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CONFIGURATION SPACES ON PUNCTURED MANIFOLDS

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Dedicated to our colleague and good friend Andrzej Granas

Abstract. The object here is to study the following question in the homotopy theory of configuration spaces of a general manifold M: When is the fibration $\mathbb{F}_{k+1}(M) \to \mathbb{F}_r(M)$, $r < k+1$, fiber homotopically trivial? The answer to this question for the special cases when M is a sphere or euclidean space is given in [4]. The key to the solution of the problem for compact manifolds M is the study of an associated question for the punctured manifold $M-q$, where q is a point of M. The fact that $M-q$ admits a nonzero vector field plays a crucial role. Also required are investigations into the Lie algebra $\pi_*(\mathbb{F}_{k+1}(M))$, with special attention to the punctured case $\pi_*(\mathbb{F}_k(M-q))$. This includes the so-called Yang–Baxter equations in homotopy, taking into account the homotopy group elements of M itself as well as the classical braid elements.

1. Introduction

Let M be a smooth simply connected manifold of dimension $n+1$ and denote by $\mathbb{F}_{k+1}(M)$ the configuration space of $(k+1)$-tuples in M. Recall that

$$\mathbb{F}_{k+1}(M) = \{(x_1, \dots , x_{k+1}) \mid x_i \neq x_j\} \subset M^{\times(k+1)}.$$

Configuration spaces play a crucial role in Analysis, primarily in problems of "$(k+1)$-body type" ([3], [6], [7]). In the Fadell–Husseini monograph ([4]), we studied the homotopy and homology theory of the special cases $\mathbb{F}_{k+1}(M)$ where M is a (punctured) euclidean space or a sphere. Our recent studies of configuration spaces of general manifolds indicate that knowledge of $\mathbb{F}_{k+1}(M)$

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relies heavily upon the punctured configuration space $\mathbb{F}_{k+1}(M - q_1)$ where q_1 is a point in M. For example, suppose that $Q_r = \{q_1, \dots, q_r\}$, $r \geq 1$, is a set of r distinct points in M. Then, because $M - Q_1$ admits a non-zero vector field it follows readily that

$$\pi_*(\mathbb{F}_k(M - Q_1)) \cong \bigoplus_{r=1}^{k} \pi_*(M_r)$$

where M_r stands for $M - Q_r$. Then the fibration

$$\mathbb{F}_k(M_1) \to \mathbb{F}_{k+1}(M) \to M$$

illustrates the dependence of $\mathbb{F}_{k+1}(M)$ on $\mathbb{F}_k(M_1)$.

In this note we study, after some preliminaries, the graded Lie algebra $\pi_*(\mathbb{F}_{k+1}(M))$ with special attention to $\pi_*(\mathbb{F}_k(M_1))$. This includes the so-called Yang–Baxter equations in homotopy, taking into account the homotopy group elements of M itself as well as the classical braid elements. As an application, we consider the question: When is the fibration

$$\mathbb{F}_{k-r}(M_{r+1}) \to \mathbb{F}_k(M_1) \to \mathbb{F}_r(M_1), \quad r < k,$$

fiber homotopically trivial? We then employ the results to answer the same question for the fibration

$$\mathbb{F}_{k-r+1}(M_1) \to \mathbb{F}_{k+1}(M) \to \mathbb{F}_r(M), \quad r < k+1.$$

2. Preliminaries

Our general assumption throughout (unless otherwise indicated) will be that the manifold M is smooth, simply connected (and connected) and of dimension $n+1 \geq 3$. However, most of the results will only require topological manifolds, but assuming smoothness simplifies the exposition. The case of dimension 2, i.e. surfaces, may also be studied by the methods in this note and in the Fadell–Husseini monograph ([4]), but also requires special attention because of the lack of simple connectivity and will appear in another work ([5]).

Let D denote a closed $(n+1)$-ball in M and denote by V its interior. Identify V with euclidean space $\mathbb{R}^{n+1}$. Keeping the notation from [4], let e denote the unit vector $(1, 0, \dots, 0) \in \mathbb{R}^{n+1} = V$, put

$$q_1 = (0, \dots, 0), \quad q_i = q_1 + 4(i-1)e, \quad \text{for } 1 \leq i \leq k+1,$$

and let

$$Q_i = \{q_1, \dots, q_i\}, \quad i \geq 1, \quad Q_0 = \emptyset, \quad \text{for } 1 \leq i \leq k.$$

For $1 \leq s \neq r \leq k+1$, define $\alpha'_{rs}\colon S^n \to \mathbb{F}_{k+1}(M)$ to be the map

$$\xi \in S^n \mapsto (q_1, \dots, q_{r-1}, q_s + \xi, q_r, \dots, q_{k-1}, q_k),$$

denote by S_{rs} the image of α'_{rs}, and by $\alpha_{rs} \in \pi_n(\mathbb{F}_{k+1}(M)$ its homotopy class. Note here that $S_{rs} \subset \mathbb{F}_{k+1}(M)$. Define

$$\mathbb{F}_{k+1-r,r}(M) = \{(x_1, \dots, x_{k+1}) \mid x_i = q_i, \text{ for all } i \le r\}.$$

which we may identify with $\mathbb{F}_{k+1-r}(M - Q_r)$. Furthermore, note that $S_{rs} \subset \mathbb{F}_{k+1-t}(M_t)$, $t \le r-1$, where $M_t = M - Q_t$. Consider next the fundamental fiber sequence diagram

$$\begin{array}{rccccccccc}
 & \mathbb{F}_{k+1,0} & \longleftarrow & \dots & \longleftarrow & \mathbb{F}_{k+1-r,r} & \longleftarrow & \mathbb{F}_{k-r,r+1} & \longleftarrow & \dots \\
\mathcal{F}_{k+1}(M): & \downarrow & & & & \downarrow & & \downarrow & & \\
 & M_0 & & \dots & & M_r & & M_{r+1} & & \dots
\end{array}$$

with $0 \le r < k$, where the vertical maps $p_r\colon \mathbb{F}_{k+1-r,r}(M) \to M_r$, $r \ge 1$ are the projections such that $(q_1, \dots, q_r, x_{r+1}, \dots, x_{k+1}) \mapsto x_{r+1}$. The vertical maps are fibrations. Those after the first stage admit sections using the fact that open manifolds admit non-zero vector fields. The last term in the sequence is the single space $M_k = M - Q_k$.

3. The spaces M_r

We will need to identify the homotopy type of the the punctured space $M_{r+1} = M - Q_{r+1}$. When $r = 0$ it clear that M_1 has the homotopy type of $M - V$. For $r + 1 \ge 2$ we have the following proposition.

