \cup (x_2, b]$ such that

$$r_{1,2}(s, \overline{s}) = 0.$$

From (H5) it follows that

$$\begin{aligned} r_{1,2}(t_1^+, x_1)) &= \tau_1(u_2(t_1^+, x_1)) - t_1 = \tau_1(I_1(u_1(t_1, x_1)) - t_1 \\ &\leq \tau_1(u_1(t_1, x_1) - t_1 = r_{1,1}(t_1, x_1) = 0. \end{aligned}$$

Thus the function $r_{1,2}$ attains a nonnegative maximum at some point $(s_1, \overline{s}_1) \in (t_1, a] \times [0, x_2) \cup (x_2, b]$. Since

$$\frac{\partial^2 u_2(t,x)}{\partial t \partial x} \in F(t, x, u_2(t,x))$$

then there exist $v(t,x) \in F(t, x, u_2(t,x))$ a.e. $(t,x) \in [t_1, a] \times J_b$ such that

$$\frac{\partial^2 u_2(t,x)}{\partial t \partial x} = v(t,x), \quad (t,x) \in [t_1, a] \times J_b.$$

Then we have

$$r'_{1,2}(t,x) = \tau'_1(u_2(t,x)) \frac{\partial u_2(t,x)}{\partial t} - 1 = 0.$$

Therefore

$$\left\langle \tau'_1(u_2(s_1, \overline{s}_1)), \int_{t_1}^{s_1} v(s, \overline{s}_1)\, ds \right\rangle = 1,$$

which contradicts (H4).

Step 3. We continue this process and taking into account that $u_m := y|_{[t_m,a]\times J_b}$ is a solution to the problem

$$\frac{\partial^2 u(t,x)}{\partial t\partial x} \in F(t,x,u(t,x)), \quad \text{a.e. } t\in (t_m,a]\times(0,b], \tag{3.5}$$

$$u(t_m^+,x) = I_m(u_{m-1}(t_m^-,x)). \tag{3.6}$$

The solution u of the problem (1.1)–(1.3) is then defined by

$$u(t,x) = \begin{cases} u_1(t,x) & \text{if } t\in[0,t_1)\times J_b, \\ u_2(t,x) & \text{if } t\in[t_1,t_2)\times J_b, \\ \dots\dots\dots\dots\dots\dots \\ u_m(t,x) & \text{if } t\in[t_m,a]\times J_b. \end{cases}$$

$\square$

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