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# A NONSTANDARD DESCRIPTION OF RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We develop a nonstandard description of Retarded Functional Differential Equations which consist of a formally finite iteration of vectors. We present two applications where the new description gives explicit formulae. The classical approach in these cases only offers a method to construct the solution.

## 1. Introduction

Differential equations where the derivative at a time $t$ depends on the state before that time, so called Functional Differential Equations, and the important special case of Retarded Functional Differential Equations (RFDE), play an important role in modeling (for some examples see e.g. [7]). The theory of RFDE's is much more complicated than the theory of ODE's. This is due to the fact, that the initial value lies in a functional space, so solving RFDE becomes an infinite dimensional problem.

We will show, how using methods from Nonstandard Analysis it is possible to transform this infinite dimensional problem into a formally finite one. (For those not familiar with Nonstandard Analysis see, for example, [1] or [8]. A very good introduction is the German book by Landers and Rogge [9].) Formally finite in this context means hyper-finite, that is finite in the nonstandard sense. The key

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idea is to sample continuous functions at infinitesimal time steps. A function is thus represented by a hyper-finite vector, and solving the RFDE becomes an iteration of hyper-finite vectors.

The description one gets this way is an elegant and very intuitive one. More important still, it opens the theory of RFDE's to applications of many classical results. In particular, the characteristic equation for a linear autonomous RFDE becomes a polynomial.

The aim of this paper is a first presentation and development of the nonstandard approach. We shall show how to transform an RFDE, and that this technique works for a quite general class of RFDE. We develop the linear theory more in detail, in particular with respect to eigenvalues and eigenfunctions. Although the aim of this paper is the presentation of the new approach, rather than applications thereof, we have included two examples of new standard results. Namely explicit formulas for the decomposition with respect to eigenfunctions, and for exchanging eigenvalues. Both are straightforward applications of our description.

To our knowledge Nonstandard Analysis has not been applied to RFDE before. But the idea of discretizing functions to represent them by hyper-finite vectors is not new. Ben El Mamoune, Benoit and Lobry looked at these representations in [2]. Discretizations of PDE have also been used. In [5] Delfini and Lobry discretize the space variable to obtain a hyper-finite system of ODE describing a PDE.

The paper is organized as follows: in Section 2 we develop the nonstandard description for the general nonlinear case. In Section 3 we contemplate the linear case, followed by the linear autonomous case in Section 4. The one-dimensional linear autonomous case and the conclusion come last in Sections 5, respectively 6.

For those not accustomed with Nonstandard Analysis, we will very briefly mention the notation we use, and the most basic features.

To practically every mathematical object and property there is a corresponding one in the nonstandard universe. Typically they have the same name preceded by a "$*$". For example $[a,b[$, $\mathbb{R}$, $\mathbb{N}$, $f\colon D \to S$ become ${}^*[a,b[$, which is contained in ${}^*\mathbb{R}$, ${}^*\mathbb{N}$ and a function ${}^*f\colon {}^*D \to {}^*S$, respectively. (Apart from certain identifications the $*$-version can be thought of as the equivalence class with respect to an ultra-filter of a sequence of the object in question.) The $*$-version of a set contains the set itself, so $\mathbb{N} \subset {}^*\mathbb{N}$ and $\mathbb{R} \subset {}^*\mathbb{R}$. Both ${}^*\mathbb{N}$ and ${}^*\mathbb{R}$ are bigger than their counterparts: they contain infinite elements, and ${}^*\mathbb{R}$ also infinitesimals. If the difference between two numbers $a, b \in {}^*\mathbb{R}$ is infinitesimal, we say $a$ is infinitely close to $b$, and write $a \approx b$. If the number $a$ is finite, there is exactly one real number $c \in \mathbb{R}$ which is infinitely close to $a$, it is called the standard part of $a$: $c = {}^\circ a$.

The $*$-versions of properties are defined transferring the standard definitions to the nonstandard setting. For example, a function $f$ is $*$-continuous at a point $x_0$, if and only if for all $0 < \varepsilon \in {}^*\mathbb{R}$ there is a $0 < \delta \in {}^*\mathbb{R}$, such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$ (it should be ${}^*|\cdot|$ and ${}^* <$, but usually one does not put the "$*$" in these cases). Another example is finite. It becomes $*$-finite, or, as it is also called, hyper-finite: a set $S$ is hyper-finite, if and only if there is an internal bijection between $S$ and $\{1, \dots, N\}$, for an $N \in {}^*\mathbb{N}$. "Internal" is a somewhat more technical term. It assures the nonstandard objects behave "well", that is similar to standard objects. We don't have the space to define "internal" properly, but as a rule of thumb everything is internal, which does not depend on one of the following in its definition: finite, infinite, infinitesimal, an infinite standard set seen as a subset of a nonstandard set. If an object is not internal, it is called external. A few examples are: $\mathbb{N} = \{n \in {}^*\mathbb{N} : n \text{ is finite}\}$, $\{x \approx 0\}$, $[0,1] \subset {}^*\mathbb{R}$ (as opposed to ${}^*[0,1]$) are all external sets.

Internal objects behave similar to standard ones, because we can apply transfer to them. Transfer means, that a formal sentence is true in the standard universe if and only if its starred version is true in the nonstandard one. To make this precise, we would have to define the formal language and the starred version of a formula in this language, which is not possible within the scope of this article. Most standard results can be transfered. For example, ${}^*\mathbb{R}$ is an ordered field (but not complete), ${}^*\mathbb{R}^M$, $M \in {}^*\mathbb{N}$, is a $*$-finite vector-space, endowed with the obvious nonstandard versions of the normal rules and operations of a finite dimensional vector-space. Another example are combinatorial formulas over $\mathbb{N}$, which are valid in a natural sense in ${}^*\mathbb{N}$.

Applying transfer to internal sets we get some useful rules. For example, an internal $*$-bounded set $A \subset {}^*\mathbb{R}$ has a supremum, a hyper-finite set $B \subset {}^*\mathbb{R}$ contains its maximum, and if an internal set contains arbitrarily large finite numbers, it also contains an infinite number. The latter is called overflow. There is an analog of it called underflow, which says, that if an internal set contains arbitrarily small infinite positive numbers, it also contains a finite one. Similar rules hold for infinitesimal/finite numbers. A common mistake is to apply these rules to external sets, where they don't hold (e.g. $\{x \approx 0\}$ is bounded but has no supremum).

Before we start with the nonstandard description, let us introduce a few notations we will be using throughout the paper.

$r > 0$ will be a fixed real number. For $I$ a real interval let $C(I, \mathbb{R}^d)$, $d \in \mathbb{N} \setminus \{0\}$, denote the space of continuous functions from $I$ into $\mathbb{R}^d$ with the supremum norm. If $I = [-r, 0]$ we just write $C = C([-r,0], \mathbb{R}^d)$. For a continuous function $x \in C([-r,T], \mathbb{R}^d)$, $x_t \in C$ denotes the restriction to $[t-r, t]$ of $x(t)$, $x_t(\theta) = x(t+\theta)\colon [-r,0] \to \mathbb{R}^d$, for $0 \le t \le T$.

Let $T_f > 0$ and $\Omega \subset C$ be open. For any continuous $f\colon [0, T_f[ \times \Omega \to \mathbb{R}^d$ and $\Phi \in \Omega$ we contemplate the RFDE

$$x'(t) = f(t, x_t), \quad t > 0, \quad x_0 = \Phi. \tag{1}$$

Note, that we do not assume $f$ to be Lipschitzian in its second argument, so we do not have in general uniqueness of the solution of (1).

Finally, for the nonstandard description, $M, N \in {}^*\mathbb{N} \setminus \mathbb{N}$ will be fixed hyper-finite natural numbers with $M/N \approx r$.

## 2. The general case

We want to represent continuous functions by hyper-finite vectors in order to be able to describe the solution of equation (1) by a formally finite iteration of vectors. We start by specifying what we mean by representing a function:

Definition 1. Let $I \subset \mathbb{R}$ be an interval with endpoints $a < b$. Fix two hyper-finite natural numbers $\widetilde{M}, \widetilde{N} \in {}^*\mathbb{N} \setminus \mathbb{N}$ with $\widetilde{M}/\widetilde{N} \approx b - a$. For any $x = (x_1, \dots, x_d)^t \in C(I, \mathbb{R}^d)$ and internal $Y = (y_0, \dots, y_{-\widetilde{M}+1})^t \in {}^*\mathbb{R}^{d\widetilde{M}}$, we say $Y$ *represents* $x$, and write $Y \triangleq x$, if the following holds:

> if $j \in \{0, \dots, -\widetilde{M}+1\}$ is such, that $b + {}^\circ(j/\widetilde{N}) \in I$, then $y_j = (y_{j,1}, \dots, y_{j,d})^t$ satisfies $y_j \approx x(b + {}^\circ(j/\widetilde{N}))$, i.e. $y_{j,l} \approx x_l(b + {}^\circ(j/\widetilde{N}))$ for all $l = 1, \dots, d$ .

We have chosen the unusual notation $Y = (y_0, y_{-1}, \dots, y_{-\widetilde{M}+1})^t$ because we think of the index as (infinitesimal) time steps.

It is clear, that we can represent complex-valued functions in the same way, and also functions defined on an infinite interval. Only that in the latter case one has to choose another "starting point" if $b = \infty$.

Note also, that every continuous function can be represented this way, for example by $y_j = {}^*x(b + j/\widetilde{N})$, since $x$ is continuous if and only if ${}^*x(t) \approx x(t_0)$ for all $t \approx t_0$. But not all vectors $Y \in \mathbb{R}^{d\widetilde{M}}$ represent continuous functions. A necessary and sufficient condition is the following:

If $j, l \in \{0, \dots, -\widetilde{M}+1\}$, $j/\widetilde{N} \approx l/\widetilde{N}$ and $b + {}^\circ(j/\widetilde{N}) \in I$, then $y_j \approx y_l$.

It is easy to see, that this condition is necessary. It is sufficient, because $y_j \approx y_l$ for $j/\widetilde{N} \approx l/\widetilde{N}$ allows to define a function by $x(t) := {}^\circ(y_j)$, for $j/\widetilde{N} \approx b - t \in I$.

We want to solve a differential equation, and derivatives can be approximated by $\Delta x/\Delta t$, where we can use as an infinitesimal time step $1/N$. This quotient can give an iterative way to construct approximations $y_n$ to the solution (as in the numerical Euler method). So, if we started with a vector $Y \in {}^*\mathbb{R}^{dM}$ representing a function $x_t \in C$, we could linearly join the points $(t + j/N, y_j)$, $j = 0, \dots, -M+1$ to get a ${}^*$continuous function, apply ${}^*f$ to it, to get the new approximation, which would be the first component of a new vector, the others just being shifted copies of the old one. Thus we would get an iteration of vectors, where each one would represent the solution $x_t \in C$ of equation (1) for a certain time $t$. Only that this method is to restrictive. We can change ${}^*f$ slightly and still the resulting vectors represent the solution. This freedom of choosing a (internal) $G$ "infinitely close to" $f$ will be important later on. In this context "infinitely close to" means the following:

Definition 2. Let $f$, $M$, $N$ as above and

$$S = \{Y \in {}^*\mathbb{R}^{dM} : \text{ there exists } \Phi \in \Omega,\ Y \overset{\triangle}{=} \Phi\}.$$

$G\colon \{n/N : n \in {}^*\mathbb{N}\} \times {}^*\mathbb{R}^{dM} \to {}^*\mathbb{R}^d$ is called *infinitely close to* $f$ if and only if it is internal and

$${}^\circ G(n/N, Y) = f({}^\circ(n/N), \Phi)$$

for all ${}^\circ(n/N) \in [0, T_f[$ and $Y \in S$ representing $\Phi \in \Omega$.

Now we are able to state how solving the RFDE (1) can be transformed into an iteration by representing functions by hyper-finite vectors:

Proposition 1. *Let $\Phi \in \Omega \subset C$, $\Omega$ open, $r, T_f > 0$, $f\colon [0, T_f[ \times \Omega \to \mathbb{R}^d$ continuous, $M, N \in {}^*\mathbb{N} \setminus \mathbb{N}$, $M/N \approx r$, and $G\colon \{n/N : n \in {}^*\mathbb{N}\} \times {}^*\mathbb{R}^{dM} \to {}^*\mathbb{R}^d$ infinitely close to $f$. Assume $Y_0 = (y_0, \dots, y_{-M+1})^t \overset{\triangle}{=} \Phi \in \Omega$. Then $Y_n = (y_n, \dots, y_{n-M+1})^t \in {}^*\mathbb{R}^{dM}$ defined by:*

$$y_n = y_{n-1} + \frac{1}{N} G\Big(\frac{n-1}{N}, (y_{n-1}, \dots, y_{n-M})^t\Big),$$

*or equivalently, with $Y'_n = (y_n, \dots, y_{n-M+2})^t \in {}^*\mathbb{R}^{d(M-1)}$*

$$Y_n = \begin{pmatrix} y_{n-1} + \frac{1}{N} G((n-1)/N, Y_{n-1}) \\ Y'_{n-1} \end{pmatrix}$$

*is well defined for $0 < n < N\beta$, where $\beta > 0$ is a real number. For $0 \le t = {}^\circ(n/N) < \beta$, $Y_n$ represents a function $x_t \in C$. The function $x \in C([-r, \beta[, \mathbb{R}^d)$ we get this way is a solution of the* RFDE (1) *on* $[0, \beta[$.

Proof. We will proceed in three steps: in the first one we show $y_n$ to be defined for $n$ big enough, in the second we prove that $y_n$ represents a continuous function $x_t$, and in the last one, that the resulting function $x(t)$ solves the equation (1). The proof we get is at the same time an existence prove for equation (1).

For the first step we use a simple a priori estimate, valid (at least) for $n/N \approx 0$, i.e. for infinitesimal times, to show existence of $y_n$ for these $n$. Then an overflow argument extends the existence up to a real time $\beta > 0$.

We claim that for (fixed) $n/N \approx 0$: (a) $y_n$ is defined and (b)

$$\|y_n - y_{n-1}\|_\infty \le \frac{1 + \|f(0, \Phi)\|_\infty}{N}. \tag{2}$$

We prove this by induction over $n$, which is allowed since all entities involved ($Y_0$, $G$, (a) and (b)) are internal. Indeed (a) and (b) hold for $n = 1$, and assuming them for $0 < j \leq n$ we see, that $y_n \approx y_0 \approx \Phi(0)$, hence $Y_n \stackrel{\triangle}{=} \Phi$, so $G(n/N, Y_n)$ is defined and $^\circ G(n/N, Y_n) = f(0, \Phi)$. This gives the existence of $y_{n+1}$, and

$$\|y_{n+1} - y_n\|_\infty = \left\|\frac{1}{N} G\left(\frac{n}{N}, Y_n\right)\right\|_\infty < \frac{1}{N}(\|f(0,\Phi)\|_\infty + 1)$$

and (b) holds for $n + 1$ too.

The claim has been proven, and (a) and (b) hold for all $n/N \approx 0$. But the set of all $n$, for which $y_n$ exists and (2) holds, is internal, hence by overflow there is a real $\beta > 0$, such that $y_n$ exists and satisfies (2) for all $0 < n < N\beta$.

To prove the second step is easy. (2) implies $y_n \approx y_l$ for all $n/N \approx l/N$, $n, l < N\beta$. Thus $Y_n$ represents a function $x_t$. It is really the restriction $x_t$ of a real function $x(t)$, as defined before, because there is a shift in the definition of $Y_n$ with respect to $Y_{n-1}$.

For the third step – namely $x(t)$ is a solution of the RFDE (1) for $0 < t < \beta$ – let $0 < n$ and $t = {}^\circ(n/N) < \beta$.We have

$$\begin{aligned} x(t) \approx y_n &= y_{n-1} + \frac{1}{N} G\left(\frac{n-1}{N}, Y_{n-1}\right) \\ &= y_0 + \frac{1}{N}\sum_{j=0}^{n-1} G\left(\frac{j}{N}, Y_j\right) \approx y_0 + \frac{1}{N}\sum_{j=0}^{n-1} {}^*f\left(\frac{j}{N}, {}^*x_{j/N}\right) \\ &\approx \Phi(0) + \int_0^t f(s, x_s)\, ds, \end{aligned}$$

where in the second but last step we used that $G$ is infinitely close to $f$, so that

$$G\left(\frac{j}{N}, Y_j\right) \approx f\left({}^\circ\left(\frac{j}{N}\right), x_{{}^\circ(j/N)}\right) \approx {}^*f\left({}^\circ\left(\frac{j}{N}\right), x_{{}^\circ(j/N)}\right) \approx {}^*f\left(\frac{j}{N}, {}^*x_{j/N}\right)$$

and $\{\|G(j/N, Y_j) - {}^*f(j/N, x_{j/N})\|_\infty : 0 \leq j \leq n-1\}$ as a hyper-finite internal set assumes its maximum.

Since in the above expression we have reals on both sides we have equality:

$$x(t) = \Phi(0) + \int_0^t f(s, x_s)\, ds.$$

$x(t)$ is a solution of (1) follows immediately. □

The proof of Proposition 1 is very elementary. For a more restricted class of RFDE's, one could prove this proposition simply by transfer. Indeed, the iteration describing the RFDE is nothing more than the numerical Euler-method for solving differential equations with an infinitesimal step width. Hence, if this method converges, by transfer the points $y_n$ lie infinitely close to the solution. (For more details on the Euler method see for example [4].)