PROPOSITION 3.1. *Let M be a manifold of dimension $n+1 \ge 2$, and $Q \subset M$ a discrete subset of $r+1$ elements such that $Q \subset V \subset D \subset M$. Then there is a homotopy equivalence*

$$(M - V) \vee (S_1 \vee \dots \vee S_r) \to (M - Q),$$

where $S_1, \dots, S_r$ are n-dimensional spheres.

PROOF. We give only a sketch of the proof. We may assume that one of the points $q \in Q$ is at the center of the ball D and the remaining r points are in the annular region between the ball D' of radius $1/2$ and the boundary ∂D (see Figure 1). Then if $Q' = Q - q$ put

$$X_r = D - \operatorname{int} D' - Q'.$$

It is easy to see that a deformation of the region A in Figure 1 induces a deformation retraction of X_r onto the subspace ∂D union the boundaries of the balls B_j in Figure 1. Furthermore, since the latter deformation is fixed on ∂D, it extends to a deformation of $(M - Q)$ onto the subspace $(M - V) \cup \partial B_1 \cup \dots \cup \partial B_r$. $\square$

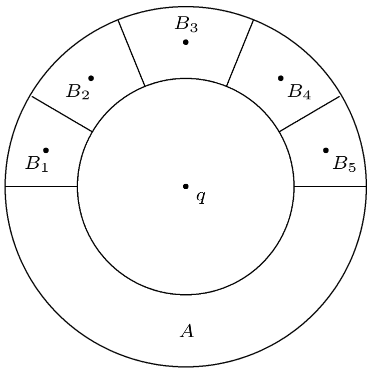

Figure 1. Deformation of X_r

Corollary 3.1. *Assume that M is simply connected and $n > 1$. Then the space $\mathbb{F}_{k+1-r,r}(M)$ is simply connected, where $0 \leq r \leq k$.*

Proof. First one proves that each M_r is simply connected for all $r \geq 1$: as $n+1 \geq 3$, this follows easily for dimensional reasons. Next one proves the desired assertion, using the long exact sequences of the fibrations of diagram $\mathcal{F}_k(M)$. $\square$

In terms of the spheres S_{rs} associated with the braid elements α_{rs}, Proposition 3.1 takes the following form.

Colloraly 3.2. *For each $r > 1$ there is a homotopy equivalence*

$$M_r \simeq M_1 \vee (S_{r+1,2}) \vee \ldots \vee (S_{r+1,r}).$$

Using the homotopy long exact sequence of a fibration, and by virtue of the existence of sections, we have the following decomposition of $\pi_*(\mathbb{F}_k(M_1))$ in terms of the $\pi_*(M_r)$.

Colloraly 3.3. *Assume that* $\dim M = n + 1 > 2$. *Then there is an isomorphism*

$$\pi_*(\mathbb{F}_k(M_1)) \cong \bigoplus_{r=1}^{k} \pi_*(M_r).$$

One would like to describe $\pi_n(M_r)$ in terms of $\pi_n(S_{r*})$ and $\pi_n(M_1)$, where $S_{r*} = S_{r2} \vee \ldots \vee S^n_{rr-1}$. In order to achieve that we need describe the relative cellular structure of $(M \times S_{r*}, M \vee S_{r*})$.

Theorem 3.1. *Assume that* $\dim M = n + 1 > 2$. *Then there is an isomorphism*

$$\pi_n(\mathbb{F}_k(M_1)) \cong \pi(M_1) \oplus \bigoplus_{r=2}^{k} (\pi_n(M_1)) \oplus (\pi_n(S_{r+1*})),$$

where S_{r+1*} stands for $S_{r+1,2} \vee \ldots \vee S_{r+1,r}$.

Proof. Observe that because M_1 is simply connected, it follows that $M_1 \simeq K$, where K is a CW complex of which the 2-skeleton $K^{(2)}$ is a bouquet of two dimensional spheres. Hence

$$M_1 \times S_{r*} \simeq (K \vee S_{r*}) \cup \bigcup_i e_i^{n+2} \cup \dots$$

where e_i^{n+2} ranges over the $(n+2)$-dimensional cells of $K^{(2)} \times S_{r*}$. Consequently,

$$\pi_i(M_1 \times S_{r*}, M_1 \vee S_{r*}) = 0,$$

for $1 \leq i \leq (n+1)$ and therefore $\pi_n(M_1 \vee S_{r*}) \to \pi_n(M_1 \times S_{r*})$ is an isomorphism. The latter clearly implies the theorem. □

Note the following distribution of the braid elements α_{rs} in the above theorem. α_{21} is in the first term $\pi_n(M_1)$. However, beyond that, in the general term $(\pi_n(M_1) \oplus (\pi_n(S_{r+1*}))$, we see that $\alpha_{r+1,1}$ is in $\pi_n(M_1)$, while $\alpha_{r+1,s} \in (\pi_n(S_{r+1*}))$, $2 \leq s \leq r$. We now describe how homotopy elements $\delta \in \pi_m(M_1)$ contribute to $\pi_n(\mathbb{F}_k(M_1))$. Let $\delta' \colon S^m \to M_1$ denote a map representing δ.

Definition 3.1. Denote by δ_r, the homotopy class in $\pi_m(\mathbb{F}_k(M_1))$ of the map

$$\xi \mapsto (q_1, \dots, q_{r-1}, \delta'(\xi), q_{r+1}, \dots, q_k) \in (\mathbb{F}_k(M_1))$$

where $2 \leq r \leq k+1$.

Thus one may think of δ_r as the element $\delta \in \pi_m(\mathbb{F}_k(M_1))$ inserted at the r-th level. In a similar fashion, an element $\delta \in \pi_m(M)$ determines elements δ_r, $1 \leq r \leq k+1$, in $\pi_m(\mathbb{F}_{k+1}(M))$.

Theorem 1.2 implies the following theorem.

Theorem 3.2. *The elements*

$$\{\alpha_{rs} \mid 1 \leq s < r \leq k+1\} \cup \{\delta_r \mid 2 \leq r \leq k+1,\ \delta \in \pi_n(M_1)\}$$

generate the group $\pi_n(\mathbb{F}_k(M_1))$.

4. Invariance under permutations

The symmetric group Σ_{k+1} acts freely on $\mathbb{F}_{k+1}(M)$ by permuting the coordinate indices $(1, \dots, k+1)$. This action induces an action on the homotopy groups $\pi_*(\mathbb{F}_{k+1}(M))$.