Our Proposition 1 resembles the Stroboscopy Theorem of Benoit in [1], which has been formulated for ODE's and which, roughly stated, says the following. Given a sampling $(t_n, y_n)$, $0 \le n \le n_0$, where the time steps are infinitesimal $(t_n - t_{n-1} \approx 0)$, and $(y_n - y_{n-1})/(t_n - t_{n-1}) \approx f(y_{n-1})$, then $(y_{n_0}, \dots, y_0)$ represents a function $x(t)$ which is a solution of the ODE $\dot{x} = f(x)$.

Proposition 1 gives only a local solution. Of course, if $Y_n$ represents a function in $\Omega$ for all $0 < n < N\beta$, i.e. if $x_\beta \in \Omega$, then one can apply Proposition 1 again to extend the interval of existence.

We will always assume $\beta$ in Proposition 1 to be maximal, i.e. there is no continuation of the solution $x(t)$ on any interval containing $[0, \beta[$ ($\beta = \infty$ is allowed).

Example 1. (i) If $f(t, x_t) = g(\int_{-r}^0 x_t(\theta)\, d\theta)$, $g$ a continuous function, then it is possible to choose $G(n/N, Y) = {}^*g((1/N) \sum_{j=0}^{M-1} y_{-j})$. Indeed, if $Y = (y_0, \dots, y_{-M+1})^t$ represents a continuous function $\Phi$, then the sum is infinitely close to the integral over $\Phi$, hence $g$ being continuous, this $G$ is infinitely close to $f$.

(ii) If $\tau_1, \dots, \tau_m \in [-r, 0]$ are fixed numbers, $g$ is continuous, and $f(t, x_t) = g(x_t(\tau_1), \dots, x_t(\tau_m))$, then it is possible to choose

$$G(n/N, Y_n) = {}^*g(y_{n_1}, \dots, y_{n_m}),$$

where $n_i/N \approx \tau_i$ for all $i$.

## 3. The linear case

In this section we will assume $f$ to be defined on $\mathbb{R} \times C$ and to be linear, so we get the equation

$$x'(t) = L(t)x_t = \int_{-r}^0 d[\eta(t, \theta)]x(t + \theta), \quad t > 0, \tag{3}$$

where the $d \times d$ matrix function $\eta(t, \theta)$ is measurable and of bounded variation in $\theta$ on $[-r, 0]$ for each $t \ge 0$. Furthermore, we assume there is a function $m\colon \mathbb{R}_+ \to \mathbb{R}$, Lebesgue integrable on each compact set, such that

$$\mathrm{Var}_{[-r,0]}\eta(t, \cdot\,) \le m(t).$$

We then have a global unique solution $x\colon [-r, \infty[ \to \mathbb{R}^d$ of equation (3) (see e.g. [7, Theorem 1.1, Chapter 6]).

With the remark after Proposition 1, we can assume $Y_n$ defined in this proposition to exist for all $n \in {}^*\mathbb{N}$, and to represent $x_t$, if $n/N \approx t$ is finite. Note, however, that we assume the existence of a solution on the whole of $\mathbb{R}$ only for convenience, everything works also for a function $f$ defined only for $t \in [0, T_f[$.

We are going to have a closer look at the properties of the iteration describing an RFDE introduced in Proposition 1 in the special case of a linear $f$.

Of course, we want the map $G$ mentioned in the proposition to share this linearity. So we start by defining a special linear $G_f$. Subsequently we will give sufficient conditions for how one can change a linear $G$ describing the solution of equation (3) without loosing this property.

To any $Y = (y_0, \dots, y_{-M+1})^t \in {}^*\mathbb{R}^{dM}$ we want to assign a $^*$continuous function $P(Y)\colon {}^*[-r,0] \to {}^*\mathbb{R}^d$. We do this by joining linearly in ${}^*\mathbb{R} \times {}^*\mathbb{R}^d$ the points $(j/N, y_j)$, for all $-r \le j/N \le 0$, and $(-r, y_{-M+1})$. Given a similar set of points in $\mathbb{R} \times \mathbb{R}^d$ and joining them linearly, we get a graph of a continuous function. Hence in our case we get a $*$-continuous function defined on ${}^*[-r,0]$, to which we can apply ${}^*f$.

Define $G_f$ by

$$G_f(n/N, Y) := {}^*f(n/N, P(Y)). \tag{4}$$

If $Y \stackrel{\triangle}{=} \Phi \in C$, then $P(Y)(\theta) \approx {}^*\Phi(\theta)$ for all $\theta \in {}^*[-r,0]$, and the continuity of $f$ implies ${}^*f(n/N, P(Y)) \approx f({}^\circ(n/N), \Phi)$ for $n/N \approx t \in \mathbb{R}$. Hence $G_f$ is infinitely close to $f$, and we can apply Proposition 1.

For fixed $n \in \mathbb{N}$ the linear $y_n \mapsto (y_n + (1/N)G(n/N, y_n), y_n \dots, y_{n-M+2})^t$ can be represented by a matrix, say $A_{f,n} \in {}^*\mathbb{R}^{dM\times dM}$. If $0,\ E_d,\ L_{j,n} \in {}^*\mathbb{R}^{d\times d}$, $0 \le j \le M-1$, $0$ and $E_d$ are the 0-matrix and the unit-matrix, respectively, we can write

$$A_{f,n} = \begin{pmatrix} E_d + \frac{1}{N}L_{f,0,n} & \frac{1}{N}L_{f,1,n} & \frac{1}{N}L_{f,2,n} & \cdots & \frac{1}{N}L_{f,M-1,n} \\ E_d & 0 & 0 & \cdots & 0 \\ 0 & E_d & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & E_d & 0 \end{pmatrix}_{dM\times dM} \tag{5}$$

We summarize what we just did in.

PROPOSITION 2. *Let $f$ be as in equation* (3)*, $G_f$ and $A_{f,n}$ be as in* (4)*, resp.* (5)*. Then for any $\Phi \in C$, $\Phi \stackrel{\triangle}{=} Y_0 \in {}^*\mathbb{R}^{dM}$ the iteration*

$$Y_n = A_{f,n-1} Y_{n-1}, \quad n > 0$$

*solves equation* (3) *in the sense, that for $n/N \approx t \in \mathbb{R}$ we have $Y_n \stackrel{\triangle}{=} x_t$, and $x \in C([-r,\infty[, \mathbb{R}^d)$ is a solution of* (3)*. We will call any $A_n$ for which above iteration solves* (3) (*in the sense of Proposition* 1) *a describing matrix of equation* (3)*, and say that the corresponding iteration describes this* RFDE*. If $K_t = \|f(t,\cdot)\|_\infty$, then for $n/N \approx t$, $L_{f,j,n} = (L_{f,j,n,k,l})_{1\le k,l\le d}$,*

$$^\circ\!\left(\max_{1\le k\le d} \sum_{j=0}^{M-1} \sum_{l=1}^{d} |L_{f,j,n,k,l}|\right) \le \limsup_{\tau\to t} K_\tau. \tag{6}$$

*Furthermore, if* $A_n \in {}^*\mathbb{R}^{dM\times dM}$ *is an arbitrary describing matrix of* (3)*, then it has the same form as* $A_{f,n}$*, namely as in* (5) *with matrix-coefficients* $L_{j,n} = (L_{j,n,k,l})_{1\le k,l\le d}$*,* $0 \le j \le M-1$*. These coefficients satisfy for* $n/N \approx t$

$$^\circ\Big(\max_{1\le k\le d}\sum_{j=0}^{M-1}\sum_{l=1}^{d}|L_{j,n,k,l}|\Big) \ge K_t. \tag{7}$$

*Also* $\sum_{j=0}^{M-1}\|L_{j,n}\|_\infty$ *is finite for these* $n$.

Proof. We have already shown everything but the inequality (6) and the conclusions concerning $A_n$. To show (6) fix $n \in {}^*\mathbb{N}$, $n/N \approx t \in \mathbb{R}$, then

$$\max_{1\le k\le d}\sum_{j=0}^{M-1}\sum_{l=1}^{d}|L_{f,j,n,k,l}| = \|G_f(n/N,\,\cdot\,)\|_\infty \le \|{}^*f(n/N,\,\cdot\,)\|_\infty\cdot\|P\|_\infty$$

and $^\circ\|{}^*f(n/N,\,\cdot\,)\|_\infty \le \limsup_{\tau\to t} K_\tau$ implies (6).

If $A_n$ corresponds to a $G(n/N,\,\cdot\,)$ in Proposition 1, then $Y_{n+1} - Y_n = (1/N)G(n/N, Y_n)e_1$ and $A_n$ is as in (5).

$\sum_{j=0}^{M-1}\|L_{j,n}\|_\infty$ has to be finite. Otherwise, let $y_{-j} \in {}^*\mathbb{R}^d$ be such, that $\|y_{-j}\|_\infty = 1$, $\|L_{j,n}y_{-j}\|_\infty = \|L_{j,n}\|_\infty$. $\sum_{j=0}^{M-1}\|L_{j,n}\|_\infty$ infinite implies, there is a $k_0 \in \{1,\dots,d\}$, such that $\sum_{j=0}^{M-1}|\sum_{l=1}^{d}L_{j,n,k_0,l}y_{-j,l}|$ is infinite. Choosing the right sign of $y_{-j}$ we see, that

$$\sum_{j=0}^{M-1}\Big|\sum_{l=1}^{d}L_{j,n,k_0,l}y_{-j,l}\Big| = \sum_{j=0}^{M-1}\sum_{l=1}^{d}L_{j,n,k_0,l}y_{-j,l}$$

is infinite, and

$$\widetilde{Y} = \frac{1}{\sum_{j=0}^{M-1}\sum_{l=1}^{d}L_{j,n,k_0,l}y_{-j,l}}(y_0,\dots,y_{-M+1})$$

satisfies: $\widetilde{Y} \overset{\triangle}{=} \Phi \equiv 0$ and $G(n/N,\widetilde{Y}) \not\approx 0$. This is a contradiction to $G$ being infinitely close to $f$.

To show the inequality (7), let $1 > \varepsilon > 0$ (in $\mathbb{R}$) and choose an $x \in C$ such that $\|x\|_\infty = 1$ and $\|f(t,x)\|_\infty \ge (1-\varepsilon)\|f(t,\,\cdot\,)\|_\infty$. With $Y \overset{\triangle}{=} x$ and $n/N \approx t$ we get

$$^\circ\|G(n/N,Y)\|_\infty = \|f(t,x)\|_\infty \ge (1-\varepsilon)\|f(t,\,\cdot\,)\|_\infty = (1-\varepsilon)K_t$$

and (7) follows immediately. □

In the last inequality of Proposition 2 we can have a strict greater. Indeed, the next lemma will make clear, that $\sum_{j=0}^{M-1}\|L_{j,n}\|_\infty$ can be arbitrarily large in $\mathbb{R}$.

We now come to the question of how a describing matrix $A_n$ can be changed. Lemma 1 gives two ways to change the $L_{j,n}$, later (see Lemma 6) we will give a necessary and sufficient condition for two describing matrices of the same autonomous RFDE, which involves the behavior of the describing matrix $A$ on a certain set.

LEMMA 1. *For $L_n = (L_{0,n}, \dots, L_{M-1,n}) \in {}^*\mathbb{R}^{d\times dM}$, $n \in {}^*\mathbb{N}$, such that the resulting $G$ is internal, let $A_n(L_n)$ denote the matrix as in* (5). *Let $Y_0 = \widetilde{Y}_0 \triangleq \Phi \in C$ and construct for $L_n$, $\widetilde{L}_n$ sequences*

$$Y_{n+1} = A_n(L_n)Y_n, \quad \widetilde{Y}_{n+1} = A_n(\widetilde{L}_n)\widetilde{Y}_n \quad \text{for } n \in {}^*\mathbb{N}.$$

*If for arbitrary finite $n_0/N$, there is a $K_{n_0} \in \mathbb{R}$, such that for all $0 \le n \le n_0$ we have $\sum_{j=0}^{M-1} \|L_{j,n}\|_\infty \le K_{n_0}$, and if $\widetilde{L}_n$ satisfies one of the following conditions:*

(i) $\sum_{j=0}^{M-1} \|\widetilde{L}_{j,n} - L_{j,n}\|_\infty = C_n \approx 0$, *for all finite $n/N$,*

(ii) *for each $n$, $n/N$ finite, there are a $\theta = \theta(n) \in [-r, 0]$ and finitely many $0 \le j_1 < \dots < j_m \le M-1$, ${}^\circ(j_l/N) = \theta$, for all $1 \le l \le m = m(n)$, $\widetilde{L}_{j,n} = L_{j,n}$ for $j \in \{0, \dots, M-1\} \setminus \{j_1, \dots, j_m\}$, $\widetilde{L}_{j,n} - L_{j,n}$ is finite for $j \in \{j_1, \dots, j_m\}$ and*

$$\sum_{j=0}^{M-1} \widetilde{L}_{j,n} - L_{j,n} = \sum_{l=1}^{m} \widetilde{L}_{j_l,n} - L_{j_l,n} \approx 0,$$

*then $\widetilde{Y}_n \approx Y_n$ for finite $n/N$. In particular, if in this case $Y_n$ represents a solution $x_t$, then $\widetilde{Y}_n$ represents $x_t$ too.*

Roughly stated Lemma 1 says: one can change (for a fixed time $n$) all coefficients $L_{j,n}$ by infinitesimal amounts, if the sum of the absolute changes remains infinitesimal. Or one can change a finite number of coefficients which correspond to a fixed real time (all $j$ such that $j/N \approx \theta \in [-r, 0]$) by finite amounts, provided the sum of the changes (including signs) is infinitesimal. Of course, both techniques can be combined and applied any finite number of times.

PROOF. We show, that $Y_n \approx \widetilde{Y}_N$ for all $0 \le n \le n_0$, for any fixed $n_0$, $n_0/N$ finite.

Let $\delta_{j,n} = \widetilde{L}_{j,n} - L_{j,n}$, for $0 \le n \le n_0$, and $n_0/N \approx t_0 \in \mathbb{R}$.

$$\begin{aligned}
\widetilde{Y}_{n+1} &= A_n(\widetilde{L}_n)\widetilde{Y}_n \\
&= \left( A_n(L_n) + \frac{1}{N} \begin{pmatrix} \delta_{0,n} & \cdots & \delta_{M-1,n} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \right) (Y_n + (\widetilde{Y}_n - Y_n)) \\
&= Y_{n+1} + A_n(L_n)(\widetilde{Y}_n - Y_n) + \frac{1}{N} \sum_{j=0}^{M-1} \delta_{j,n} \widetilde{y}_{n-j} e_1.
\end{aligned}$$

Note, that $\sum_{j=0}^{M-1} \|L_{j,n}\|_\infty \le K_{n_0}$ implies $\|A_n\|_\infty \le 1 + K_{n_0}/N$. Note also, that wlog we can assume $K_{n_0} > 0$.

Proof that condition (i) is sufficient. The set $\{C_n : 0 \le n \le n_0\}$ is hyperfinite, hence it has a maximal element, say $C$, and $C \approx 0$.

$$\begin{aligned}
\|\widetilde{Y}_{n+1} - Y_{n+1}\|_\infty &\le \frac{1}{N}\sum_{j=0}^{M-1} \|\delta_{j,n}\widetilde{y}_{n-j}\|_\infty + \left(1 + \frac{K_{n_0}}{N}\right)\|\widetilde{Y}_n - Y_n\|_\infty \\
&\le \frac{C_n}{N}\|\widetilde{Y}_n\|_\infty + \left(1 + \frac{K_{n_0}}{N}\right)\|\widetilde{Y}_n - Y_n\|_\infty \\
&\le \frac{C}{N}\|Y_n\|_\infty + \left(1 + \frac{C+K_{n_0}}{N}\right)\|\widetilde{Y}_n - Y_n\|_\infty \\
&\le \underbrace{2\frac{C}{N}\max_{-r\le t\le t_0}\|x(t)\|_\infty}_{=:\delta/N} + \left(1 + \frac{C+K_{n_0}}{N}\right)\|\widetilde{Y}_n - Y_n\|_\infty \\
&\le \frac{\delta}{N}\sum_{n=0}^{n_0}\left(1 + \frac{C+K_{n_0}}{N}\right)^n \\
&= \frac{\delta}{N}\frac{1 - ((1+(C+K_{n_0})/N)^N)^{(n_0+1)/N}}{1-(1+(C+K_{n_0})/N)} \le \frac{\delta(e^{K_0 t_0}-1)}{C+K_{n_0}}.
\end{aligned}$$

Since $\delta \approx 0$ we have $\widetilde{Y}_{n+1} \approx Y_{n+1}$.

Proof that condition (ii) is sufficient. Let $\delta_n := \max\{\|\delta_{j,n}\|_\infty : 0 \le j \le M-1\}$ (exists, because $L_n$ and $\widetilde{L}_n$ are internal), and $j^{(n)} \in \{0,\dots,-M+1\}$ such that $j^{(n)}/N \approx \theta(n)$, and the set $\{j^{(n)} : 0 \le n \le n_0\}$ is internal. We have

$$\begin{aligned}
\|\widetilde{Y}_{n+1} - Y_{n+1}\|_\infty \le{}& \frac{1}{N}\left\|\sum_{j=0}^{M-1}\delta_{j,n}\widetilde{y}_{n-j}\right\|_\infty + \left(1+\frac{K_{n_0}}{N}\right)\|\widetilde{Y}_n - Y_n\|_\infty \\
\le{}& \frac{1}{N}\left\|\sum_{l=1}^{m}\delta_{j_l,n}y_{n-j^{(n)}}\right\|_\infty + \frac{1}{N}\left\|\sum_{l=1}^{m}\delta_{j_l,n}(y_{n-j_l} - y_{n-j^{(n)}})\right\|_\infty \\
&+ \frac{1}{N}\left\|\sum_{l=1}^{m}\delta_{j_l,n}(\widetilde{y}_{n-j_l} - y_{n-j_l})\right\|_\infty \\
&+ \left(1+\frac{K_{n_0}}{N}\right)\|\widetilde{Y}_n - Y_n\|_\infty.
\end{aligned}$$

Since all objects involved are internal, the following maxima exist:

$$\begin{aligned}
j_{\max} &= \max\{|j_l - j^{(n)}| : 1 \le l \le m(n),\ 0 \le n \le n_0\} \\
\delta^{(1)} &= 2\max_{-r\le t\le t_0}\|x(t)\|_\infty \max_{0\le n\le n_0}\left\|\sum_{l=1}^{m(n)}\delta_{j_l,n}\right\|_\infty, \\
\delta^{(2)} &= \max_{0\le n\le n_0}\left\{\sum_{l=1}^{m(n)}\|\delta_{j_l,n}\|_\infty\right\} \\
&\quad \cdot \max\{\|y_{l_1} - y_{l_2}\|_\infty : |l_1 - l_2| \le j_{\max},\ -M+1 \le l_1,\ l_2 \le n_0\},
\end{aligned}$$

$$\delta^{(3)} = \max_{0\le n\le n_0} \left\{ \sum_{l=1}^{m(n)} \|\delta_{j_l,n}\|_\infty \right\}.$$

We get

$$\|\widetilde{Y}_{n+1} - Y_{n+1}\|_\infty \le \frac{\delta^{(1)} + \delta^{(2)}}{N} + \left(1 + \frac{\delta^{(3)} + K_{n_0}}{N}\right)\|\widetilde{Y}_n - Y_n\|_\infty,$$

and keeping in mind $\delta^{(1)} \approx 0 \approx \delta^{(2)}$, $\delta^{(3)}$ finite, we can conclude the proof as in the former case. □

An easy example to illustrate Lemma 1 is the following:

Example 2. Each of the following $M \times M$ matrices describes the same RFDE $x'(t) = x(t-1)$, $t > 0$:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \frac{1}{N} \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & \cdots & 0 & \frac{2}{N} & \frac{-1}{N} \\ 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \cdots & 0 & 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 + \frac{1}{N^2} & \frac{1}{N^2} & \cdots & \frac{1}{N^2} & \frac{1}{N} + \frac{1}{N^2} \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

## 4. The linear autonomous case

In this section we assume the same conditions on $f$ as in the former section, with additionally $L$ being independent on $t$. That is we contemplate the linear autonomous RFDE

$$x'(t) = Lx_t = \int_{-r}^{0} d[\eta(\theta)]x(t+\theta) \quad \text{for } t > 0. \tag{8}$$

Proposition 2 states that in this case there is a matrix

$$A = \begin{pmatrix} E + \frac{L_0}{N} & \frac{L_1}{N} & \cdots & \cdots & \frac{L_{M-1}}{N} \\ E & 0 & \cdots & \cdots & 0 \\ 0 & E & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & E & 0 \end{pmatrix} \in {}^*\mathbb{R}^{dM\times dM} \tag{9}$$

which generates the describing sequence $Y_n = A^n Y_0$, $n > 0$, for initial value $Y_0 \stackrel{\triangle}{=} \Phi \in C$. As a comment, note that $A$ is not compact as defined in [2], although we shall see later, that it behaves very well.

By Proposition 2 we have

$$\sum_{j=0}^{M-1} \|L_j\|_\infty \leq K \in \mathbb{R}. \tag{10}$$

Let henceforth $K$ denote this bound (but note that $K$ depends on $L_j$, not only on the RFDE (8)).

The iteration $Y_n = A^n Y_0$ can be completely described if one knows the eigenvalues and (generalized) eigenvectors of $A$. The special form of $A$ in (9) allows an explicit formula for the characteristic polynomial as well as the (generalized) eigenvectors:

Lemma 2. *Let $A \in {}^*\mathbb{R}^{dM\times dM}$ be as in* (9)*. For $\lambda \in {}^*\mathbb{C}$ define*

$$B(\lambda) := (\lambda^M - \lambda^{M-1})E - \frac{1}{N}\sum_{j=0}^{M-1} \lambda^{M-1-j} L_j \in {}^*\mathbb{C}^{d\times d} \tag{11}$$

*and let $B'(\lambda)$ denote the component wise derivative of $B(\lambda)$. The characteristic polynomial of $A$ is*

$$p(\lambda) = \det(B(\lambda)). \tag{12}$$

*$v_0 = v_0(\lambda_0)$ is eigenvector to an eigenvalue $\lambda_0 \in {}^*\mathbb{C}$ if and only if it has the form*

$$v_0 = \begin{pmatrix} \lambda_0^{M-1} w_0 \\ \lambda_0^{M-2} w_0 \\ \vdots \\ \lambda_0 w_0 \\ w_0 \end{pmatrix} \in {}^*\mathbb{C}^{dM}. \tag{13}$$

*$v_1 = v_1(\lambda_0)$ is a generalized eigenvector of order 1 if and only if it has the form*

$$v_1 = \begin{pmatrix} (M-1)\lambda_0^{M-2} w_0 \\ (M-2)\lambda_0^{M-3} w_0 \\ \vdots \\ 2\lambda_0 w_0 \\ w_0 \\ 0 \end{pmatrix} + \begin{pmatrix} \lambda_0^{M-1} w_1 \\ \lambda_0^{M-2} w_1 \\ \vdots \\ \lambda_0 w_1 \\ w_1 \end{pmatrix} \in {}^*\mathbb{C}^{dM},$$

*where $0 \neq w_0$, $w_1 \in {}^*\mathbb{C}^d$ satisfy $B(\lambda_0)w_0 = 0 = B'(\lambda_0)w_0 + B(\lambda_0)w_1$.*

Proof. A straightforward computation shows: $(A - \lambda E)v_0 = 0 \Rightarrow v_0$ is as stated in (13). In particular, if $\lambda_0$ is an eigenvalue, $p(\lambda)$ defined in (12) has a root at $\lambda_0$.

A similar computation shows, that $(A - \lambda_0 E)v_1 = v_0$, $v_0$ as in (13), $v_1 = (v_{1,M-1}, \dots, v_{1,0})^t \in {}^*\mathbb{C}^{dM}$, implies

$$v_{1,j} = j\lambda_0^{j-1} w_0 + \lambda_0^j v_{1,0}, \quad 1 \le j \le M-1,$$
$$w_0 \lambda_0^{M-1} = v_{1,M-1}(1 - \lambda_0) + \frac{1}{N} \sum_{j=0}^{M-1} L_j v_{1,M-1-j}.$$

Both equations together give

$$v_1 = \begin{pmatrix} (M-1)\lambda_0^{M-2} w_0 \\ (M-2)\lambda_0^{M-3} w_0 \\ \vdots \\ 2\lambda_0 w_0 \\ w_0 \\ 0 \end{pmatrix} + \begin{pmatrix} \lambda_0^{M-1} v_{1,0} \\ \lambda_0^{M-2} v_{1,0} \\ \vdots \\ \lambda_0 v_{1,0} \\ v_{1,0} \end{pmatrix}$$

and

$$B'(\lambda_0) w_0 = -B(\lambda_0) v_{1,0}$$

which proves $v_1$ to have the desired form.

Now assume for a moment, that there are only simple eigenvalues, i.e. there are $dM$ distinct eigenvalues. Then $p(\lambda)$ defined above has $dM$ roots, and being a normalized polynomial of degree $dM$ it has to be the characteristic polynomial of $A$. By continuity this remains true if we no longer have simple eigenvalues. Thus $p(\lambda)$ is the characteristic polynomial and the proof is complete. $\square$

Note, that it is possible to derive a formula for the case of generalized eigenvectors of order m too, but since this is tedious and we don't need it we have not done it.

Henceforth we will denote with $p(\lambda)$, $B(\lambda)$ and $v = v(\lambda)$ always the characteristic polynomial and matrix of $A$, and its eigenvectors, respectively. We will also say this not only with respect to to $A$ but to the RFDE (8) they describe. But note, that in the latter case neither $p(\lambda)$ nor $B(\lambda)$ nor $v$ are uniquely determined. We will show later how they might differ (see Lemmas 5 and 6).

$Y_n = A^n Y_0$ can be described explicitly depending only on $n$, the eigenvalues and generalized eigenvectors of $A$, if one has a formula for the coefficients in the representation of $Y_0$ with respect to a basis of generalized eigenvectors. This can be done if there is a basis of eigenvectors. But this we can assume to be the case (using Lemma 1).

LEMMA 3. *Let $\lambda_1, \dots, \lambda_{dM}$ be the eigenvalues of $A$ with corresponding eigenvectors $v_1, \dots, v_{dM} \in {}^*\mathbb{C}^{dM}$, $w_1, \dots, w_{dM} \in {}^*\mathbb{C}^d$ as in* (13). *Assume, that the eigenvectors form a basis of ${}^*\mathbb{C}^{dM}$. Let $B(\lambda)$ be as in* (11), *and define $b_j \in {}^*\mathbb{C}^{d\times d}$ by $B(\lambda) = \sum_{j=0}^{M} \lambda^{M-j} b_j$. Then for each $1 \le j \le dM$, there is* *a* $\widetilde{w}_j^t \in {}^*\mathbb{C}^d$*, such that* $\widetilde{w}_j B'(\lambda_j) w_j \neq 0$ *and* $\widetilde{w}_j \perp B(\lambda_j)({}^*\mathbb{C}^d)$*. Given such* $\widetilde{w}_j$*, any* $Y = (y_0, \dots, y_{-M+1})^t \in {}^*\mathbb{C}^{dM}$ *has a unique representation*

$$Y = \sum_{j=1}^{dM} \alpha_j v_j,$$

*where for* $j = 1, \dots, dM$

$$\begin{aligned}(14) \quad \alpha_j = \frac{\widetilde{w}_j K(Y, \lambda_j)}{\widetilde{w}_j B'(\lambda_j) w_j} &= \frac{\widetilde{w}_j \sum_{l=0}^{M-1} \sum_{k=0}^{l} b_k \lambda_j^{l-k} y_{-l}}{\widetilde{w}_j B'(\lambda_j) w_j} \\ &= \frac{\widetilde{w}_j (y_0 + \sum_{l=1}^{M-1} (\lambda_j^l - \lambda_j^{l-1}) y_{-l} - (1/N) \sum_{l=1}^{M-1} \sum_{k=0}^{l-1} \lambda_j^{l-1-k} L_k y_{-l})}{\widetilde{w}_j B'(\lambda_j) w_j}.\end{aligned}$$

Proof. First we show, that such vectors $\widetilde{w}_j^t \in {}^*\mathbb{C}^d$ exist. For this it is sufficient to show, that $B'(\lambda_j) w_j \notin B(\lambda_j)({}^*\mathbb{C}^d)$.

If this were not true, i.e. there were a $w \in {}^*\mathbb{C}^d, B(\lambda_j) w = B'(\lambda_j) w_j$, then Lemma 2 would imply the existence of a generalized eigenvector of order 1 to the eigenvalue $\lambda_j$, hence we could not have a basis of eigenvectors of $A$.

To prove formula (14) for the coefficients it is sufficient to show $\widetilde{w}_{j_0} K(v_j, \lambda_{j_0}) = 0$ for $j_0 \neq j$, and $K(v_j, \lambda_j) = B'(\lambda_j) w_j$, $j_0, j = 1, \dots, dM$. Indeed, for $j_0 \neq j$,

$$\begin{aligned}\widetilde{w}_{j_0} K(v_j, \lambda_{j_0}) &= \widetilde{w}_{j_0} \sum_{k=0}^{M-1} \sum_{l=k}^{M-1} b_k \lambda_{j_0}^{l-k} \lambda_j^{M-1-l} w_j \\ &= \frac{\widetilde{w}_{j_0}}{\lambda_j - \lambda_{j_0}} \sum_{k=0}^{M-1} b_k (\lambda_j^{M-k} - \lambda_{j_0}^{M-k}) w_j \\ &= \frac{\widetilde{w}_{j_0}}{\lambda_j - \lambda_{j_0}} (B(\lambda_j) - B(\lambda_{j_0})) w_j = 0\end{aligned}$$

and

$$K(v_j, \lambda_j) = \sum_{k=0}^{M-1} b_k (M-k) \lambda_j^{M-1-k} w_j = B'(\lambda_j) w_j. \qquad \square$$

Remark 1. Every $\widetilde{w}_j$ of Lemma 3 is an eigenvector to the eigenvalue 0 of $B^t(\lambda_j)$. If $\lambda_j$ is a simple root, then any eigenvector $0 \neq \widetilde{w}_j$ of $B^t(\lambda_j)$ to the eigenvalue 0 suffices in Lemma 3.

Proof. We only have to show $\widetilde{w}_j B'(\lambda_j) w_j \neq 0$, if $\lambda_j$ is a simple root of $p(\lambda)$. So assume wlog $\widetilde{w}$ and $w$ to be eigenvectors to the eigenvalue 0 of $B^t(\lambda)$ and $B(\lambda)$, respectively. We can write ${}^*\mathbb{C}^d = \mathrm{lin}\,\{\widetilde{w}^t, \widetilde{v}_2, \dots, \widetilde{v}_d\}$, where $\mathrm{lin}\,\{\widetilde{v}_2, \dots, \widetilde{v}_d\} = B(\lambda)({}^*\mathbb{C})$, i.e. there are independent $v_2, \dots, v_d$, such that $\widetilde{v}_l = B(\lambda) v_l$. There are numbers $\alpha_1, \dots, \alpha_d$, such that $B'(\lambda) w = \alpha_1 \widetilde{w}^t + \sum_{l=2}^{d} \alpha_l \widetilde{v}_l$.

Now, if $\widetilde{w}B'(\lambda)w = 0$, then

$$0 = \widetilde{w}B'(\lambda)w - \widetilde{w}B(\lambda)\sum_{l=2}^{d}\alpha_l v_l = \widetilde{w}\alpha_1\widetilde{w}^t$$

and $\alpha_1 = 0$ follows. Hence in this case $0 = B'(\lambda)w + B(\lambda)(-\sum_{l=2}^{d}\alpha_l v_l)$, and with Lemma 2 $\lambda$ is an eigenvalue of order at least two, where we assumed it to be a simple one. $\square$

Now, changing the $L_j$ slightly, we can assume to have a basis of eigenvectors and decompose ${}^*\mathbb{C}^{dM}$ as in Lemma 3. It would be nice if eigenvectors would represent functions, but this cannot be expected because we have too many of them. Indeed, if for example $\lambda = 0$ is an eigenvalue, the corresponding eigenvector is $(0,\dots,0,w)^t \in {}^*\mathbb{R}^{dM}$, $w \in {}^*\mathbb{R}^d$, which does not represent any real function. On the other hand, any contribution due to this eigenvector decays very rapidly. The next lemma shows that this is typical in the sense, that either an eigenvector represents a function or decays very fast.

LEMMA 4. *Let $\lambda_0 = (1+\varepsilon_0/N)e^{i\varphi_0/N}$, $\varepsilon_0,\varphi_0 \in {}^*\mathbb{R}$, $\varphi_0/N \in [-\pi,\pi]$, be a root of $p(\lambda)$ with $\varepsilon_0$ positive or finite. Then $N(\lambda_0 - 1)$ is finite and ${}^\circ\varepsilon_0 \le K$, where $K$ is the bound in* (10)*. The corresponding eigenvector $v_0$ defined in* (13) *represents a function:*

$$v_0 \triangleq e^{\mu_0(\theta+r)}\xi_0\colon [-r,0] \to \mathbb{C}^d,$$

*where $w_0 \approx \xi_0 \in \mathbb{C}^d$, and $N(\lambda_0 - 1) \approx \mu_0$, or equivalently $\varepsilon_0 + i\varphi_0 \approx \mu_0$.*

PROOF. First note, that if $\varepsilon_0$ and $\varphi_0$ finite, then for $n/N \approx t \in \mathbb{R}$:

$$\lambda_0^n = ((1+\varepsilon_0/N)^N)^{n/N}e^{i\varphi_0 n/N} \approx e^{\varepsilon_0 t}e^{i\varphi_0 t}.$$

Hence in this case $v_0 = (\lambda_0^{M-1}w_0,\dots,\lambda_0 w_0, w_0)^t \in {}^*\mathbb{C}^{dM}$ represents $e^{\mu_0(\theta+r)}\xi_0$.