Proposition 4.1. *The braid elements α_{rs} satisfy the following relations relative to the action of Σ_{k+1}. If $1 \leq s \neq r \leq k+1$, then*

$$\alpha_{sr} = (-1)^{n+1}\alpha_{rs} \quad \textit{and} \quad \pi_n(\sigma)(\alpha_{rs}) = \alpha_{\sigma r \sigma s}.$$

Proof. These relations are verified in [4] for $M = \mathbb{R}^{n+1}$. However, the inclusion map $\mathbb{R}^{n+1} \to V \subset M$ induces a homomorphism which carries these relations into $\pi_n(\mathbb{F}_{k+1}(M))$. $\square$

Adapting the above relations to the punctured manifold M_1 will require certain modifications because a typical point of $\mathbb{F}_k(M_1)$ has the form $(q_1, x_2, \dots, x_{k+1})$ and the permutation group is restricted to Σ_k based on the indices $(2, \dots, k+1)$. Σ_{k+1} is not allowed to act on the elements $\alpha_{r1}, \dots, \alpha_{k+1,1}$. However, each α_{r1} may be considered as a δ in $\pi_n(\mathbb{F}_k(M_1))$.

Definition 4.1. Set $\delta_{r1} = \alpha_{r1}$, for $2 \le r \le k+1$, in $\pi_n(\mathbb{F}_k(M_1))$.

We now consider relations satisfied by elements of $\pi_n(\mathbb{F}_k(M_1))$ under the action of Σ_k. The first part of the following proposition is an immediate consequence of the above proposition. While the latter part is a simple exercise.

Proposition 4.2. *The generating elements of $\pi_n(\mathbb{F}_k(M_1))$ satisfy the following under the action of Σ_k.*

(i) *If $2 \le s \ne r \le k+1$, then $\alpha_{sr} = (-1)^{n+1}\alpha_{rs}$ and $\pi_n(\sigma)(\alpha_{rs}) = \alpha_{\sigma r \sigma s}$.*

(ii) *If $\delta \in \pi_n(M_1)$ with corresponding $\delta_r \in \pi_n(\mathbb{F}_k(M_1))$, $2 \le r \le k+1$, then $\pi_n(\sigma)(\delta_r) = \delta_{\sigma r}$. In particular, $\pi_n(\sigma)(\delta_{r1}) = \delta_{\sigma r,1}$.*

5. The Yang–Baxter relations

The following theorem is useful when computing Whitehead products. We state it for the general case $\pi_*(\mathbb{F}_{k+1}(M))$.

Theorem 5.1. *For all $\sigma \in \Sigma_{k+1}$, the following (Yang–Baxter) relations hold in $\pi_*(\mathbb{F}_{k+1}(M))$:*

(i) $[\alpha_{\sigma 2 \sigma 1}, \alpha_{\sigma 3 \sigma 1} + \alpha_{\sigma 3 \sigma 2}] = 0$, *for $k+1 \ge 3$,*

and

(ii) $[\alpha_{\sigma 2 \sigma 1}, \alpha_{\sigma 4 \sigma 3}] = 0$, *for $k+1 \ge 4$,*

in $\pi_*(\mathbb{F}_k(M_\infty))$.

Proof. The proof of Proposition 4.1 applies here. $\square$

In order to establish the Yang–Baxter relations for the punctured manifold M_1, we observe that the following relations are valid in $\pi_*(\mathbb{F}_{k+1}(\mathbb{R}^{n+1}))$.

(i) $[\alpha_{21}, \alpha_{31} + \alpha_{32}] = 0$,

(ii) $[\alpha_{31}, \alpha_{21} + (-1)^{n+1}\alpha_{32}] = 0$,

(iii) $[\alpha_{21}, \alpha_{43}] = 0$,

(iv) $[\alpha_{32}, \alpha_{42} + \alpha_{43}] = 0$,

(v) $[\alpha_{42}, \alpha_{32} + (-1)^{n+1}\alpha_{43}] = 0$,

(vi) $[\alpha_{32}, \alpha_{54}] = 0$.

These relations remain valid in $\pi_*(\mathbb{F}_k(\mathbb{R}^{n+1}-0))$ since since the latter injects into $\pi_*(\mathbb{F}_{k+1}(\mathbb{R}^{n+1}))$. The inclusion map $(\mathbb{R}^{n+1})-0 \to (V-Q_1) \subset M_1$ induces a homomorphism which carries these relations into $\pi_*(\mathbb{F}_k(M_1)$. Applying Σ_k based on the indices $\{2,\ldots,k+1\}$ and recalling that $\delta_{r1}=\alpha_{r1}$, we have the following theorem.

THEOREM 5.2. *The Yang–Baxter relations in $\pi_*(\mathbb{F}_k(M_1))$ for the punctured manifold M_1 are given below, where σ belongs to the permutation group Σ_k based on the indices $\{2,\ldots,k+1\}$.*

(i) $[\delta_{\sigma 2,1}, \delta_{\sigma 3,1} + \alpha_{\sigma 3,\sigma 2}] = 0,$
(ii) $[\delta_{\sigma 3,1}, \delta_{\sigma 2,1} + (-1)^{n+1}\alpha_{\sigma 3,\sigma 2}] = 0,$
(iii) $[\delta_{\sigma 2,1}, \alpha_{\sigma 4,\sigma 3}] = 0,$
(iv) $[\alpha_{\sigma 3,\sigma 2}, \alpha_{\sigma 4,\sigma 2} + \alpha_{\sigma 4,\sigma 3}] = 0,$
(v) $[\alpha_{\sigma 4,\sigma 2}, \alpha_{\sigma 3,\sigma 2} + (-1)^{n+1}\alpha_{\sigma 4,\sigma 3}] = 0,$
(vi) $[\alpha_{\sigma 3,\sigma 2}, \alpha_{\sigma 5,\sigma 4}] = 0.$

Before stating Whitehead product relations involving the elements δ_r, we recall one the basic tools for recognizing when Whitehead products are zero in $\pi_*(\mathbb{F}_{k+1}(M))$.

Let $\alpha \in \pi_m(\mathbb{F}_{k+1}(M))$ and let $\alpha'\colon (S^m,*) \to (\mathbb{F}_{k+1}(M,*)$ denote a based map representing α. If $\alpha' = (\alpha'_1,\ldots,\alpha'_{k+1})$ has the property that α'_j is constant except for $i \neq j$, we say that α' is concentrated in the i-th coordinate.

PROPOSITION 5.1. *Let $\alpha \in \pi_m(\mathbb{F}_{k+1}(M))$ and $\beta \in \pi_n(\mathbb{F}_{k+1}(M))$ with representatives α' and β' concentrated in the i-th and j-th coordinates, $i \neq j$. Then,* $[\alpha,\beta]=0$.

An immediate application of this proposition yields the following relation.

PROPOSITION 5.2. *Let $\delta \in \pi_m(M_1)$. Then,*

$$[\delta_{\sigma 2}, \alpha_{\sigma 4 \sigma 3}] = 0 \quad \textit{for all } \sigma \in \Sigma_k.$$

To prove our next relation we need some preparation and we will not distinguish here between the notation for a map and its homotopy class. Proceeding as in [4, Chapter III, Section 5], denote by $p_\delta\colon E(\delta) \to S^m$ the pull-back of the tangent bundle $T(M_1)$ of M_1 by a map $\delta\colon S^m \to M_1$.