Using a diagonal matrix $D \in {}^*\mathbb{R}^{dM}$ with entries $E, cE, c^2E,\dots,c^{M-1}E$, $E \in \mathbb{R}^{d\times d}$ the unit-matrix, and applying Gershgorin's Principle to $D^{-1}AD$, we find, that all eigenvalues $\lambda$ satisfy at least one of the following inequalities $(L_j = (L_{j,l,k})_{1\le l,k\le d})$:

$$\left|\lambda - 1 - \frac{1}{N}L_{0,l_0,l_0}\right| \le \frac{1}{N}\sum_{k=1}^{d}\sum_{j=1}^{M-1}|c^j L_{j,l_0,k}| + \frac{1}{N}\sum_{\substack{k=1\\k\ne l_0}}^{d}|L_{0,l_0,k}|, \quad 1 \le l_0 \le d,$$

$$|\lambda| \le c^{-1}.$$

For $c = 1$, we get $|\lambda| \le 1$ or

$$|\lambda - 1| \le \frac{1}{N}\sum_{k=1}^{d}\sum_{j=0}^{M-1}|L_{j,l_0,k}| \le \frac{1}{N}\sum_{j=0}^{M-1}\|L_j\|_\infty \le \frac{K}{N},$$

and $N(|\lambda_0|-1)=\varepsilon_0\le K$ follows. For $c=(1-m/N)^{-1}$, $m\in\mathbb{N}$, we get $|\lambda|\le 1-m/N$ or

$$\begin{aligned}|\lambda-1| &\le \frac{1}{N}\sum_{k=1}^{d}\sum_{j=0}^{M-1}\left(1-\frac{m}{N}\right)^{-j}|L_{j,l_0,k}| \le \frac{K}{N}\left(1-\frac{m}{N}\right)^{-M+1}\\ &= \frac{K}{N}(e^{rm}+\text{infinitesimal}),\end{aligned}$$

and for $\varepsilon_0$ finite follows, $N(\lambda-1)$ is finite too. $\square$

We are interested in the standard solutions of RFDE. The next lemma links eigenfunctions of the RFDE (8) with the nonstandard description:

Lemma 5. *Let $\mu\in\mathbb{C}$, $\xi\in\mathbb{C}^d\setminus\{0\}$ and*

$$S_\mu=\{\lambda=(1+\varepsilon/N)e^{i\varphi/N}\in{}^*\mathbb{C} : {}^\circ\varepsilon=\operatorname{Re}\mu,\ {}^\circ\varphi=\operatorname{Im}\mu\}.$$

*The following are equivalent:*

(i) *$z(t)=e^{\mu(t+r)}\xi$ is a solution of equation (8),*
(ii) *there are $\lambda\in S_\mu$, $w\approx\xi$ such that $B(\lambda)w=0$,*
(iii) *for all $\lambda\in S_\mu$, $w\approx\xi : NB(\lambda)w\approx 0$,*
(iv) *there are $\lambda\in S_\mu$, $w\approx\xi : NB(\lambda)w\approx 0$,*
(v) *for every $\lambda_0\in S_\mu$, $w\approx\xi$, there exist $\Delta_j\in{}^*\mathbb{C}^{d\times d}$, $j=0,\dots,M-1$, $\sum_{j=0}^{M-1}\|\Delta_j\|_\infty\approx 0$, such that using $\widetilde{L}_j=L_j+\Delta_j$ to define an internal $\widetilde{B}(\lambda)$, we have $\widetilde{B}(\lambda_0)w=0$.*

Proof. (ii)$\Rightarrow$(iv) is trivial.

(iv)$\Rightarrow$(v). Let $\lambda_1$, $w_1$ be as in (iv) and fix $\lambda_0\in S_\mu$, $w_0\approx\xi$. We will change only $L_{M-1}$. That is we set $\Delta_0=\ldots=\Delta_{M-2}=0$, and will find a $\Delta_{M-1}=\Delta_{M-1}(\lambda_0,w_0)$, $\|\Delta_{M-1}\|_\infty\approx 0$, such that $\widetilde{B}(\lambda_0)w_0=0$.

Now if $NB(\lambda_0)w_0\approx 0$, then letting $i_0$ such that $|w_{0,i_0}|=\max\{|w_{0,j}| : j=1,\dots,d\}$, and setting $\Delta_{M-1}$ equal to the matrix, consisting of zeros and only the $i_0$-th column equal to

$$-\frac{N}{w_{0,i_0}}B(\lambda_0)w_0\in{}^*\mathbb{C}^d,$$

we get $(\Delta_{M-1}/N)w_0=-B(\lambda_0)w_0$, and thus $\widetilde{B}(\lambda_0)w_0=0$. Also

$$\|\Delta_{M-1}\|_\infty=\frac{N}{|w_{0,i_0}|}\|B(\lambda_0)w_0\|_\infty\approx 0.$$

So we only have to show $NB(\lambda_0)w_0\approx 0$. But

$$NB(\lambda_0)w_0=N(B(\lambda_0)-B(\lambda_1))w_0+NB(\lambda_1)(w_0-w_1)+NB(\lambda_1)w_1$$

and, for $B(\lambda) = (b_{i,j}(\lambda))_{1\le i,j\le d}$, $\lambda \in S_\mu$,

$$\begin{aligned}|b'_{i,j}(\lambda)| &\le |M\lambda^{M-1} - (M-1)\lambda^{M-2}|\delta_{ij} + \frac{1}{N}\sum_{l=0}^{M-1}(M-1-l)|\lambda^{M-2-l}\|L_{l,i,j}| \\ &\le |\lambda^{M-2}|\left(N|\lambda-1|\frac{M}{N}+1\right)\delta_{i,j} + \frac{M}{N}(1+|\lambda|^M)\sum_{l=0}^{M-1}|L_{l,i,j}| \\ &\le (1+e^{r\operatorname{Re}\mu})(|\mu|r+2)\delta_{ij} + r(2+e^{r\operatorname{Re}\mu})\sum_{l=0}^{M-1}|L_{l,i,j}|,\end{aligned}$$

thus

$$\begin{aligned}&N\|B(\lambda_0)-B(\lambda_1)\|_\infty \\ &\qquad \le dN|\lambda_0-\lambda_1|\left[(1+e^{rRe\ \mu})(|\mu|r+2) + r(2+e^{rRe\ \mu})\sum_{l=0}^{M-1}\|L_l\|_\infty\right] \\ &\qquad = \text{infinitesimal}\cdot\text{finite}.\end{aligned}$$

Similarly one can show $N\|B(\lambda_1)\|_\infty$ to be finite, and we have $NB(\lambda_1)(w_0-w_1) \approx 0$. This together with the condition on $\lambda_1$ and $w_1$ show indeed $NB(\lambda_0)w_0 \approx 0$.

(v)$\Rightarrow$(i) follows immediate from Lemma 4.

(i)$\Rightarrow$(iii). Let $\lambda_0 \in S_\mu$, $w_0 \approx \xi$ and $z(t)$ as in (i). Then $z_0 \triangleq v_0 = (\lambda_0^{M-1}w_0, \dots, \lambda_0 w_0, w_0)^t$. We set $Y_0 := v_0$. An easy induction shows

$$\lambda_0^n v_0 - A^n Y_0 = \sum_{j=0}^{n-1}\lambda_0^j A^{n-1-j}\begin{pmatrix}B(\lambda_0)w_0\\0\\\vdots\\0\end{pmatrix}, \quad n \ge 1.$$

Since $\lambda_0^n v_0 \triangleq z_t \triangleq Y_n = A^n Y_0$, for $n/N \approx t \in \mathbb{R}$, we get

$$0 \approx \sum_{j=0}^{n-1}\lambda_0^j A^{n-1-j}\begin{pmatrix}B(\lambda_0)w_0\\0\\\vdots\\0\end{pmatrix}. \tag{15}$$

Let $\operatorname{proj}: {}^*\mathbb{C}^{dM} \to {}^*\mathbb{C}^d$ denote the projection onto the first $d$ coordinates. Then $\sum_{j=0}^{M-1}\|L_j\|_\infty \le K$ implies for any $Y \in {}^*\mathbb{C}^{dM}$ and $0 \le j \le n-1$:

$$\begin{aligned}&\|\lambda^j \operatorname{proj}(A^{n-1-j}Y - Y)\|_\infty \\ &\qquad \le |\lambda|^j\sum_{l=1}^{n-1-j}\|\operatorname{proj}(A(A^{l-1}Y) - A^{l-1}Y)\|_\infty \le |\lambda|^j\sum_{l=1}^{n-1-j}\frac{K}{N}\|A^{l-1}Y\|_\infty \\ &\qquad \le |\lambda|^j\frac{K}{N}\sum_{l=1}^{n-1-j}\left(1+\frac{K}{N}\right)^{l-1}\|Y\|_\infty \le |\lambda|^j\left(\left(1+\frac{K}{N}\right)^{n-j}-1\right)\|Y\|_\infty.\end{aligned}$$

There is a $n_1 \in {}^*\mathbb{N}$, $n_1/N \not\approx 0$, such that

$$\|\lambda_0^j \mathrm{proj}\,(A^{n-1-j}Y - Y)\|_\infty \leq \frac{1}{2}\|Y\|_\infty \quad \text{for all } 0 \leq j \leq n-1 \leq n_1.$$

Eventually decreasing $n_1$ slightly, we can assume $Re\ \lambda_0^j > 2/3$ for all $0 \leq j \leq n_1$. Using this in (15) we have

$$\begin{aligned}
0 &\approx \mathrm{proj}\left(\sum_{j=0}^{n_1-1} \lambda_0^j A^{n_1-1-j} \begin{pmatrix} B(\lambda_0)w_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right) \\
&= \sum_{j=0}^{n_1-1} \lambda_0^j \mathrm{proj}\left(A^{n_1-1-j} \begin{pmatrix} B(\lambda_0)w_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} B(\lambda_0)w_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right) + \sum_{j=0}^{n_1-1} \lambda_0^j B(\lambda_0)w_0 \\
&= \frac{n_1}{2}X + n_1 C_1 B(\lambda_0)w_0,
\end{aligned}$$

where $X \in {}^*\mathbb{C}^d$, $\|X\|_\infty \leq \|B(\lambda_0)w_0\|_\infty$ and $C_1 \in {}^*\mathbb{C}$, $|C_1| \geq 2/3$.

Now we get easily $\|B(\lambda_0)w_0\|_\infty \leq \text{infinitesimal}/n_1$ and $n_1/N \not\approx 0$ yields (iii).

(iii)$\Rightarrow$(ii). If $L_{M-1}$ gets changed to $L_{M-1} + \Delta$, then let $B(\lambda, \Delta)$, $\lambda_j(\Delta)$ and $w_j(\Delta)$ be defined accordingly. Fix $\lambda_0 \in S_\mu$, $w_0 \approx \xi$. As in the proof of (iv)$\Rightarrow$(v), there is a $\Delta \approx 0$ such that $B(\lambda_0, \Delta)w_0 = 0$.

Now assume, that for no $\lambda \in S_\mu$, $w \approx \xi$ we have $B(\lambda)w = 0$. Then there is an internal path $\Gamma(t)$ joining $\Delta$ and $0 \in {}^*\mathbb{C}^{d\times d}$, which induces continuous $\lambda(t)$, $B(\lambda, \Gamma(t))$, $w(t)$, $|w(t)| \equiv |w_0|$, satisfying $B(\lambda(t),\ \Gamma(t))w(t) = 0$ for all $0 \leq t \leq 1$, and $\lambda(0) = \lambda_0$, $B(\lambda,\ \Gamma(0)) = B(\lambda, \Delta)$, $w(0) = w_0$. Since by assumption we can't have $\lambda(1) \in S_\mu$ together with $w(1) \approx \xi$, the path $(\lambda(t), w(t))$ leaves $S_\mu \times \{w : w \approx \xi\}$.

We already know that (ii) implies (i), which applied to this situation gives us an infinite number of $\widetilde{\mu}$, $\widetilde{\xi}$ forming solutions of equation (8) as in (i). These $\widetilde{\mu}$ and $\widetilde{\xi}$ are in a neighbourhood of $\mu$, resp. $\xi$, which cannot be. $\square$

We can now use Lemma 5 to give a necessary and sufficient condition for two matrices $B(\lambda)$ as defined in Lemma 2 to belong to the same RFDE. At the same time we show, that if $\sum_{j=0}^{M-1} \|L_j\|_\infty$ finite, then $A$ (and $B(\lambda)$) describes an RFDE. Since, by Proposition 2, this sum is bounded if $A$ describes an RFDE, we have an equivalence: $\sum_{j=0}^{M-1} \|L_j\|_\infty$ is finite, if and only if $A$ describes a linear autonomous RFDE.

Lemma 6. *Let*

$$B(\lambda) = (\lambda^M - \lambda^{M-1})E - \frac{1}{N}\sum_{j=0}^{M-1} \lambda^{M-1-j} L_j \in {}^*\mathbb{R}^{d\times d}$$

*be internal and assume* $\sum_{j=0}^{M-1} \|L_j\|_\infty \le K \in \mathbb{R}$. *Then* $B(\lambda)$ *induces a bounded linear operator* $L\colon C \to \mathbb{R}^d$,

$$L\Phi = \int_{-r}^{0} d[\eta(\theta)]\Phi(\theta),$$

*where* $\eta(\theta)$, *defined in* (16) *below is of bounded variation.* $B(\lambda)$ *is the characteristic matrix defined in Lemma* 2 *for the equation* $x'(t) = Lx_t$, $L$ *as above. Moreover, if there are two matrices* $B_1$, $B_2$ *as described above, then there are equivalent:*

(i) $B_1(\lambda)$ *and* $B_2(\lambda)$ *are characteristic matrices to the same* (*linear autonomous*) RFDE: $x'(t) = Lx_t$, $t \ge 0$,

(ii) *if* $\lambda \in S := \{z \in {}^*\mathbb{C} : z = (1+\varepsilon/N)e^{i\varphi/N},\ \varepsilon, \varphi$ *finite*$\}$, *then*

$$\sum_{j=0}^{M-1} \lambda^{M-1-j} L_{1,j} \approx \sum_{j=0}^{M-1} \lambda^{M-1-j} L_{2,j}.$$

PROOF. We could prove in an abstract way that $B(\lambda)$ defines a linear bounded operator, but we prefer to construct $\eta(\theta)$ explicitly. For $-r \le \theta \le 0$ define $\eta(\theta)$ by

$$\eta(\theta) = \begin{cases} 0 & \theta \ge 0, \\ -{}^\circ\Big(\displaystyle\sum_{j=0}^{n} L_j\Big) & \text{where } n/N \le -\theta \text{ is maximal}, \ -r < \theta < 0, \\ -{}^\circ\Big(\displaystyle\sum_{j=0}^{M-1} L_j\Big) & \theta \le -r. \end{cases} \tag{16}$$

$\eta(\theta)$ is of bounded variation:

Let $-r = \theta_0 < \ldots < \theta_n = 0$ with corresponding $0 < n_{n-1} < \ldots < n_0 = M-1$, $n_n := -1$. Then

$$\sum_{l=1}^{n} \|\eta(\theta_l) - \eta(\theta_{l-1})\|_\infty = \sum_{l=1}^{n} \Big\| {}^\circ\Big( \sum_{j=n_l+1}^{n_{l-1}} L_j \Big) \Big\|_\infty \le {}^\circ\Big( \sum_{j=0}^{M-1} \|L_j\|_\infty \Big) \le K.$$

So we have a linear bounded operator $L\colon C \to \mathbb{R}^d$, $L\Phi = \int_{-r}^{0} d[\eta(\theta)]\Phi(\theta)$. To show, that $B(\lambda)$, respectively the corresponding matrix $A$ as in [9], describes the RFDE, let $C \ni \Phi \triangleq Y \in {}^*\mathbb{R}^{dM}$. Also let $\varepsilon > 0$ (in $\mathbb{R}$) and take a division $-r = \theta_0 < \ldots < \theta_n = 0$ such that

$$\Big\| \int_{-r}^{0} d[\eta(\theta)]\Phi(\theta) - \sum_{j=1}^{n} (\eta(\theta_j) - \eta(\theta_{j-1}))\Phi(\theta_j) \Big\|_\infty < \frac{\varepsilon}{2}$$

and (assume without loss of generality $K > 0$)

$$\|\Phi(t) - \Phi(s)\|_\infty < \frac{\varepsilon}{2K} \quad \text{for all } t, s \in [\theta_{j-1}, \theta_j],\ j = 1, \ldots, n.$$

Choosing $n_j$ in (16) to match $\theta_j$ $(n_n = -1)$

$$\begin{aligned}\Big\| \int_{-r}^{0} d[\eta(\theta)]\Phi(\theta) - \sum_{j=1}^{M-1} L_j y_{-j} \Big\|_\infty &\le \frac{\varepsilon}{2} + \Big\| \sum_{j=1}^{n} \sum_{l=n_j+1}^{n_{j-1}} L_l \Phi(\theta_j) - \sum_{j=0}^{M-1} L_j y_{-l} \Big\|_\infty \\ &\le \frac{\varepsilon}{2} + \sum_{j=1}^{n} \sum_{l=n_j+1}^{n_{j-1}} \|L_l\|_\infty \|\Phi(\theta_j) - y_{-l}\|_\infty \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2K} \sum_{j=0}^{M-1} \|L_j\|_\infty \le \varepsilon\end{aligned}$$

and $A$ indeed describes the RFDE.

Now we prove the equivalence (i)$\Rightarrow$(ii). Assume there is a $\lambda_0 \in S$, such that $\sum_{l=0}^{M-1}(L_{1,l} - L_{2,l})\lambda_0^{M-1-l} \not\approx 0$. An easy calculation shows: $N\|B_1(\lambda_0)\|_\infty$ is finite if $\operatorname{Im}\lambda_0 = 0$, then $\operatorname{Im} B_1(\lambda_0) = 0$ and if $\lambda_0 \approx 0$, then $\operatorname{Im} B_1(\lambda_0)/\operatorname{Im}\Lambda_0$ is finite. Hence there are finite $\Delta_1, \Delta_2 \in {}^*\mathbb{R}^{d\times d}$ such that $\lambda_0\Delta_1 + \Delta_2 = NB_1(\lambda_0)$.