The tangent bundle $T(M_1)$ admits a nonzero section, $v\colon M_1 \to T(M_1)$ and hence it is equivalent to $\xi \oplus o^1$, where the trivial bundle corresponds to the nonzero tangent vector field v. The pull-back bundle has an induced splitting and the charateristic map for the bundle has the form $\eta\colon S^{m-1} \to O(n)$. Let $SE(\delta)$ and $ST(M_1)$ denote the associated sphere bundles. Then the homotopy type of $SE(\delta)$ is given by $(S^m_v \vee S^n) \cup_\mu (D^{n+m})$, where μ is given by $[i_m, i_n] + J(\eta)$, where J is the "J-homomorphism". Here S^m_v is the image of the cross section in $SE(\delta)$

and i_m and i_n are generators in $\pi_m(S_v^m)$ and $\pi_n(S^n)$, respectively (see [11], [10] and [13]). We will also make use of the exponential map: $\exp W \to M_1$, where W is a suitable neigbourhood of the 0-section of the tangent bundle $T(M_1)$.

Theorem 5.3. *Let $\delta \in \pi_m(M_1))$. For $2 \le r < k$, there is a map $\phi: SE(\delta) \to \mathbb{F}_k(M_1)$, $k \ge 3$, which implies the relation*

$$[\delta_{r+1} + \delta_{r+2}, \alpha_{r+2,r+1}] + \zeta = 0$$

in $\pi_* \mathbb{F}_k(M_1)$, where ζ is the image of $J(\eta)$ induced by ϕ. Furthermore, when the fibration $p: \mathbb{F}_k(M_1) \to \mathbb{F}_r(M_1)$ is fiber homotopically trivial, $\zeta = 0$. In this case the relation becomes

$$[\delta_{\sigma(r+1)} + \delta_{\sigma(r+2)}, \alpha_{\sigma(r+2)\sigma(r+1)}] = 0,$$

$\sigma \in \Sigma_{k-r+1}$, based on $r+1, \dots, k+1$.

Proof. Denote a point of $SE(\delta)$ by (ξ_1, ξ_2) with $\xi_1 \in S^m$, $\xi_2 \in ST_{\delta(\xi_1)}(M)$. Define a map $\psi: SE(\delta) \to \mathbb{F}_{r+2}(M_1)$ by

$$\psi((\xi_1, \xi_2)) = (q_2, \dots, q_r, \delta(\xi_1), \exp_{\delta(\xi_1)}(\xi_2)).$$

where we are assuming, without loss of generality, that (ξ_1, ξ_2) is in W. A simple calculation shows that

$$\pi_*(\psi)([i_m, i_n]) = [\delta_{r+1} + \delta_{r+2}, \alpha_{r+2,r+1}]$$

and by definition $\zeta = \pi_*(\psi)(J(\eta))$. Hence,

$$[\delta_{r+1} + \delta_{r+2}, \alpha_{r+2,r+1}] + \zeta = 0.$$

Now, to obtain the map ϕ, let s denote a cross section of the bundle $\mathbb{F}_k(M_1) \to \mathbb{F}_{r+1}(M_1)$ induced by the vector field v. The desired map ϕ is given by the composition $\phi = s \circ \psi$. Next, let $\overline{\delta}: S^m \to \mathbb{F}_r(M_1)$ be given by $\overline{\delta}(\xi_1) = (q_2, \dots, q_r, \delta(\xi_1))$. Furthermore, let $p^*: \mathbb{F}_k^*(M_1) \to S^m$ denote the pull-back of $p: \mathbb{F}_k(M_1) \to \mathbb{F}_r(M_1)$. Then we have the diagram

$$\begin{array}{ccccc} SE(\delta) & \xrightarrow{\phi^*} & \mathbb{F}_k^*(M_1) & \xrightarrow{\delta^*} & \mathbb{F}_k(M_1) \\ \downarrow & & \downarrow & & \downarrow \\ S^m & \xrightarrow[\mathrm{id}]{} & S^m & \xrightarrow[\overline{\delta}]{} & \mathbb{F}_r(M_1) \end{array}$$

Note that $\phi^*((\xi_1, \xi_2)) = ((\xi_1, \phi((\xi_1, \xi_2))$ and $\phi = \delta^* \circ \phi^*$. Note also that there is a characteristic map ξ^* from S^{m-1} into the space of homotopy equivalences of the fiber $\mathbb{F}_{k-r}(M_{r+1})$. Therefore, if $\mathbb{F}_k(M_1) \to \mathbb{F}_r(M_1)$ is fiber homotopically trival we see that

$$\zeta = \pi(\phi)(J(\eta)) = \pi(\delta^* \circ \phi^*)(J(\eta)) = \pi(\delta^*)(J(\xi^*)) = 0.$$

This suffices to prove the second part of the theorem. □

When the manifold M itself admits a nonzero vector field, an analogue of Theorem 5.3 obtains with M replacing M_1 and other appropriate notational changes. In particular, $SE(\delta) \to S^m$ is the pull-back of the sphere tangent bundle $T(M)$ of M by a map $\delta\colon S^m \to M$ and η is the characteristic map of $SE(\delta) \to S^m$.

Theorem 5.4. *Suppose M admits a nonzero vector field and $\delta \in \pi_m(M))$. For $2 \leq r < k+1$, there is a map $\phi\colon SE(\delta) \to \mathbb{F}_{k+1}(M), k+1 \geq 3$, which implies the relation*

$$[\delta_r + \delta_{r+1}, \alpha_{r+1,r}] + \zeta = 0$$

in $\pi_*(\mathbb{F}_{k+1}(M))$, where ζ is the image of $J(\eta)$ induced by ϕ. Furthermore, when the fibration $p\colon \mathbb{F}_{k+1}(M) \to \mathbb{F}_r(M)$ is fiber homotopically trivial, $\zeta = 0$. In this case the relation becomes

$$[\delta_{\sigma r} + \delta_{\sigma(r+1)}, \alpha_{\sigma(r+1),\sigma r}] = 0, \quad \sigma \in \Sigma_{k-r}.$$

6. Wedge representations

In the consideration of our main application in the next section, we will need to describe a space over $(E/G) \vee Y$ as a suitable bouquet, where $\mathrm{p}\colon E \to B$ is a principal G-bundle and G is a topological group. We will restrict our attention to the special case when E has the form $E = G * G * \ldots * G * \ldots$, the Milnor join ([12]) of n copies of G, where n may be infinite and $B = E/G$. A more general result will be found in [5]. We will work in the category of spaces whose topology is compactly generated.

Observe that $\mathrm{p} \times \mathrm{id}\colon E \times Y \to B \times Y$ is again a principal G-bundle, where Y is a pointed space. Denote by $\mathrm{p}\colon E_w \to \mathrm{B} \vee Y$ the principal bundle induced by the inclusion map $B \vee Y \to B \times Y$. Our objective is the homotopy type of E_w.

Fix g_2^0 in G and consider the subset of $G * G$ in E, given by

$$\{(1-t)g_1 + tg_2^0 \mid g_1 \in G,\ 0 \leq t \leq 1\}$$

which represents a cone $cG = G \times I/G \times 1$ This cone has the property that its projection is a suspension $S^1 \wedge G \subset B$. Choose as base point b_1 in B, the projection of the base of the cone in E.

Theorem 6.1. *There is a homotopy equivalence $\phi\colon E_w \to E \vee (G \wedge Y) \vee Y$.*

Proof. Let y_0 denote a base point of Y and let $p((cG, G)) = (S^1 \wedge G, b_1)$. Observe that

$$E_w = (E \times \{y_0\}) \cup (G \times Y), \quad G = (E \times \{y_0\}) \cap (G \times Y).$$

Put

$$K = (E \times \{y_0\}), \quad L = (cG \times \{y_0\}) \cup (G \times Y).$$

Observe that $K \cup L = E_w$ and $K \cap L = cG \times \{y_0\}$.

As cG is contractible, it follows readily that the natural projections

$$\begin{aligned}(K \cup L) &\to (K \cup L)/cG \times y_0,\\ K &\to K/cG \times y_0,\\ L &\to L/cG \times y_0,\end{aligned}$$

are homotopy equivalences. Note that $(K \cup L)/cG \times y_0 = (K/cG \times y_0) \vee (L/cG \times y_0)$. Also note that

$$L/cG \times y_0 = (G \times Y)/G \times y_0 \simeq (G \wedge Y) \vee Y.$$

Thus $E_w = (E \times \{y_0\}) \cup (G \times Y)$ is homotopy equivalent to $E \vee (G \wedge Y) \vee Y$.$\square$

The induced map $\mathrm{p}_w \circ \phi^{-1}\colon (G \wedge Y) \to B \vee Y$, where ϕ^{-1} is a homotopy inverse for ϕ, brings into play the Whitehead product. Let $\alpha \in \pi_{m-1}(G)$. Then the suspension of α may be regarded as a map into B since $S^1 \wedge G \subset B$. We refer to it as the suspension of α in B

Corollary 6.1. *Let $s^1(\alpha) \in \pi_m(B)$ be the suspension (in B) of $\alpha \in \pi_{m-1}(G)$. Then, $\pi_*(\mathrm{p}_w)$ is injective and*

$$\pi_{n+m-1}(\mathrm{p} \circ \phi^{-1})(\alpha \wedge \beta) = [s^1(\alpha), \beta] \in \pi_{n+m-1}(B \vee Y)),$$

where $\beta \in \pi_n(Y)$.

Proof. First observe that the fiber G remains contractible in E_w which implies that $\pi_*(\mathrm{p}_w)$ is injective. Next, consider the composite map

$$\psi\colon (D^m, S^{m-1}) \times S^n \longrightarrow (cG, G) \times S^m \xrightarrow{\subset} (E, E_w \times S^n) \longrightarrow (E, E_w \times Y)$$

induced by α, β and the natural imbeddings. Observe that it takes the subspaces $(D^m \times \{e_0\}$ and $(S^{m-1} \times S^n)$ to E_w. Note that

$$D^m \times S^n = ((S^{m-1} \times S^n) \cup D^m \times \{e_0\} \cup_\nu D^{n+m} = L \cup_\nu D^{n+m},$$

where the attaching map

$$\nu\colon (D^m \times \partial D^{n+m}) \cup (\partial D^m \times D^{n+m}) \to (D^m \times \{e_0\}) \cup (S^{m-1} \times S^n)$$

is the identity on D^m and collapses ∂D^{m+n} to $\{e_0\}$. Thus we see that the obstruction to deforming ψ into E_w is exactly the homotopy class of ν. Therefore p takes ν to the obstruction of deforming $\mathrm{p}(D^m, S^{m-1}) \times Y$ onto $B \vee Y$. But the latter is the Whitehead product $[s^1(\alpha), \beta]$. $\square$

7. When is $\mathbb{F}_k(M_1) \to \mathbb{F}_r(M_1)$ trivial?

In [4] we determined when the projection

$$\mathrm{proj}_{kr}\colon \mathbb{F}_k(M_1) \to \mathbb{F}_r(M_1)$$

is fiber homotopically trivial (abbreviated f.h.t.) in the case when $M = \mathbb{R}^{n+1}$ or S^{n+1} . Below, we consider the more general case. In our first result we assume that M is any 1-connected (as usual), smooth manifold (not necessarily closed) of dimension $n+1$.

Theorem 7.1. *A necessary condition that the fibration*

$$\mathrm{proj}_{kr}\colon \mathbb{F}_k(M_1) \to \mathbb{F}_r(M_1), \quad (x_1, \dots, x_k) \mapsto (x_1, \dots, x_r), \quad r \geq 2,$$

is fiber homotropically trivial is that $n = 3$ or 7, and $r = 2$.

Proof. Assume that $\mathrm{proj}_{kr}\colon \mathbb{F}_k(M_1) \to \mathbb{F}_r(M_1)$ is f.h.t. We first show that $n = 3$ or 7.

Let $\{\alpha_{st} \mid 2 \leq t < s \leq k{+}1\} \cup \{\delta_{s1} \mid 2 \leq s \leq k{+}1\} \cup \{\delta_s \mid 2 \leq k{+}1\}$ denote the generators of $\pi_*(\mathbb{F}_k(M_1))$. Regard the subset $\{\alpha_{st} \mid 2 \leq t < s \leq r+1\} \cup \{\delta_{s1} \mid 2 \leq s \leq r+1\} \cup \{\delta_s \mid 2 \leq s \leq r+1\}$ as the generators of $\pi_*(\mathbb{F}_r(M_1))$. In both cases, the set involving the elements δ_{sj} may be incorporated into the set containing the elements δ_s.