Define $\widetilde{B}_j(\lambda) = B_j(\lambda) - (\lambda\Delta_1 + \Delta_2)/N$, $j = 1, 2$, then both $\widetilde{B}_j(\lambda)$ arise from the equation

$$x'(t) = Lx_t + {}^\circ(\Delta_1 + \Delta_2)x(t-r). \tag{17}$$

But $\widetilde{B}_1(\lambda_0) = 0$, $N\widetilde{B}_2(\lambda_0) = -\sum_{j=0}^{M-1}(L_{1,j} - L_{2,j})\lambda_0^{M-1-j} \not\approx 0$, and with Lemma 5 the first equation implies $e^{\mu_0\theta}\xi$ is a solution of (17) for arbitrary $\xi \in \mathbb{C}^d$, the latter that there is a $\xi \in \mathbb{C}^d$, such that $e^{\mu_0\theta}\xi$ is not a solution. This contradiction proves our claim.

(ii)$\Rightarrow$(i). $B_1$ and $B_2$ define two bounded linear operators $L_1, L_2\colon C \to \mathbb{R}^d$. The condition in (ii) implies $L_1(e^{\mu\theta}\xi) = L_2(e^{\mu\theta}\xi)$ for all $\mu \in \mathbb{C}$, $\xi \in \mathbb{R}^d$. But $\operatorname{lin}\{\xi e^{\mu\theta} : \mu \in \mathbb{R},\ \xi \in \mathbb{R}^d\}$ is dense in $C$, so $L_1$ and $L_2$ have to be equal on the whole of $C$. □

We know, that if an eigenvalue is "big", then the eigenvector represents a function, and if it is "small", the contribution due to this eigenvector decays rapidly. In principle we could have an infinite number of eigenvalues very close to each other, or we could have infinite coefficients, or the contribution of all eigenvectors to eigenvalues having an absolute value less than a given constant would give something big, and in each of this cases the respective partial sums would not represent a function. We want the possibility to decompose $C$ with respect to functions which are represented by eigenvectors, so we need to know if above mentioned cases really occur. Lemma 5 essentially says, that we need not worry. The contribution due to all eigenvalues having an absolute value less than a given constant $1 + \varepsilon/N$, $\varepsilon$ finite, is bounded by the exponential $e^{\varepsilon t}$ (with a finite coefficient). And the sum over all contributions coming from eigenvalues which correspond to the same exponent $\mu \in \mathbb{C}$, grows less rapidly than $e^{(\operatorname{Re}\mu+\varepsilon)t}$, $\varepsilon > 0$ an arbitrary real number. There can still be infinite coefficients, even in the case of "big" eigenvalues, but they cancel each other to give something finite (see also Lemma 9 in the one-dimensional case).

We need a technical lemma before stating mentioned results in Lemma 8.

LEMMA 7. *Let $\Gamma \subset {}^*\mathbb{C}$ be a closed simple positively oriented curve. Assume that no eigenvalue of $A$ lies on $\Gamma$ and that in its interior lie only simple roots of $p(\lambda)$, say $\lambda_1, \dots, \lambda_m$. Assume there is a basis of eigenvectors $v_1, \dots, v_{dM}$ of $A$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_{dM}$. Hence $Y_0 \in {}^*\mathbb{C}^{dM}$ has a representation $Y_0 = \sum_{j=1}^{dM} \alpha_j v_j$, $\alpha_j$ as in* (14). *Then, for $n \in {}^*\mathbb{N}$,*

$$\frac{1}{2\pi i}\int_\Gamma \lambda^n B^{-1}(\lambda) K(Y_0,\lambda) d\lambda = \sum_{j=1}^{m} \alpha_j \lambda_j^n w_j. \tag{18}$$

PROOF. It is sufficient to show (18) for $m = 1$, i.e. there is only one simple root $\lambda_1$ in the interior of $\Gamma$. $w_1$ and $\widetilde{w}_1$ are eigenvectors to the eigenvalue 0 of $B(\lambda_1)$ respectively $B^t(\lambda_1)$, so we can extend them to form a basis $w_1, \dots, w_d$, resp. $\widetilde{w}_1, \dots, \widetilde{w}_d$, such that

$$\begin{pmatrix} \widetilde{w}_1 \\ \vdots \\ \widetilde{w}_d \end{pmatrix} B(\lambda)(w_1 \dots w_d) = \widetilde{W} B(\lambda) W = \begin{pmatrix} d_{1,1}(\lambda) & \cdots & d_{1,d}(\lambda) \\ \vdots & \ddots & \vdots \\ d_{d,1}(\lambda) & \cdots & d_{d,d}(\lambda) \end{pmatrix} = D(\lambda),$$

and for $\lambda \to \lambda_1$ $D(\lambda)$ tends to the Jordan normal form of $B(\lambda_1)$, i.e. $d_{j,j}(\lambda) \to d_{j,j}(\lambda_1) \neq 0$, $j = 2, \dots, d$, $d_{j,j+1}(\lambda) \to d_{j,j+1}(\lambda_1) \in \{0, 1\}$, $j = 2, \dots, d-1$, and all other entries $d_{i,j}(\lambda)$ tend to 0. Indeed, $\lambda_1$ being a simple root of $\det(B(\lambda))$ implies 0 being a simple eigenvalue of $B(\lambda_1)$, and $d_{1,1}(\lambda_1)$ has a simple root at $\lambda_1$.

For $\lambda \neq \lambda_1$ on $\Gamma$ or in its interior, $D^{-1}(\lambda) = (c_{i,j}(\lambda))$ exists. Taking into account, that all $c_{i,j}(\lambda)$ are meromorphic functions, it is straightforward to show, that $c_{1,1}(\lambda)$ has a simple pole at $\lambda_1$ and all other $c_{i,j}(\lambda)$ are holomorphic at $\lambda_1$. Moreover, $c_{1,1}(\lambda) d_{1,1}(\lambda) \to 1$ $(\lambda \to \lambda_1)$, hence $\operatorname{res}_{\lambda_1} c_{1,1}(\lambda) = (d'_{1,1}(\lambda_1))^{-1}$. Now

$$\begin{aligned} \frac{1}{2\pi i}\int_\Gamma \lambda^n B^{-1}(\lambda) K(Y_0,\lambda)\, d\lambda &= \frac{1}{2\pi i}\int_\Gamma \lambda^n W D^{-1}(\lambda) \widetilde{W} K(Y_0,\lambda)\, d\lambda \\ &= \lambda_1^n W \begin{pmatrix} (d'_{1,1}(\lambda_1))^{-1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \widetilde{W} K(Y_0,\lambda_1) \\ &= \frac{{\lambda_1}^n}{d'_{1,1}(\lambda_1)} (\widetilde{w}_1 K(Y_0,\lambda_1)) w_1. \end{aligned}$$

On the other hand, $D'(\lambda) = \widetilde{W} B'(\lambda) W$ implies $d'_{1,1}(\lambda) = \widetilde{w}_1 B'(\lambda_1) w_1$ and (18) follows immediately with (14). $\square$

Lemma 8. *Assume $p(\lambda)$ to have only simple roots $\lambda_1, \dots, \lambda_{dM}$, so that we have a basis of eigenvectors $v_1, \dots, v_{dM} \in {}^*\mathbb{C}^{dM}$ with corresponding $w_1, \dots, w_{dM} \in {}^*\mathbb{C}^d$ as in* (13). *Let $\widetilde{\lambda} = (1 + \widetilde{\varepsilon}/N)e^{i\widetilde{\varphi}/N} \in {}^*\mathbb{C}$, $\widetilde{\varepsilon}$, $\widetilde{\varphi}$ finite, and $Y_0 \in {}^*\mathbb{C}^{dM}$ be a vector. Write $Y_0 = \sum_{j=1}^{dM} \alpha_j v_j$ as in Lemma* 3.

(i) *For $0 < \rho_1 \in \mathbb{R}$ small enough*

$$\widetilde{y}_n := \sum_{\substack{j=0 \\ N(\lambda_j - \widetilde{\lambda}) \approx 0}}^{dM} \alpha_j \lambda_j^{M-1+n} w_j = \sum_{\substack{j=0 \\ N|\lambda_j - \widetilde{\lambda}| < \rho_1}}^{dM} \alpha_j \lambda_j^{M-1+n} w_j$$

*satisfies for $-M+1 \le n$, $n/N$ finite,*

$$\|\widetilde{y}_n\|_\infty \le \widetilde{C}\left(1 + \frac{\widetilde{\varepsilon} + \rho_1}{N}\right)^n \|Y_0\|_\infty \tag{19}$$

*for a $\widetilde{C} \in \mathbb{R}$. In particular, $\widetilde{y}_n$ is finite for these $n$, if $Y_0$ has finite components.*

(ii) *Let $\rho_2 \in \mathbb{R}$. Assume there are no roots of $p(\lambda)$ with $N(|\lambda| - 1) \approx \rho_2$. Then*

$$\widehat{y}_n := \sum_{\substack{j=0 \\ |\lambda_j| < 1 + \rho_2/N}}^{dM} \alpha_j \lambda_j^{M-1+n} w_j$$

*satisfies for $-M+1 \le n$, $n/N$ finite*

$$\|\widehat{y}_n\|_\infty \le \widehat{C}\left(1 + \frac{\rho_2}{N}\right)^n \|Y_0\|_\infty \tag{20}$$

*for a $\widehat{C} \in \mathbb{R}$. In particular, $\widehat{y}_n$ is finite for these $n$, if $Y_0$ has finite components.*

Proof. We will show (i) and (ii) by contour-integration. Let $K$ be as the bound in (10) and assume wlog $K > 0$. If $\Gamma \subset {}^*\mathbb{C}$ is a closed simple positively oriented curve which does not contain any roots of $p(\lambda)$ we have by Lemma 7, for $n \in {}^*\mathbb{N}$,

$$\sum_{j:\lambda_j \text{ in interior of } \Gamma} \alpha_j \lambda_j^n w_j = \frac{1}{2\pi i} \int_\Gamma \lambda^n B^{-1}(\lambda) K(Y_0, \lambda)\, d\lambda.$$

We shall bound the contour-integral to show (19) and (20), with $\Gamma$ a suitable circle.

Proof of part (i). First note, that by Lemma 5 there cannot be eigenvalues $\lambda$ with $0 \ne {}^\circ(N(\lambda - \widetilde{\lambda}))$ arbitrarily small, hence for $0 < \rho_1 \in \mathbb{R}$ small enough: $N|\lambda_j - \widetilde{\lambda}| < \rho_1 \Leftrightarrow N(\lambda_j - \widetilde{\lambda}) \approx 0$, and both sums in (i) are equal. In particular, we can choose $\Gamma$ to be the circle $|\lambda - \widetilde{\lambda}| = \rho_1/(2N)$, and for all $\lambda \in \Gamma$, $0 \ne \xi \in \mathbb{C}^d$ we have $NB(\lambda)\xi \not\approx 0$. Let

$$\delta_1 := \min\{\|B(\lambda)w\|_\infty : \lambda \in \Gamma,\ w \in {}^*\mathbb{C}^d,\ \|w\|_\infty = 1\},$$

then $N\delta_1 \not\approx 0$ and $\|B^1(\lambda)\|_\infty \le 1/\delta_1$.

To bound $K(Y_0, \lambda)$, note that for $\lambda \in \Gamma$

$$
\begin{aligned}
\|K(Y_0,\lambda)\|_\infty &= \left\| \sum_{l=0}^{M-1} (\lambda^l - \lambda^{l-1}) y_{-l} - \frac{1}{N} \sum_{l=1}^{M-1} \sum_{k=1}^{l} L_{k-1} \lambda^{l-k} y_{-l} \right\|_\infty \\
&\le M(1+|\lambda|^{M-1})|\lambda - 1| \|Y_0\|_\infty + \frac{1}{N}(1+|\lambda|^{M-2})(M-1)\|Y_0\|_\infty K \\
&\le C_1 \|Y_0\|_\infty
\end{aligned}
$$

for a constant $C_1 \in \mathbb{R}$. Now we get the estimate on $\widetilde{y}_n$:

$$
\begin{aligned}
\|\widetilde{y}_n\|_\infty &= \left\| \sum_{\substack{j=0 \\ N|\lambda_j - \widetilde{\lambda}| < \rho_1}}^{dM} \alpha_j \lambda_j^{M-1+n} w_j \right\|_\infty \\
&= \frac{1}{2\pi} \left\| \int_{|\lambda - \widetilde{\lambda}| = \rho_1/(2N)} \lambda_j^{M-1+n} B^{-1}(\lambda) K(Y_0, \lambda)\, d\lambda \right\|_\infty \\
&\le \frac{\rho_1}{2N} \left(1 + \frac{\widetilde{\varepsilon}}{N} + \frac{\rho_1}{2N}\right)^{M-1+n} \frac{1}{\delta_1} C_1 \|Y_0\|_\infty
\end{aligned}
$$

and $N\delta_1 \not\approx 0$ gives (19).

Proof of part (ii). It is sufficient to show (20) for $\|Y_0\| = 1$, which we will assume for the rest of this proof. First we prove (20) for $-M+1 \le n \le 2M$ using part (i) of this lemma. There are only finitely many eigenvalues $\mu$ of the RFDE (8) with $\operatorname{Re}\mu \ge \rho_2$. For each of them

$$
\sum_{\substack{j=0 \\ N(\lambda_j - 1) \approx \mu}}^{dM} \alpha_j v_j
$$

is finite by part (i). Hence

$$
\sum_{\substack{j=0 \\ \operatorname{Re} N(\lambda_j - 1) \ge \rho_2}}^{dM} \alpha_j \lambda_j^{M-1+n} w_j, \qquad \widehat{y}_n = y_n - \sum_{\substack{j=0 \\ \operatorname{Re} N(\lambda_j - 1) \ge \rho_2}}^{dM} \alpha_j \lambda_j^{M-1+n} w_j
$$

are finite too, for $-M+1 \le n \le 0$, and thus also $\widehat{Y}_n = A^n(\widehat{y}_0, \dots, \widehat{y}_{-M+1})^t$, for $0 \le n \le 2M$. This shows, together with $\|Y_0\|_\infty = 1$, that (20) holds for these $n$.

For the rest of the proof let $n_0 \in {}^*\mathbb{N}$ be fixed, $n_0/N$ finite and $n_0 > 2M$. Let $\Gamma$ be the circle $|\lambda| = 1 + \rho_2/N$. We can use the same technique as in the first part only on part of $\Gamma$, so divide the circle into $\Gamma_1$, the part from $e^{i\varphi_1}$ to $e^{i(2\pi - \varphi_1)}$, and $\Gamma_2$ the part from $e^{-i\varphi_1}$ to $e^{i\varphi_1}$, where

$$
\varphi_1 = \frac{4dK}{N} \max\left\{1, \left(1 + \frac{\rho_2}{N}\right)^{-M+1}\right\}.
$$

Note for later use, that $0 \not\approx N\varphi_1$ is finite and, for $\varphi_1 \le \varphi \le \pi$,

$$\left|\left(1+\frac{\rho_2}{N}\right)e^{i\varphi}-1\right| \ge \frac{\varphi_1}{2} = 2\frac{dK}{N}\max\left\{1, \left(1+\frac{\rho_2}{N}\right)^{-M+1}\right\}. \tag{21}$$

To prove (20) we need to show

$$\left\| \int_{\Gamma_j} \lambda^{M-1+n_0} B^{-1}(\lambda) K(Y_0,\lambda)\, d\lambda \right\|_\infty \le \widehat{C}_j \left(1+\frac{\rho_2}{N}\right)^{n_0} \tag{22}$$

for $j = 1, 2$. We start with $\Gamma_2$. As in part (i) we can define

$$\delta_2 := \min\{\|B(\lambda)w\|_\infty : \lambda \in \Gamma_2,\ w \in {}^*\mathbb{C}^d,\ \|w\|_\infty = 1\}$$

and by choice of $\varphi_1$ and $\rho_2$, we can apply Lemma 5 to $\Gamma_2$ to find $N\delta_2 \not\approx 0$, and $\|B^{-1}(\lambda)\| \le 1/\delta_2$.

Also as in (i), there is a constant $C_2 \in \mathbb{R}$ such that $\|K(Y_0,\lambda)\|_\infty \le C_2\|Y_0\|_\infty$ for all $\lambda \in \Gamma_2$. Thus

$$\begin{aligned}\left\| \int_{\Gamma_2} \lambda^{M-1+n_0} B^{-1}(\lambda) K(Y_0,\lambda)\, d\lambda \right\|_\infty& \\ &\le 2\varphi_1\left(1+\frac{\rho_2}{N}\right)^{M+n_0}\frac{1}{\delta_2}C_2 \le C_3\left(1+\frac{\rho_2}{N}\right)^{n_0}\end{aligned}$$

for a constant $C_3 \in \mathbb{R}$. To prove (22) for $\Gamma_1$ we need a lot more technical stuff. The reason is, that we have to take into account the cancelation which happens while integrating on the circle far away from $1 + \rho_2/N$.