Let $\alpha'_{r+1,2}\colon S^n \to \mathbb{F}_r(M_1)$ be the representative of $\alpha_{r+1,2}$, and denote its image by $S_{r+1,2}$. Denote the restriction of $\mathrm{proj}_{k,r}\colon \mathbb{F}_k(M_1) \to \mathbb{F}_r(M_1)$ to $S_{r+1,2}$ by $\mathrm{proj}^*\colon \mathbb{F}^*_k(M_1) \to S_{r+1,2}$. It is a fibration with $\mathbb{F}_{k-r}(M_{r+1})$ as fiber. Suppose that

$$\begin{array}{ccc} S_{r+1,2} \times \mathbb{F}_{k-r}(M_{r+1}) & \xrightarrow{\phi} & \mathbb{F}^*_k(M_1) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{proj}_{k,r}} \\ S_{r+1,2} & \xrightarrow[\mathrm{id}]{} & S_{r+1,2} \end{array}$$

is a homotopy equivalence over $S_{r+1,2}$. Note that ϕ can be adjusted, if necessary, so that $\phi | \mathbb{F}_{k-r}(M_{r+1})\colon \mathbb{F}_{k-r}(M_{r+1}) \to \mathbb{F}_{k-r}(M_{r+1})$ is the identity. As the configuration spaces here are all simply connected, we have the direct sum decomposition

$$\pi_n(\mathbb{F}^*_k(M_1)) \cong \pi_n((S_{r+1,2}) \oplus \left(\bigoplus_{s=r+2}^{k+1} \pi_n(M_1 \vee \left(\bigvee_{t=2}^{s-1} S_{st} \right)\right). \tag{1}$$

The morphism $\pi_n(\phi)$ takes $\alpha_{r+1,2}$ to an element of the form

$$\alpha_{r+1,2} + \sum_{s=r+1}^{k+1} (\beta_{s*} + \delta_s),$$

where each $\beta_{s*} \in \pi_n(S_{s2} \vee \ldots \vee S_{ss-1})$ and $\delta_s \in \pi_n(M_1)$. Since $[\alpha_{k+1,2}, \alpha_{r+1,2}] = 0$ in $\pi_{2n-1}(S_{r+1,2} \times \mathbb{F}_{k-r}(M_{r+1})$, and because $\pi_*(\phi)$ preserves Whitehead products, it follows that

$$(2) \qquad \phi_*[\alpha_{k+1,2}, \alpha_{r+1,2}] = [\alpha_{k+1,2}, \alpha_{r+1,2}] + \sum_{s=r+2}^{k+1} [\alpha_{k+1,2}, \beta_{s*}] + \sum_{s=r+2}^{k+1} [\alpha_{k+1,2}, \delta_s] = 0.$$

The Yang–Baxter relations of Theorem 5.2 and the second relation of Theorem 5.3 imply that the elements $[\alpha_{k+1,2}, \alpha_{r+1,2}]$, $[\alpha_{k+2,2}, \beta_{s*}]$ and $[\alpha_{k+1,2}, \delta_s]$ are in $\pi_{2n-1}(M_1 \vee (S_{k+1,2} \vee \ldots \vee S_{k+1,k}))$. Applying the obvious retraction ρ from $M_k = M_1 \vee (S_{k+1,2} \vee \ldots \vee S_{k+1,k})$ to $(S_{k+1,2} \vee \ldots \vee S_{k+1,k})$, let $\overline{\beta}_{k+1}$ denote the image of

$$\sum_{s=r+2}^{k+1} [\alpha_{k+1,2}, \beta_{s*}] + \sum_{s=r+2}^{k+1} [\alpha_{k+1,2}, \delta_s]$$

under $\pi_{2n-1}(\rho)$. $\overline{\beta}_{k+1}$ has the form

$$\overline{\beta}_{k+1} = \sum_{t=2}^{k} c_{k+1,t} \alpha_{k+1,t}$$

with $c_{k+1,t} \in \mathbb{Z}$. Therefore (2) becomes

$$(3) \quad [\alpha_{k+1,2}, \alpha_{r+1,2}] + c_{k+1,2}[\alpha_{k+1,2}, \alpha_{k+1,2}] + \ldots + c_{k+1,k}[\alpha_{k+1,2}, \alpha_{k+1,k}] = 0.$$

Now, employ the Yang–Baxter relation

$$[\alpha_{k+1,2}, \alpha_{r+1,2} + (-1)^{n+1} \alpha_{k+1,3}] = 0$$

and replace $[\alpha_{k+1,2}, \alpha_{r+1,2}]$ by its value in terms of the Whitehead product $[\alpha_{k+1,2}, \alpha_{k+1,3}]$ to obtain the following version of (3)

$$(4) \quad c_{k+1,2}[\alpha_{k+1,2}, \alpha_{k+1,2}] + (c_{k+1,3} - (-1)^{n+1})[\alpha_{k+1,2}, \alpha_{k+1,3}] + \sum_{t=4}^{r+1} c_{k+1,t}[\alpha_{k+1,2}, \alpha_{k+1,t}] = 0.$$

Note that the preceding formula (4) is valid in the Lie subalgebra $\pi_*((S_{k+1,2} \vee \ldots \vee S_{k+1,k}))$.

Next, recall that Hilton's theorem ([9]) gives, for each s such that $(r+2) \leq s \leq k+1$, the direct sum decomposition

$$\pi_{2n-1}(S_{k+1,2} \vee \ldots \vee S_{k+1,k}) \cong \bigg(\bigoplus_{i=2}^{k} \pi_{2n-1}(S_{k+1,i}) \bigg) \oplus \bigg(\bigoplus_{w} \pi_{2n-1}(S_w) \bigg),$$

where w ranges over all Whitehead products of weight 2 on the set of symbols $\{\alpha_{k+1,s} \mid 2 \leq s < k+1\}$ ([13]). Observe that the various Whitehead products in (4) belong to different summands in the Hilton formula. Thus we obtain the equations

$$(5)\qquad \begin{cases} \text{(i)} & c_{k+1,2}[\alpha_{k+1,2}, \alpha_{k+1,2}] = 0, \\ \text{(ii)} & (c_{k+1,3} - (-1)^{n+1})[\alpha_{k+1,2}, \alpha_{k+1,3}] = 0, \\ \text{(iii)} & c_{k+1,t}[\alpha_{k+1,2}, \alpha_{k+1,t}] = 0, \end{cases}$$

where $t > 3$ in (iii) above. Now, note that $w = [\alpha_{k+1,2}, \alpha_{k+1,3}] \in \pi_{2n-1}(S^w)$ is a basic product. Therefore, it defines a summand in the Hilton Theorem. Also note that it is of infinite order. This clearly implies that $c_{k+1,3} = (-1)^{n+1}$.

Next, starting with the fact $[\alpha_{k+1,3}, \alpha_{r+1,2}] = 0 \in \pi_{2n-1}(S^n_{r+1,2} \times \mathbb{F}_{k-r,r}(M_1))$, apply the same argument as that given above using the Yang–Baxter relation $[\alpha_{k+1,3}, \alpha_{r+1,2} + \alpha_{k+1,2}] = 0$ to obtain $(1 + c_{k+1,2})[\alpha_{k+1,3}, \alpha_{k+1,2}] = 0$. The Hilton Theorem again applies and $c_{k+1,2} = -1$. Hence, (i) of (5) implies that $[\alpha_{k+1,2}, \alpha_{k+1,2}] = 0$ and for a generator ι_n of S^n $[\iota_n, \iota_n] = [\alpha_{r+2,2}, \alpha_{r+2,2}]$. Therefore, $[\iota_n, \iota_n] = 0$ and S^n is an H-space. An appication of the celebrated theorem of J. F. Adams ([1], [2]) shows that $n = 3$ or $n = 7$.