A first step is an estimate on $B^{-1}(\lambda)$. For $\lambda \in \Gamma_1$, $\arg(\lambda) \le \pi$ and $w \in {}^*\mathbb{C}^d$ we have

$$\begin{aligned}\|B(\lambda)w\|_\infty &= \max_{1\le j\le d}\left\{\left|\lambda^{M-1}(\lambda-1)w_j - \frac{1}{N}\sum_{k=1}^{d}\sum_{l=0}^{M-1}\lambda^{M-1-l}L_{l,j,k}w_k\right|\right\} \\ &\ge \max_{1\le j\le d}\left\{|\lambda|^{M-1}|\lambda-1||w_j| - \frac{d}{N}\max\{1,|\lambda|^{M-1}\}K\|w\|_\infty\right\} \\ &\ge \frac{1}{2}|\lambda|^{M-1}|\lambda-1|\|w\|_\infty\end{aligned}$$

using (21). Hence for these $\lambda$

$$\|B^{-1}(\lambda)\|_\infty \le 2|\lambda|^{-M+1}|\lambda-1|^{-1}. \tag{23}$$

In a second step we reduce integration over an interval of length $2\pi/(n+1-M)$ to one over half of the interval. This is the cancelation we mentioned earlier, which happens for $(n-M)/N \not\approx 0$. Assume $n > M$ satisfies this condition, $n/N$

finite, and $\varphi_1 \le \varphi_2 < \varphi_2 + 2\pi/(n+1-M) \le \pi$, then

$$
\begin{aligned}
(24)\quad & \int_{\substack{|\lambda|=1+\rho_2/N\\ \varphi_2\le\arg(\lambda)\le\varphi_2+2\pi/(n+1-M)}} \lambda^n B^{-1}(\lambda)\,d\lambda \\
&= i\int_{\varphi_2}^{\varphi_2+2\pi/(n+1-M)} \Big(1+\frac{\rho_2}{N}\Big)^{n+1} e^{i\varphi(n+1)} B^{-1}\Big(\Big(1+\frac{\rho_2}{N}\Big)e^{i\varphi}\Big)\,d\varphi \\
&= i\Big(1+\frac{\rho_2}{N}\Big)^{n+1} \int_{\varphi_2}^{\varphi_2+\pi/(n+1-M)} \Big(e^{i\varphi(n+1)} B^{-1}\Big(\Big(1+\frac{\rho_2}{N}\Big)e^{i\varphi}\Big) \\
&\qquad + e^{i(n+1)(\varphi+\pi/(n+1-M))} B^{-1}\Big(\Big(1+\frac{\rho_2}{N}\Big)e^{i(\varphi+\pi/(n+1-M))}\Big)\Big)\,d\varphi \\
&= i\Big(1+\frac{\rho_2}{N}\Big)^{n+1} \int_{\varphi_2}^{\varphi_2+\pi/(n+1-M)} B^{-1}\Big(\Big(1+\frac{\rho_2}{N}\Big)e^{i(\varphi+\pi/(n+1-M))}\Big) \\
&\qquad \cdot (*) \cdot B^{-1}\Big(\Big(1+\frac{\rho_2}{N}\Big)e^{i\varphi}\Big)\,d\varphi
\end{aligned}
$$

where

$$
\begin{aligned}
(25)\quad (*) &= e^{i\varphi(n+1)} B\Big(\Big(1+\frac{\rho_2}{N}\Big)e^{i(\varphi+\pi/(n+1-M))}\Big) \\
&\quad + e^{i(n+1)(\varphi+\pi/(n+1-M))} B\Big(\Big(1+\frac{\rho_2}{N}\Big)e^{i\varphi}\Big) \\
&= \Big(1+\frac{\rho_2}{N}\Big)^{M} e^{i(n+1+M)\varphi} e^{iM\pi/(n+1-M)} (1+e^{i\pi})E \\
&\quad - \Big(1+\frac{\rho_2}{N}\Big)^{M-1} e^{i(n+M)\varphi} \\
&\quad \cdot e^{i(M-1)\pi/(n+1-M)} (1+e^{i(n+2-M)\pi/(n+1-M)})E \\
&\quad - \frac{1}{N}\sum_{j=0}^{M-1} L_j \Big(1+\frac{\rho_2}{N}\Big)^{M-1-j} e^{i(n+M-j)\varphi} \\
&\quad \cdot (e^{i(M-1-j)\pi/(n+1-M)} + e^{i(n+1)\pi/(n+1-M)}) \\
&= -\Big(1+\frac{\rho_2}{N}\Big)^{M-1} e^{i(n+M)\varphi} e^{i(M-1)\pi/(n+1-M)} (1-e^{i\pi/(n+1-M)})E \\
&\quad - \frac{1}{N}\sum_{j=0}^{M-1} L_j \Big(1+\frac{\rho_2}{N}\Big)^{M-1-j} e^{i(n+M-j)\varphi} \\
&\quad \cdot (e^{i(M-1-j)\pi/(n+1-M)} + e^{i(n+1)\pi/(n+1-M)}).
\end{aligned}
$$

Now, using (23) and putting (25) back into (24), we have

$$
\begin{aligned}
&\Big\| \int_{\substack{|\lambda|=1+\rho_2/N\\ \varphi_2\le\arg(\lambda)\le\varphi_2+2\pi/(n+1-M)}} \lambda^n B^{-1}(\lambda)\,d\lambda \Big\|_\infty \\
&\le 4\Big(1+\frac{\rho_2}{N}\Big)^{n+1} \int_{\varphi_2}^{\varphi_2+\pi/(n+1-M)} \Big(1+\frac{\rho_2}{N}\Big)^{-2M+2}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \frac{(1+\rho_2/N)^{M-1}|1-e^{i\pi/(n+1-M)}| + (2K/N)\max\{1,(1+\rho_2/N)^{M-1}\}}{|(1+\rho_2/N)e^{i(\varphi+\pi/(n+1-M))}-1||(1+\rho_2/N)e^{i\varphi}-1|}\,d\varphi \\
&\le \left(1+\frac{\rho_2}{N}\right)^{n+1}\int_{\varphi_2}^{\varphi_2+\pi/(n+1-M)} \frac{C_4}{N} \\
&\quad\cdot \left[\left(1+\left(1+\frac{\rho_2}{N}\right)^2 - 2\left(1+\frac{\rho_2}{N}\right)\cos\left(\varphi+\frac{\pi}{n+1-M}\right)\right)\right. \\
&\quad\cdot \left.\left(1+\left(1+\frac{\rho_2}{N}\right)^2 - 2\left(1+\frac{\rho_2}{N}\right)\cos(\varphi)\right)\right]^{-1/2} d\varphi \\
&\le \left(1+\frac{\rho_2}{N}\right)^{n+1}\frac{C_4}{N} \\
&\quad\cdot \int_{\varphi_2}^{\varphi_2+\pi/(n+1-M)} \left(\frac{1}{1+(1+\rho_2/N)^2-2(1+\rho_2/N)\cos(\varphi+\pi/(n+1-M))}\right. \\
&\quad+ \left.\frac{1}{1+(1+\rho_2/N)^2-2(1+\rho_2/N)\cos(\varphi)}\right) d\varphi
\end{aligned}
$$

for a $C_4 \in \mathbb{R}$, where in the second but last step we used $(n-M)/N \not\approx 0$.

We apply this technique to a contour-integral over $\Gamma_1$. To be able to do this, let $j_1 = j_1(n)$ be the number of intervals of length $2\pi/(n+1-M)$ in the interval $[\varphi_1, \pi]$, i.e. the maximal hyper-finite number satisfying $j_1 \le (\pi-\varphi_1)(n+1-M)/2\pi$. Then

$$
\begin{aligned}
&\left\|\int_{\Gamma_1} \lambda^n B^{-1}(\lambda)\,d\lambda\right\|_\infty \\
&= 2\left\|\operatorname{Im}\int_{\substack{|\lambda|=1+\rho_2/N \\ \varphi_1\le\arg(\lambda)\le\pi}} \lambda^n B^{-1}(\lambda)\,d\lambda\right\|_\infty \\
&\le 2\sum_{j=0}^{j_1}\left\|\int_{\substack{|\lambda|=1+\rho_2/N \\ \varphi_1+j2\pi/(n+1-M)\le\arg(\lambda)\le\varphi_1+(j+1)2\pi/(n+1-M)}} \lambda^n B^{-1}(\lambda)\,d\lambda\right\|_\infty \\
&\quad + 2\left\|\int_{\substack{|\lambda|=1+\rho_2/N \\ \pi-2\pi/(n+1-M)\le\arg(\lambda)\le\pi}} \lambda^n B^{-1}(\lambda)\,d\lambda\right\|_\infty \\
&\le 2\sum_{j=0}^{j_1}\left(1+\frac{\rho_2}{N}\right)^{n+1}\frac{C_4}{N} \\
&\quad\cdot\int_{\varphi_1+j2\pi/(n+1-M)}^{\varphi_1+(2j+1)\pi/(n+1-M)}\left(\frac{1}{1+(1+\rho_2/N)^2-2(1+\rho_2/N)\cos(\varphi+\pi/(n+1-M))}\right. \\
&\quad+\left.\frac{1}{1+(1+\rho_2/N)^2-2(1+\rho_2/N)\cos(\varphi)}\right)d\varphi + \frac{8\pi}{n+1-M}\left(1+\frac{\rho_2}{N}\right)^{n+2-M} \\
&\le 4\left(1+\frac{\rho_2}{N}\right)^{n+1}\frac{C_4}{N}\int_{\varphi_1}^{\pi+\pi/(n+1-M)}\frac{d\varphi}{1+(1+\rho_2/N)^2-2(1+\rho_2/N)\cos(\varphi)} \\
&\quad+\frac{8\pi(1+\rho_2/N)^{n+2-M}}{n+1-M}
\end{aligned}
$$

$$\leq 4\left(1+\frac{\rho_2}{N}\right)^{n+1}\frac{C_4}{N}\left\{\begin{matrix} 2\pi N/\rho_2 & \text{if } \rho_2 \neq 0, \\ \cos(\varphi_1/2)+1 & \text{if } \rho_2 = 0, \end{matrix}\right\} + \frac{8\pi(1+\rho_2/N)^{n+2-M}}{n+1-M}$$
$$\leq C_5\left(1+\frac{\rho_2}{N}\right)^n,$$

where, keeping in mind $N\varphi_1$ is finite but not infinitesimal, $C_5 \in \mathbb{R}$. Since $Y_n = A^n Y_0 = \sum_{j=1}^{dM} \alpha_j \lambda_j^n v_j$, we have $\alpha_j(Y_{M-1}) = \lambda_j^{M-1}\alpha_j(Y_0)$. We now prove the inequality (22):

$$\begin{aligned}
&\left\| \int_{\Gamma_1} \lambda^{M-1+n_0} B^{-1}(\lambda) K(Y_0,\lambda)\, d\lambda \right\|_\infty \\
&= \left\| \int_{\Gamma_1} \lambda^{n_0} B^{-1}(\lambda) K(Y_{M-1}\lambda)\, d\lambda \right\|_\infty \\
&= \left\| \int_{\Gamma_1} \lambda^{n_0} B^{-1}(\lambda) \biggl( \sum_{l=0}^{M-1} \lambda^l y_{M-1-l} - \sum_{l=1}^{M-1} \lambda^{l-1} y_{M-1-l} \right. \\
&\qquad \left. - \frac{1}{N} \sum_{k=1}^{M-1} L_{k-1} \sum_{l=k}^{M-1} \lambda^{l-k} y_{M-1-l} \biggr)\, d\lambda \right\|_\infty \\
&\leq \sum_{l=0}^{M-2} \left\| \int_{\Gamma_1} \lambda^{n_0+l} B^{-1}(\lambda)(y_{M-1-l} - y_{M-2-l})\, d\lambda \right\|_\infty \\
&\quad + \left\| \int_{\Gamma_1} \lambda^{M-1+n_0} B^1(\lambda) y_0\, d\lambda \right\|_\infty \\
&\quad + \frac{1}{N} \sum_{k=1}^{M-1} \sum_{l=k}^{M-1} \left\| \int_{\Gamma_1} \lambda^{n_0+l-k} B^{-1}(\lambda) \right\|_\infty \|L_{k-1}\|_\infty \|Y_{M-1}\|_\infty \\
&\leq \sum_{l=0}^{M-2} C_6 \left(1+\frac{\rho_2}{N}\right)^{n_0+l} \max_{1\leq j\leq M-1} \{\|y_j - y_{j-1}\|_\infty\} \\
&\quad + C_7 \left(1+\frac{\rho_2}{N}\right)^{M-1+n_0} \|Y_0\|_\infty \\
&\quad + \frac{C_5}{N} \sum_{k=1}^{M-1} \|L_{k-1}\|_\infty \sum_{l=k}^{M-1} \left(1+\frac{\rho_2}{N}\right)^{n_0+l-k} \|Y_{M-1}\|_\infty \\
&\leq [C_8 M \max_{1\leq j\leq M-1} \{\|y_j - y_{j-1}\|_\infty\} + C_9 + C_{10}\|Y_{M-1}\|_\infty] \left(1+\frac{\rho_2}{N}\right)^{n_0},
\end{aligned}$$

where all $C_6, \ldots, C_{10} \in \mathbb{R}$.

For $j \geq 1$ we have $\|y_j - y_{j-1}\|_\infty \leq (K/N)\|(y_{j-1}, \ldots, y_{j-M})\|_\infty$, hence also

$$\max_{1\leq j\leq M-1} \|y_j - y_{j-1}\|_\infty \leq \frac{K}{N}(\|Y_{M-1}\|_\infty + \|Y_0\|_\infty)$$

and $\|Y_{M-1}\|_\infty \leq (1+K/N)^{M-1}\|y_0\|_\infty$ being finite implies (22). □

## 5 The one-dimensional case

In this section we contemplate a linear autonomous RFDE in one dimension, that is equation (8) with $d = 1$:

$$x'(t) = Lx_t = \int_{-r}^{0} x(t+\theta)\, d\eta(\theta). \tag{26}$$

In this case the iteration becomes $Y_n = AY_{n-1} \in {}^*\mathbb{C}^M$, where

$$A = \begin{pmatrix} 1+\frac{L_0}{N} & \frac{L_1}{N} & \cdots & \cdots & \frac{L_{M-1}}{N} \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in {}^*\mathbb{R}^{M\times M} \tag{27}$$

$L_j \in {}^*\mathbb{R}$ and by Proposition 2 $\sum_{j=0}^{M-1} |L_j| \le K \in \mathbb{R}$ as before. The characteristic polynomial is

$$p(\lambda) = \lambda^M - \lambda^{M-1} - \frac{1}{N}\sum_{j=0}^{M-1} L_j \lambda^{M-1-j}$$

and the eigenvectors are of the form

$$v = v(\lambda) = \begin{pmatrix} \lambda^{M-1} \\ \vdots \\ \lambda \\ 1 \end{pmatrix} \in {}^*\mathbb{C}^M.$$

If we assume $\lambda_1, \dots, \lambda_M$ to be the pairwise distinct eigenvalues of $A$ with corresponding eigenvectors $v_1, \dots, v_M$, then $Y = (y_0, \dots, y_{-M+1})^t = \sum_{j=1}^M \alpha_j v_j$ is equivalent to

$$\alpha_j = \frac{K(Y,\lambda_j)}{p'(\lambda_j)} = \frac{\sum_{l=0}^{M-1}\sum_{k=0}^{l} b_k \lambda_j^{l-k} y_{-l}}{p'(\lambda_j)}, \quad j = 1, \dots, M, \tag{28}$$

where $p(\lambda) = \sum_{k=0}^{M} b_k \lambda^{M-k}$. (28) is just formula (14) for the one-dimensional case, with $\widetilde{w}_j = w_j = 1$. A straightforward calculation shows, for a multiple root $\lambda_0$ of order $m_0$ we have generalized eigenvectors

$$v_{0,m} = \begin{pmatrix} \binom{M-1}{m}\lambda_0^{M-1-m} \\ \binom{M-2}{m}\lambda_0^{M-2-m} \\ \vdots \\ \binom{1}{m}\lambda_0^{1-m} \\ \binom{0}{m}\lambda_0^{-m} \end{pmatrix} \in {}^*\mathbb{C}^M, \quad 0 \le m \le m_0 - 1,$$

and, for $n \in {}^*\mathbb{N}$,

$$A^n v_{0,m} = \begin{pmatrix} \binom{n+M-1}{m}\lambda_0^{n+M-1-m} \\ \binom{n+M-2}{m}\lambda_0^{n+M-2-m} \\ \vdots \\ \binom{n+1}{m}\lambda_0^{n+1-m} \\ \binom{n}{m}\lambda_0^{n-m} \end{pmatrix} \in {}^*\mathbb{C}^M, \quad 0 \le m \le m_0 - 1.$$

It is also easy to see that for $N(\lambda_0 - 1) \approx \mu_0 \in \mathbb{C}$ ($\Leftrightarrow \lambda_0 \in S_{\mu_0}$ as defined in Lemma 5)

$$\frac{1}{N^m}\binom{n}{m}\lambda_0^{n-m} \approx \frac{t^m}{m!}e^{\mu_0 t}, \quad t = {}^\circ\!\left(\frac{n}{N}\right),$$

or in other words: suitably normalized generalized eigenvectors represent (generalized) eigenfunctions of equation (26).

In Lemma 10 we will incorporate the last remark into Lemma 5, but before we can do this we need a way to express eigenfunctions of higher order by a linear combination of eigenvectors belonging to eigenvalues "near" to each other.

Lemma 9. *For a fixed $1 \le m_0 \le M$ let $\lambda_1, \dots, \lambda_{m_0}$ be pairwise distinct eigenvalues of $A$, with corresponding eigenvectors $v_1, \dots, v_{m_0}$. Define $\nu_{j,l} \in {}^*\mathbb{C}^M$ by*

$$\begin{aligned} \nu_{1,l} &= v_l && \text{for } 1 \le l \le m_0, \\ \nu_{j,l} &= \frac{1}{N(\lambda_l - \lambda_{j-1})}(\nu_{j-1,l} - \nu_{j-1,j-1}) && \text{for } 2 \le j \le l \le m_0. \end{aligned}$$

*Then*

$$\nu_{j,j} = \frac{1}{N^{j-1}} \sum_{m=1}^{j} \frac{v_m}{\prod_{\substack{k=1\\k\neq m}}^{j} (\lambda_m - \lambda_k)} \quad \text{for } 1 \le j \le m_0. \tag{29}$$

*In particular* $\operatorname{lin}\{v_1, \dots, v_{m_0}\} = \operatorname{lin}\{\nu_{1,1}, \dots, \nu_{m_0,m_0}\}$.

*If* $\sum_{j=1}^{m_0} \alpha_j v_j = \sum_{j=1}^{m_0} \beta_j \nu_{j,j}$, *then*

$$\beta_j = N^{j-1} \sum_{l=j}^{m_0} \prod_{k=1}^{j-1} (\lambda_l - \lambda_k)\alpha_l \quad \text{for } 1 \le j \le m_0. \tag{30}$$

*If additionally there is a $\mu \in \mathbb{C}$, such that*

$$\lambda_j \in S_\mu = \left\{\lambda = \left(1 + \frac{\varepsilon}{N}\right)e^{i\varphi/N} \in {}^*\mathbb{C} : {}^\circ\varepsilon = \operatorname{Re}\mu,\ {}^\circ\varphi = \operatorname{Im}\mu\right\}, \quad 1 \le j \le m_0,$$

*then, for $0 \le n/N \approx t \in \mathbb{R}$,*

$$A^n \nu_{j,j} \triangleq \frac{(t+r+\theta)^{j-1}}{(j-1)!} e^{\mu(t+r+\theta)} \colon [-r, 0] \to \mathbb{C} \quad \textit{for } 1 \le j \le m_0. \tag{31}$$

Proof. First we derive an expression for $\nu_{j,l}$, $1 \le j \le l \le m_0$:

$$\nu_{j,l} = \frac{1}{N^{j-1}}\left(\sum_{m=1}^{j-1} \frac{v_m}{\prod_{\substack{k=1\\k\neq m}}^{j-1}(\lambda_m-\lambda_k)(\lambda_m-\lambda_l)} + \frac{v_l}{\prod_{k=1}^{j-1}(\lambda_l-\lambda_k)}\right). \tag{32}$$

We prove (32) by induction over $j$. The case $j=1$ is trivial. Assume (32) holds for $1 \le j$. Then for $j+1 \le l \le m_0$:

$$\begin{aligned}\nu_{j+1,l} = {} & \frac{1}{N^j(\lambda_l-\lambda_j)}\Bigg[\sum_{m=1}^{j-1}\left(\frac{v_m}{\prod_{\substack{k=1\\k\neq m}}^{j-1}(\lambda_m-\lambda_k)(\lambda_m-\lambda_l)} - \frac{v_m}{\prod_{\substack{k=1\\k\neq m}}^{j}(\lambda_m-\lambda_k)}\right)\\ & + \frac{v_l}{\prod_{k=1}^{j-1}(\lambda_l-\lambda_k)} - \frac{v_j}{\prod_{k=1}^{j-1}(\lambda_j-\lambda_k)}\Bigg]\\ = {} & \frac{1}{N^j}\left[\sum_{m=1}^{j}\frac{v_m}{\prod_{\substack{k=1\\k\neq m}}^{j}(\lambda_m-\lambda_k)(\lambda_m-\lambda_l)} + \frac{v_l}{\prod_{k=1}^{j}(\lambda_l-\lambda_k)}\right]\end{aligned}$$

and (32) has been proven. (32) implies $\operatorname{lin}\{v_1,\ldots,v_{m_0}\} = \operatorname{lin}\{\nu_{1,1},\ldots,\nu_{m_0,m_0}\}$ and (29).

To prove (30) we use (29) and again an induction over $j = m_0,\ldots,1$. For $j = m_0$

$$\alpha_{m_0}v_{m_0} = \beta_{m_0}\frac{1}{N^{m_0-1}}\frac{v_{m_0}}{\prod_{k=1}^{m_0-1}(\lambda_{m_0}-\lambda_k)}$$

and (30) follows.

For $j \Rightarrow j-1$

$$\begin{aligned}\alpha_{j-1}v_{j-1} = {} & v_{j-1}\cdot\Bigg(\beta_{j-1}\frac{1}{N^{j-2}\prod_{k=1}^{j-2}(\lambda_{j-1}-\lambda_k)}\\ & + \sum_{l=j}^{m_0}\beta_l\frac{1}{N^{l-1}\prod_{\substack{k=1\\k\neq j-1}}^{l}(\lambda_{j-1}-\lambda_k)}\Bigg)\end{aligned}$$

and

$$\begin{aligned}\beta_{j-1} &= N^{j-2}\prod_{k=1}^{j-2}(\lambda_{j-1}-\lambda_k)\alpha_{j-1} - \sum_{l=j}^{m_0}\beta_l\frac{1}{N^{l-j+1}}\frac{1}{\prod_{k=j}^{l}(\lambda_{j-1}-\lambda_k)}\\ &= N^{j-2}\prod_{k=1}^{j-2}(\lambda_{j-1}-\lambda_k)\alpha_{j-1} - \sum_{l=j}^{m_0}\frac{N^{l-1}}{N^{l-1-j+2}}\sum_{m=l}^{m_0}\frac{\prod_{k=1}^{l-1}(\lambda_m-\lambda_k)}{\prod_{k=j}^{l}(\lambda_{j-1}-\lambda_k)}\alpha_m\\ &= N^{j-2}\prod_{k=1}^{j-2}(\lambda_{j-1}-\lambda_k)\alpha_{j-1} - \sum_{m=j}^{m_0}\alpha_m\sum_{l=j}^{m}N^{j-2}\frac{\prod_{k=1}^{l-1}(\lambda_m-\lambda_k)}{\prod_{k=j}^{l}(\lambda_{j-1}-\lambda_k)}.\end{aligned}$$

We claim, for $j \le m \le m_0$

$$\sum_{l=j}^{m} \frac{\prod_{k=1}^{l-1}(\lambda_m - \lambda_k)}{\prod_{k=j}^{l}(\lambda_{j-1} - \lambda_k)} = -\prod_{k=1}^{j-2}(\lambda_m - \lambda_k). \tag{33}$$

This immediately yields (30). Setting $\lambda_{j-1} = \widetilde{\lambda}$, the former is equivalent to

$$\begin{aligned}
&\sum_{l=j}^{m} \frac{\prod_{k=j-1}^{l-1}(\lambda_m - \lambda_k)}{\prod_{k=j}^{l}(\widetilde{\lambda} - \lambda_k)} = -1 \\
\Leftrightarrow &\sum_{l=j}^{m} \prod_{k=j-1}^{l-1}(\lambda_m - \lambda_k) \prod_{k=l+1}^{m}(\widetilde{\lambda} - \lambda_k) = -\prod_{k=j}^{m}(\widetilde{\lambda} - \lambda_k) \\
\Leftrightarrow &\sum_{l=j}^{m} \prod_{k=j}^{l-1}(\lambda_m - \lambda_k) \prod_{k=l+1}^{m}(\widetilde{\lambda} - \lambda_k) = \prod_{k=j}^{m-1}(\widetilde{\lambda} - \lambda_k) && (34) \\
\Leftrightarrow &\sum_{l=j+1}^{m} \prod_{k=j}^{l-1}(\lambda_m - \lambda_k) \prod_{k=l+1}^{m}(\widetilde{\lambda} - \lambda_k) = \prod_{k=j+1}^{m-1}(\widetilde{\lambda} - \lambda_k)(\lambda_m - \lambda_j) \\
\Leftrightarrow &\sum_{l=j+1}^{m} \prod_{k=j+1}^{l-1}(\lambda_m - \lambda_k) \prod_{k=l+1}^{m}(\widetilde{\lambda} - \lambda_k) = \prod_{k=j+1}^{m-1}(\widetilde{\lambda} - \lambda_k)
\end{aligned}$$

but the last equation is like (34), so applying these steps various times we see (33) is equivalent to

$$(\widetilde{\lambda} - \lambda_m) + (\lambda_m - \lambda_{m-1}) = \widetilde{\lambda} - \lambda_{m-1}$$

and the claim has been proven.

Now assume $\lambda_m \in S_\mu$ for all $m$. Fix a $\lambda_0 \in S_\mu$, and define $\delta_m = \lambda_m - \lambda_0$, $1 \le m \le m_0$. Then $N\delta_m \approx 0$. With formula (29) one can reduce the proof of (31) to the proof of

$$\frac{1}{N^{j-1}} \sum_{m=1}^{j} \frac{\lambda_m^{M-1+n}}{\prod_{\substack{k=1\\k\neq m}}^{j}(\lambda_m - \lambda_k)} \approx \frac{({}^\circ(n/N) + r)^{j-1}}{(j-1)!} e^{\mu({}^\circ(n/N)+r)},$$

for $-M+1 \le n$, $n/N$ finite, and $1 \le j \le m_0$. Using the fact that $N^{-j}\binom{n}{j}\lambda_0^{nj} \approx ({}^\circ(n/N))^j e^{\mu\,{}^\circ(n/N)}/j!$ for finite $j$ and $n/N$, it suffices to show

$$\frac{1}{N^{j-1}} \sum_{m=1}^{j} \frac{\lambda_m^{M-1+n}}{\prod_{\substack{k=1\\k\neq m}}^{j}(\delta_m - \delta_k)} - \frac{1}{N^{j-1}}\binom{M-1+n}{j-1}\lambda_0^{M+n-j} \approx 0$$

for $-M+1 \le n$, $n/N$ finite and $1 \le j \le m_0$. If $j = 1$ we have equality, so assume $1 < j \le m_0$.

We claim for these $n$ and $j$

$$\sum_{m=1}^{j} \frac{\lambda_m^{M-1+n}}{\prod_{\substack{k=1\\k\neq m}}^{j}(\delta_m-\delta_k)} = \binom{M-1+n}{j-1}\lambda_0^{M+n-j} + \sum_{k_1=j}^{M-1+n}\binom{M-1+n}{k_1}\lambda_0^{M-1+n-k_1} \cdot \sum_{k_2=j-2}^{k_1-1}\delta_1^{k_1-1-k_2}\sum_{k_3=j-3}^{k_2-1}\delta_2^{k_2-1-k_3}\dots\sum_{k_j=0}^{k_{j-1}-1}\delta_{j-1}^{k_{j-1}-1-k_j}\delta_j^{k_j}. \tag{35}$$

Indeed, both sides of (35) are the leading coefficient of the polynomial interpolating the function $(\lambda_0+x)^{M+1-n}$ in $\delta_1,\dots,\delta_j$: the left-hand side is the expression we get by the Lagrange formula, and the right-hand side is due to Newton's formula. To see the latter, note that for pairwise different $\delta_{l_1},\dots,\delta_{l_j}$

$$[\delta_{l_1}\dots\delta_{l_j}] := \binom{M-1+n}{j-1}\lambda_0^{M+n-j} + \sum_{k_1=j}^{M-1+n}\binom{M-1+n}{k_1}\lambda_0^{M+n-1-k_1} \cdot \sum_{k_2=j-2}^{k_1-1}\delta_{l_1}^{k_1-1-k_2}\dots\sum_{k_j=0}^{k_{j-1}-1}\delta_{l_{j-1}}^{k_{j-1}-1-k_j}\delta_{l_j}^{k_j}$$

satisfies the inductive rule for Newton's formula, i.e.

$$[\delta_{l_1}] = \sum_{k_1=0}^{M-1+n}\binom{M-1+n}{k_1}\lambda_0^{M+n-1-k_1}\delta_{l_1}^{k_1} = \lambda_{l_1}^{M-1+n}$$

and

$$[\delta_{l_1}\dots\delta_{l_{j+1}}] = \frac{[\delta_{l_1}\dots\delta_{l_j}] - [\delta_{l_1}\dots\delta_{l_{j-1}},\delta_{l_{j+1}}]}{\delta_{l_j}-\delta_{l_{j+1}}}.$$

(Note, that the order of the entities in the square brackets is of no importance for the Newton interpolation.)

With the claim above, and defining $\delta=\max\{|\delta_0|,\dots,|\delta_{m_0}|\}$, $c_n=\max\{|\lambda_0^j| : 0\le j\le M-1+n\}$, we get for $1\le j\le m_0$, $-M+1\le n$, $n/N$ finite

$$N^{-j+1}\left|\sum_{m=1}^{j}\frac{\lambda_m^{M-1+n}}{\prod_{\substack{k=1\\k\neq m}}^{j}(\delta_m-\delta_k)} - \binom{M-1+n}{j-1}\lambda_0^{M+n-j}\right| \le \frac{c_n}{N^{j-1}}\sum_{k_1=j}^{M-1+n}\binom{M-1+n}{k_1}\delta^{k_1-j+1}\sum_{k_2=j-2}^{k_1-1}\sum_{k_3=j-3}^{k_2-1}\dots\sum_{k_j=0}^{k_{j-1}-1}1 = \frac{c_n}{N^{j-1}}\sum_{k_1=j}^{M-1+n}\binom{M-1+n}{k_1}\delta^{k_1-j+1}\binom{k_1}{j-1} = \frac{c_n}{N^{j-1}}\sum_{k=j}^{M-1+n}S_k, \tag{36}$$

defining $S_k$ in the last step and using

$$\sum_{k_1=m-1}^{k_0-1}\sum_{k_2=m-2}^{k_1-1}\cdots\sum_{k_m=0}^{k_{m-1}-1}1=\binom{k_0}{m}.$$

Now for $n/N$ finite,

$$\frac{S_{k+1}}{S_k}\le\delta\frac{M-1+n-k}{k+2-j}<\delta\frac{M+n}{2}\approx 0,$$

where we used $N\delta\approx 0$. For these $n$ we conclude

$$\begin{aligned}(36)\ &\le\frac{c_n}{N^{j-1}}S_j\sum_{k=j}^{M-1+n}\left(\delta\frac{M+n}{2}\right)^{k-j}\\&\le\frac{c_n}{N^{j-1}}\frac{(M-1+n)!j}{j!(M-1+n-j)!}\delta\frac{1-(\delta(M+n)/2)^{M+n-j}}{1-\delta(M+n)/2}\\&\le\text{finite}\cdot\text{infinitesimal}\approx 0,\end{aligned}$$

which finishes the proof. □

Lemma 10. *Let $u\in\mathbb{C}$, $m_0\in\mathbb{N}$, and define as before*

$$S_\mu=\{\lambda=(1+\varepsilon/N)e^{i\varphi/N}\in{}^*\mathbb{C}:{}^\circ\varepsilon=\operatorname{Re}\mu,\ {}^\circ\operatorname{Im}\mu\}.$$

*Then following conditions are equivalent:*

(i) *for all $0\le m\le m_0$, $z_m(t)=((t+r)^m/m!)e^{\mu(t+r)}$ is a solution of equation* (26),
(ii) *$p(\lambda)$ has $m_0+1$ roots* (*counting multiplicities*) *in $S_\mu$*,
(iii) *for all $0\le m\le m_0$ and $\lambda\in S_\mu: N^{-m+1}p^{(m)}(\lambda)\approx 0$*,
(iv) *for all $0\le m\le m_0$ there exists a $\lambda_m\in S_\mu: N^{-m+1}p^{(m)}(\lambda_m)\approx 0$*,
(v) *for all $\lambda_0\in S_\mu$ there exist $\Delta_j\in{}^*\mathbb{R}$, such that $\sum_{j=0}^{M-1}|\Delta_j|\approx 0$, the internal $p_\Delta(\lambda)=p(\lambda)+(1/N)\sum_{j=0}^{M-1}\Delta_j\lambda^{M-1-j}$ is the characteristic polynomial for the disturbed matrix $A_\Delta$, and $p_\Delta(\lambda)$ has a root of order $m_0+1$ at $\lambda_0$.*

Proof. For $m_0=0$ Lemma 10 is just a special case of Lemma 5 for $d=1$. Hence we can inductively assume that all statements are equivalent for $m_0-1\ge 0$ and have to prove the equivalence for $m_0$.

(ii)⇒(i). If we change the $L_j$ slightly to get pairwise distinct eigenvalues, we can apply Lemma 9 to get the desired solution of the RFDE. Since the existence of such solutions is independent of the particular description, (i) follows from (ii).