We next prove that $r = 2$, assuming now that $n = 3$ or $n = 7$. In particular, $n + 1$ is even. Suppose to the contrary that $r > 2$ and consider equations (5). The basic product $w = [\alpha_{r+2,2}, \alpha_{r+2,t}]$, $t > 3$, generates the infinite cyclic group $\pi_{2n-1}(S_w)$. This implies that its coefficient $c_{r+1,t}$ is zero and, therefore,

$$\alpha_{r+1,2} + \overline{\beta}_{k+1} = \alpha_{r+1,3} - \alpha_{k+1,2} + \alpha_{k+1,3}.$$

Since $r > 2$, we have $k + 1 \geq 5$, so that we have available the elements $\alpha_{k+1,t}$, $t = 2, 3, 4$. Observe that, since $[\alpha_{r+1,2}, \alpha_{k+1,4}] = 0$, we have

$$[\alpha_{r+1,2} - \alpha_{k+1,2} + \alpha_{k+1,3}, \alpha_{k+1,4}] = 0.$$

Then,

$$(-1)[\alpha_{k+1,2}, \alpha_{k+1,4}] + [\alpha_{k+1,3}, \alpha_{k+1,4}] = 0.$$

Finally, since the two summands above represent distinct basic elements of weight 2, each must be zero, which is a contradiction and $r = 2$. □

Our next necessary condition involves the homomotopy groups of M_1.

Theorem 7.2. *A necessary condition that the fibration*

$$\mathrm{p}_{k,r}\colon \mathbb{F}_k(M_1) \to \mathbb{F}_2(M_1),$$

is fiber homotropically trivial is equivalent to that the homotopy groups $\pi_q(M_1) = 0$ *for* $q < n$.

Proof. If we deny the conclusion, let m denote the minimum value of m for which $\pi_m(M_1) \neq 0, 2 \leq m < n$. Furthermore, let $\pi = \pi_m(M_1)$ Using the classical method for killing homotopy groups by adding cells (see e.g. [8]), there is a $K(\pi, m)$ space X and an inclusion map $j: M_1 \to X$ which induces an isomorphism $\pi_m(j): \pi_m(M_1) \to \pi_m(X)$.

Suppose that $\delta \neq 0 \in \pi_m(M_1)$. Denote by $\delta_t \in \pi_m(M_1)$ the insertion of δ in the t-th coordinates of $\mathbb{F}_k(M_1)$. Employing Proposition 5.3, since $\mathrm{p}_{k,r}: \mathbb{F}_k(M_1) \to \mathbb{F}_2(M_1)$ is f.h.t., we have at our disposal the relation $[\delta_2 + \delta_3, \alpha_{32}] = 0$. Applying the permutation (43), we obtain $[\delta_2 + \delta_4, \alpha_{42}] = 0$. Again using f.h.t., $[\delta_2, \alpha_{42}] = 0$, which in turn implies that $[\delta_4, \alpha_{42}] = 0$. We complete the proof by showing that $[\delta_4, \alpha_{42}] \neq 0$, thereby arriving at a contradiction.

Since M_3 has the form $M_3 = M_1 \vee S_{42} \vee S_{43}$, there is a map f which takes M_3 to $K(\pi, m) \vee S^n$ with $\pi_m(f)$ taking δ_4 to δ in π and α_{42} to ι_n, the fundamental class of S^n.

Next, consider the principal G-bundle $p: E \to B$, where G is a topological group with the homotopy type of the loop space $\Omega(K(\pi, m))$, $L(\pi, m)$ is the infinite join of copies of G and B is the orbit space in the Milnor construction which is a $K(\pi, m)$. Then by Theorem 6.1 there is a fiber homotopy equivalence

$$\begin{tikzcd} E_w \arrow[r, "\phi"] \arrow[d] & (L(\pi, m) \vee (K(\pi, m-1) \wedge S^n) \vee S^n \arrow[d] \\ K(\pi, m) \vee S^n \arrow[r, "\mathrm{id}"'] & K(\pi, m) \vee S^n \end{tikzcd}$$

Let

$$s_*: \pi_m(K(\pi, m-1)) \to \pi_{m-1}(K(\pi, m))$$

denote the suspension isomorphism and let $\overline{\delta} = (s_*)^{-1}(\delta)$. Observe that $\overline{\delta} \wedge \iota_n$ is in $\pi_{n+m-1}(K(\pi, m-1) \wedge S^n)$, where ι_n is the fundamental class of S^n. The class $\overline{\delta} \wedge \iota_n$ is nontrivial because the spherical class $\overline{\delta} \wedge \iota_n \in H_{n+m-1}(K(\pi, m-1) \vee (S^n, \mathbb{Z})$ is nontrivial. The morphism

$$\pi_w(\mathrm{p}): \pi_{n+m-1}(E_w) \to \pi_{n+m-1}(K(m, \mathbb{Z}) \vee S^n)$$

takes $\overline{\delta} \wedge \iota_n$ to the Whitehead product $[\delta, \iota_n] \in \pi_{n+m-1}(K(\pi, m) \vee S^n)$. (See Corollary 6.1.) Since $\pi_n(f)([\delta_4, \alpha_{42}]) = [\delta, \iota_n] \neq 0$, we see that $[\delta_4, \alpha_{4,2}] \neq 0$ which contradicts the fact that $[\delta_4, \alpha_{43}] = 0$.

Corollary 7.1. *Let M denote a closed manifold. Then a necessary condition that the fibration*

$$\mathrm{proj}_{kr}: \mathbb{F}_k(M_1) \to \mathbb{F}_2(M_1),$$

is fiber homotropically trivial is equivalent to that M_1 is contractible.

Proof. The previous theorem implies that the homotopy groups of M_1 vanish in dimensions up to and including $n-1$. Poincare Duality in M forces $\pi_n(M_1) = 0$ and $\pi_{n+1}(M_1) = 0$ because $H_{n+1}(M_1) = 0$. Thus M_1 is contractible. □

The above Theorem 7.2 may be stated in terms of the unpunctured manifold M itself with some notational changes in the proof

Theorem 7.3. *We assume that the tangent bundle of M admits a nonzero vector field . A necessary condition that the fibration*

$$\mathrm{p}_{k+1,r} \colon \mathbb{F}_{k+1}(M) \to \mathbb{F}_2(M),$$

is fiber homotropically trivial is equivalent that the homotopy groups $\pi_q(M) = 0$ for $0 < q < n$. If M is closed we may also conclude that M_1 is contractible.

The proof is based upon Proposition 5.4 and then proceeds as in the proof of Theorem 7.2 with only notational changes. □

8. When is $\mathbb{F}_k(M) \to \mathbb{F}_r(M)$ fiber homotropically trivial? M closed

We now apply the previous results to illustrate how results on the punctured manifold M_1 apply to the question when the fibration

$$\mathrm{proj}_{k+1,r} \colon \mathbb{F}_{k+1}(M) \to \mathbb{F}_r(M)$$

is fiber homotropically trivial, where M is a closed manifold. The situation here is different from the punctured manifold case because of the lack of cross sections. First, however, we make the following observations before restricting ourselves to closed manifolds.

Proposition 8.1. *If the fibration*

$$\mathrm{proj}_{k+1,r} \colon \mathbb{F}_{k+1}(M) \to \mathbb{F}_r(M), \quad k+1 > r,\ r \geq 2.$$

is fiber homotropically trivial then the fibration

$$\mathrm{proj}_{k,r-1} \colon \mathbb{F}_k(M_1) \to \mathbb{F}_{r-1}(M_1),$$

is also fiber homotropically trivial.

Proof. Identify the fiber of the projection $\mathrm{proj}_{r,1} \colon \mathbb{F}_r(M) \to M$ at the point q_1, with $\mathbb{F}_{r-1}(M_1)$. Observe that the preimage of $\mathbb{F}_{r-1}(M_1)$ under $\mathrm{proj}_{k+1,r}$ is $\mathbb{F}_k(M_1)$. The conclusion of the proposition is then immediate. □

Theorem 8.1. *Let M denote a manifold (closed or open) of dimension $n+1 \geq 3$. Then a necessary condition that*

$$\mathrm{proj}_{k+1,r} \colon \mathbb{F}_{k+1}(M) \to \mathbb{F}_r(M), \quad k+1 \geq 4,\ r \geq 3$$

is fiber homotropically trivial is equivalent to that $n+1$ is 4 or 8, $r \leq 3$, and the homotopy groups $\pi_q(M) = 0$ for $q < n$.

Proof. Apply Propositions 8.1, 7.1 and 7.2. □

We now return to the case of closed manifolds and consider first the odd dimensional case. We will make use of the following special case for spheres from [4].

Theorem 8.2. *Suppose that* $(n+1)$ *is odd. Then, the fibration*

$$\mathrm{proj}_{k+1,r}\colon \mathbb{F}_{k+1}(S^{n+1}) \to \mathbb{F}_r(S^{n+1}), \quad k+1 \geq 3,\ r \geq 1,$$

is fiber homotropically trivial if and only if $r \leq 2$ *and* $(n+1)$ *is* 3 *or* 7.

We extend this theorem as follows.

Theorem 8.3. *Suppose that* M *is a closed manifold of odd dimension* $n+1 \geq 4$. *Then the fibration*

$$\mathrm{proj}_{k+1,r}\colon \mathbb{F}_{k+1}(M) \to \mathbb{F}_r(M), \quad k+1 \geq 3,\ r \geq 2,$$

is fiber homotropically trivial if and only if $r = 2$, *and* M *is homeomorphic to the sphere* S^7.

Proof. We need only prove the necessity because of Theorem 8.2. Assume that

$$\mathrm{proj}_{k+1,r}\colon \mathbb{F}_{k+1}(M) \to \mathbb{F}_r(M)$$

is f.h.t. Suppose $r \geq 3$. Then, by Proposition 8.1, the fibration

$$\mathrm{proj}_{k,r-1}\colon \mathbb{F}_k(M_1) \to \mathbb{F}_{r-1}(M_1)$$

is also f.h.t. and by Theorem 7.1, $n+1$ is 4 or 8 which contradicts $n+1$ being odd. Therefore, $r=2$. Now that we know that $r=2$, we apply Theorem 7.3 to conclude that M_1 is contractible and hence M has the homotopy type of S^{n+1}. The validity of the Poincaré conjecture in dimensions $n+1 \geq 4$ implies that M is homeomorphic to S^{n+1}. Applying Theorem 8.1, if $n+1 \geq 4$, then $n+1 = 7$.□

Since the Poincaré conjecture in dimension 3 remains open, we can only state the following for dimension 3.

Proposition 8.2. *Suppose that* M *is a closed* (*simply connected*) 3*-manifold. Then, if the fibration*

$$\mathrm{proj}_{k+1,r}\colon \mathbb{F}_{k+1}(M) \to \mathbb{F}_r(M), \quad k+1 \geq 3,\ r \geq 2,$$

is fiber homotropically trivial, then $r = 2$.

We now take up the even dimensional case.

Theorem 8.4. *Suppose that M is a closed manifold of even dimension $n+1 \geq 4$. Then, a necessary condition that the fibration*

$$\mathrm{proj}_{k+1,r}\colon \mathbb{F}_{k+1}(M) \to \mathbb{F}_r(M), \quad r \geq 3,$$

is f.h.t. is equivalent to that $r = 3$ and M is S^4 or S^8.

Proof. Suppose that $r \geq 3$. Then, by Proposition 8.1, the fibration

$$\mathrm{proj}_{k,r-1}\colon \mathbb{F}_k(M_1) \to \mathbb{F}_{r-1}(M_1), \quad r-1 \geq 2,$$

is also f.h.t. and, by Theorem 7.1, $n+1 = 4, 8$, $r-1 = 2$ and M_1 is contractible. This forces M to be a homotopy sphere and consequently a sphere. □

The question whether

$$\mathrm{proj}_{k+1,3}\colon \mathbb{F}_{k+1}(M) \to \mathbb{F}_3(M)$$

is fiber homotopically trivial when M is a sphere and $n = 3$ or 7 remains open.

We add the following additional information in the case of even dimensional manifolds.

Theorem 8.5. *Suppose that M is a closed manifold of even dimension $n+1 \geq 4$ which admits a nonzero vector field. Then,*

$$\mathrm{proj}_{k+1,2}\colon \mathbb{F}_{k+1}(M) \to \mathbb{F}_2(M),$$

is never fiber homotropically trivial.

Proof. Suppose the contrary. Then, by Theorem 7.3, M_1 is contractible and conseqently M is a sphere. Since $k+1 \geq 3$, we easily obtain a cross section in the fibration $\mathbb{F}_3(M) \to M$ by employing a cross section from $\mathbb{F}_2(M)$ to $\mathbb{F}_{k+1}(M)$ followed by a projection to $\mathbb{F}_3(M)$. However, the latter is fiber homotopic to the tangent sphere bundle of M. This would imply a nonzero vector field on an even sphere which is a contradiction. □

Finally, the question as to necessary conditions that

$$\mathrm{proj}_{k+1}\colon \mathbb{F}_{k+1}(M) \to M,$$

(i.e. the case $r = 1$) is f.h.t., requires study and is complicated by the fact that this fibration is f.h.t. whenever M is a compact topological group.

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