(i)⇒(iii). The proof runs along the same lines as in Lemma 5. Fix a $\lambda_0\in S_\mu$ and set for finite $n/N$:

$$V_{m_0,n}:=(v_{m_0,n},\ldots,v_{m_0,n-M+1})^t,\quad v_{m_0,j}=N^{-m_0}\binom{M-1+j}{m_0}\lambda_0^{M-1+j-m_0}.$$

Then for $t = {}^\circ(n/N)$

$$V_{m_0,n} \triangleq \frac{(t+r+\theta)^{m_0}}{m_0!} e^{\mu(t+r+\theta)}\colon [t-r,t] \to \mathbb{C}.$$

For these $n$ we have $0 \approx A^n V_{m_0,0} - V_{m_0,n}$. Now, if $e_1$ is the first unit vector,

$$AV_{m_0,n} = V_{m_0,n+1} - \frac{(\lambda_0^n p(\lambda_0))^{(m_0)}}{N^{m_0} m_0!} e_1,$$

and, for $n/N$ finite,

$$\begin{aligned}(37)\qquad 0 &\approx \frac{1}{N^{m_0} m_0!} \sum_{j=0}^{n-1} (\lambda_0^j p(\lambda_0))^{(m_0)} A^{n-1-j} e_1 \\ &= \frac{1}{N^{m_0} m_0!} p^{(m_0)}(\lambda_0) \sum_{j=0}^{n-1} \lambda_0^j A^{n-1-j} e_1 - \sum_{l=0}^{m_0-1} \binom{m_0}{l} \frac{p^{(l)}(\lambda_0)}{N^{m_0} m_0!} \\ &\quad \cdot \sum_{j=0}^{n-1} j(j-1)\dots(j-m_0+l+1)\lambda_0^{j-m_0+l} A^{n-1-j} e_1.\end{aligned}$$

Assuming by induction $N^{-l+1} p^{(l)}(\lambda_0) \approx 0$ for $0 \le l < m_0$ ($m_0$ is finite!), we see that the last sum is infinitesimal.

As in the proof of (i)⇒(iii) of Lemma 5, there is a $n_1 \in {}^*\mathbb{N}$, $n_1/N \not\approx 0$, such that for arbitrary $V \in {}^*\mathbb{C}^M$

$$|\lambda_0^j (A^{n-1-j} V - V) e_1| \le \frac{1}{2} \|V\|_\infty \quad \text{for all } 0 \le j \le n-1 \le n_1$$

and $Re\ \lambda_0^j > 2/3$ for these $j$. Using this in (37)

$$0 \approx \frac{1}{N^{m_0} m_0!} p^{(m_0)}(\lambda_0) \underbrace{\Big( \sum_{j=0}^{n_1} \lambda_0^j (A^{n_1-j} e_1 - e_1) + \sum_{j=0}^{n_1} \lambda_0^j e_1 \Big)}_{|\ ''\ | \ge 2n_1/3 - n_1/2 = n_1/6},$$

and the choice of $n_1$ implies $0 \approx N^{-m_0+1} p^{(m_0)}(\lambda_0)$.

Proof of (iii)⇒(iv) is trivial.

(iv)⇒(iii). We only have to show $N^{-m_0+1} p^{(m_0)}(\lambda) \approx 0$, for all $\lambda \in S_\mu$. So fix a $\widetilde{\lambda} \in S_\mu$, and let $\lambda_{m_0}$ as in (iv). Set $\delta := \widetilde{\lambda} - \lambda_{m_0}$, then

$$\begin{aligned} p^{(m_0)}(\widetilde{\lambda}) &= M(M-1)\dots(M-m_0+1) \sum_{j=0}^{M-m_0} \binom{M-m_0}{j} \lambda_{m_0}^{M-m_0-j} \delta^j \\ &\quad - (M-1)\dots(M-m_0) \sum_{j=0}^{M-1-m_0} \binom{M-1-m_0}{j} \lambda_{m_0}^{M-1-m_0-j} \delta^j \\ &\quad - \frac{1}{N} \sum_{l=0}^{M-1} L_l (M-1-l)\dots(M-l-m_0) \end{aligned}$$

$$
\begin{aligned}
&\quad \cdot \sum_{j=0}^{M-1-m_0-l} \binom{M-1-m_0-l}{j} \lambda_{m_0}^{M-1-m_0-j-l} \delta^j \\
&= p^{(m_0)}(\lambda_{m_0}) + \delta \Big[ M \dots (M-m_0+1)\delta^{M-m_0-1} \\
&\quad + (M-1)\dots(M-m_0+1) \\
&\quad \cdot \sum_{j=0}^{M-m_0-2} \bigg( \frac{(M-m_0)(M-m_0-1)(m_0+1+j)}{(j+1)(M-m_0-1-j)} \\
&\quad + (\lambda_{m_0}-1) \frac{M(M-m_0)(M-m_0-1)}{(j+1)(M-m_0-1-j)} \bigg) \\
&\quad \cdot \binom{M-2-m_0}{j} \lambda_{m_0}^{M-2-m_0-j} \delta^j \\
&\quad - \frac{1}{N} \sum_{l=0}^{M-1} L_l (M-1-l)\dots(M-l-m_0) \\
&\quad \cdot \sum_{j=0}^{M-2-m_0-l} \binom{M-2-m_0-l}{j} \frac{M-1-m_0-l}{j+1} \lambda_{m_0}^{M-2-m_0-j-l} \delta^j \Big] \\
&= p^{(m_0)}(\lambda_{m_0}) + \delta[**],
\end{aligned}
$$

where we define $[**]$ in the last step. Keeping in mind $N\delta \approx 0$, it is sufficient to show $N^{-m_0}[**]$ to be finite. This is not difficult. Let

$$
S_j := \binom{M-2-m_0}{j} \frac{(M-m_0)(M-m_0-1)(m_0+1+j)}{(j+1)(M-m_0-1-j)} |\lambda_{m_0}|^{M-2-m_0-j} |\delta|^j,
$$

then

$$
\begin{aligned}
&|[**]|N^{-m_0} \\
&\quad \le \left(\frac{M}{N}\right)^{m_0} |\delta|^{M-1-m_0} + \left(\frac{M}{N}\right)^{m_0-1} \frac{1}{N} \\
&\qquad \cdot \sum_{j=0}^{M-2-m_0} \left( S_j + |\lambda_{m_0}-1| M(M-m_0) \binom{M-m_0-2}{j} |\lambda_{m_0}|^{M-2-m_0-j} |\delta|^j \right) \\
&\qquad + \left(\frac{M}{N}\right)^{m_0+1} \sum_{l=0}^{M-1} |L_l| (|\lambda_{m_0}| + |\delta|)^{M-m_0-2-l} \\
&\quad \le 1 + \left(\frac{M}{N}\right)^{m_0-1} \frac{1}{N} \sum_{j=0}^{M-2-m_0} S_j + (|\mu|+1) \left(\frac{M}{N}\right)^{m_0+1} (|\lambda_{m_0}| + |\delta|)^{M-m_0-2} \\
&\qquad + \left(\frac{M}{N}\right)^{m_0+1} K \max\{(|\lambda_{m_0}| + |\delta|)^{M-m_0-2}, (|\lambda_{m_0}| + |\delta|)^{-m_0-1}\} \\
&\quad = \text{finite} + \left(\frac{M}{N}\right)^{m_0-1} \frac{1}{N} \sum_{j=0}^{M-2-m_0} S_j.
\end{aligned}
$$

But
$$\frac{S_{j+1}}{S_j} = \frac{(M-m_0-j-1)(m_0+2+j)}{(m_0+1+j)(j+2)}\frac{|\delta|}{|\lambda_{m_0}|} \le 2M|\delta| \approx 0,$$
and
$$\frac{1}{N}\sum_{j=0}^{M-2-m_0} S_j \le \frac{(M-m_0)(m_0+1)}{N}|\lambda_{m_0}|^{M-2-m_0}\sum_{j=0}^{M-2-m_0}(2M|\delta|)^j = \text{finite}$$
follows, which in turn yields $N^{-m_0}|(**)|$ to by finite too.

(iii)$\Rightarrow$(v). Fix $\lambda_0 \in S_\mu$. $\widetilde{p}(\lambda)$ having a root of order $m_0+1$ at $\lambda_0$ is equivalent to the existence of a solution $(\Delta_0, \dots, \Delta_{M-1})$ of the system of linear equations described by

$$\begin{pmatrix} \lambda_0^{M-1} & \lambda_0^{M-2} & \dots & \dots & \dots & 1 & -N^1 p(\lambda_0) \\ \frac{M-1}{N}\lambda_0^{M-2} & \frac{M-2}{N}\lambda_0^{M-3} & \dots & \dots & \frac{1}{N} & 0 & -N^0 p'(\lambda_0) \\ \frac{(M-1)(M-2)}{N^2}\lambda_0^{M-3} & \frac{(M-2)(M-3)}{N^2}\lambda_0^{M-4} & \dots & \frac{2}{N^2} & 0 & 0 & -N^{-1}p''(\lambda_0) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{(M-1)\dots(M-m_0)}{N^{m_0}}\lambda_0^{M-1-m_0} & \dots & \frac{m_0!}{N^{m_0}} & 0 & \dots & 0 & -N^{-m_0+1}p^{(m_0)}(\lambda_0) \end{pmatrix} \tag{38}$$

has a solution $(\Delta_0, \dots, \Delta_{M-1})$. Choose $0 \le j_0 < \dots < j_{m_0} \le M-1$, such that for $t_m := {}^\circ(j_m/N)$, $m = 0, \dots, m_0$ we have: $t_0 < \dots < t_{m_0}$. The system above has a solution, if there is a solution considering on the left-hand side only the columns $j_0, \dots, j_{m_0}$. But this reduced (square) matrix is infinitely close to
$$R = \begin{pmatrix} e^{\mu(r-t_0)} & e^{\mu(r-t_1)} & \cdots & e^{\mu(r-t_{m_0})} \\ (r-t_0)e^{\mu(r-t_0)} & (r-t_1)e^{\mu(r-t_1)} & \cdots & (r-t_{m_0})e^{\mu(r-t_{m_0})} \\ \vdots & \vdots & \ddots & \vdots \\ (r-t_0)^{m_0}e^{\mu(r-t_0)} & (r-t_1)^{m_0}e^{\mu(r-t_1)} & \cdots & (r-t_{m_0})^{m_0}e^{\mu(r-t_{m_0})} \end{pmatrix}.$$
This is an invertible matrix, and $(R, x) \mapsto R^{-1}x$ is continuous. By (iii), the right-hand side of (38) is infinitely close to $0 \in \mathbb{R}^{m_0+1}$. Continuity now implies (38) has a solution $\Delta_j = 0$ for all $j \in \{0, \dots, M-1\} \setminus \{j_0, \dots, j_{m_0}\}$, and $\Delta_j \approx 0$ for $j \in \{j_0, \dots, j_{m_0}\}$. Since $m_0$ is finite, (v) has been proved.

(v)$\Rightarrow$(ii). Fix $\lambda_0 \in S_\mu$ and let $\Delta_0, \dots, \Delta_{M-1}$, $\sum_{j=0}^{M-1}|\Delta_j| \approx 0$, such that $p_\Delta(\lambda)$ has a root of order $m_0+1$ at $\lambda_0$. Now let $\Gamma\colon {}^*[0,1] \to {}^*\mathbb{R}^M$ be an internal path joining $(\Delta_0, \dots, \Delta_{M-1})$ with 0. For $\tau \in {}^*[0,1]$ we have corresponding characteristic polynomials $p_{\Gamma(\tau)}(\lambda)$, and continuously depending roots $\lambda_m(\Gamma(\tau))$, $0 \le m \le m_0$. If all $\lambda_m(\Gamma(\tau)) \in S_\mu$ we are done. So assume that at least one $\lambda$ leaves $S_\mu$. But then by Lemma 5 (or Lemma 10 and $m_0 = 0$), we would have solutions $e^{\mu(\tau)(t+r)}$ of equation (26), where $\mathbb{C} \ni \mu(\tau) \neq \mu$ connects continuously with $\mu$. This cannot be, hence indeed no $\lambda_m(\tau)$ leaves $S_\mu$, and for $\tau = 1$ we have (at least) $m_0+1$ roots in $S_\mu$. $\square$

Lemma 10 implies, that in any $S_\mu$ there can only be finitely many eigenvalues of $A$. If we have exactly $m_0 \in \mathbb{N}$ eigenvalues within one given $S_\mu$, say $\lambda_1, \dots, \lambda_{m_0}$, we can change in the representation $Y = \sum_{j=1}^{M-1} \alpha_j v_j$ the part belonging to these eigenvalues. We use Lemma 9 to do this, and get $\sum_{j=1}^{m_0} \alpha_j v_j = \sum_{j=1}^{m_0} \beta_j \nu_{j,j}$. The new representation uses vectors $\nu_{j,j}$ representing linearly independent eigenfunctions instead of eigenvectors. These sums represent some standard function, and by Lemma 8 part (i), the contribution of this (partial) sum is finite, hence all coefficients $\beta_j$ have to be finite too. Note, that this is not true for the $\alpha_j$. A simple example is the case, that $N(\lambda_1 - 1) \approx N(\lambda_2 - 1)$, because then $v_1 \triangleq x(\theta) \triangleq v_2$ and $(v_1 - v_2)$ times an infinite number can still represent a function, for example $(\theta + r)e^{\mu(\theta+r)}$.

For $j = 1, \ldots, m_0$ let $\nu_{j,j}$, $\beta_j$ be defined as in (29) and (30), respectively. Then $^\circ(\beta_j \nu_{j,j})$ is the projection onto the eigenfunction $(\theta+r)^{j-1}e^{\mu(\theta+r)}/(j-1)!$. We will give now in Proposition 3 an explicit formula for these projections, using only $\mu$ and $\eta(\theta)$. The projection onto the "highest" eigenfunction is given by a quotient of integrals, the other ones are given by derivating this quotient with respect to the eigenvalue and inserting certain factors. This is a new result, because other methods only give a construction of the projections via a normalized basis for the eigenspaces of the equation and its transposed.

PROPOSITION 3. *Assume the linear autonomous one-dimensional* RFDE (26) *has an eigenvalue* $\mu \in \mathbb{C}$ *with corresponding eigenspace* $P_\mu = \operatorname{lin}\{z_1, \ldots, z_{m_0}\} \subset C([-r,0],\mathbb{C})$, *where* $z_m(\theta) = (\theta + r)^{m-1}e^{\mu(\theta+r)}/(m-1)!$, $m = 1, \ldots, m_0$. *There is a decomposition* $C([-r,0],\mathbb{C}) = P_\mu \oplus Q_\mu$, *such that the following holds:*

*If* $\{\mu \in \mathbb{C} : \mu \text{ is eigenvalue of equation (26)},\ \operatorname{Re}\mu \geq \rho\} = \{\mu_1, \ldots, \mu_q\}$ *and* $\operatorname{pr}_{\mu_j} \colon C([-r,0],\mathbb{C}) \to P_{\mu_j}$, $j = 1, \ldots, q$, *are the corresponding projections of* $C([-r,0],\mathbb{C}) = P_{\mu_j} \oplus Q_{\mu_j}$, *then for* $0 < \varepsilon \in \mathbb{R}$ *small enough*

$$\left| x(t) - \sum_{j=1}^{q} (\operatorname{pr}_{\mu_j}(\Phi))(t) \right| \leq Ce^{(\rho-\varepsilon)t}\|\Phi\|_\infty \quad \text{for all } t \geq 0,$$

*for a constant* $C \in \mathbb{R}$, $x(t)$ *the solution of* (26) *with initial value* $\Phi$.

*If we write* $\operatorname{pr}_\mu \Phi = \sum_{m=1}^{m_0} \Psi_m z_m$, *then*

$$\Psi_{m_0-m} = \frac{1}{m!} \sum_{\substack{\gamma \in \mathbb{N}^{m+1} \\ |\gamma| = m}} \frac{a_{\gamma,m}}{\prod_{j=1}^{m} \binom{m_0+\gamma_j}{\gamma_j}} \frac{N^{(\gamma_0)}(\mu) D^{(\gamma_1)}(\mu) \ldots D^{(\gamma_m)}(\mu)}{(D(\mu))^{m+1}}, \tag{39}$$

*where* $N(\mu)$, $D(\mu)$ *and* $a_{\gamma,m} \in \mathbb{R}$ *are defined by*

$$N(\mu) = \Phi(0) + \int_{-r}^{0} \mu e^{-\mu t}\Phi(t)\,dt - \int_{-r}^{0}\int_{-r}^{\theta} e^{\mu(\theta-t)}\Phi(t)\,dt\,d\eta(\theta),$$

$$D(\mu) = \frac{1}{m_0!}\left(e^{\mu r}(m_0 r^{m_0-1} + \mu r^{m_0}) - \int_{-r}^{0} (\theta + r)^{m_0} e^{\mu(\theta+r)} d\eta(\theta)\right), \tag{40}$$

$$\left(\frac{N(\mu)}{D(\mu)}\right)^{(m)} = \frac{1}{(D(\mu))^{m+1}} \sum_{\substack{\gamma\in\mathbb{N}^{m+1}\\ |\gamma|=m}} a_{\gamma,m} N^{(\gamma_0)}(\mu) D^{(\gamma_1)}(\mu)\dots D^{(\gamma_m)}(\mu).$$

Proof. As before, we describe the RFDE by $Y_0 = (y_0,\dots,y_{-M+1})^t \stackrel{\wedge}{=} \Phi$, $Y_n = AY_{n-1}$, $A$ as in (27). Without loss of generality, we can assume $A$ to have only simple eigenvalues $\lambda_1,\dots,\lambda_M$, so we get a basis of eigenvectors $v_1,\dots,v_M$ and a decomposition of ${}^*\mathbb{C}^M$ by $Y_0 = \sum_{j=1}^M \alpha_j v_j$, where $\alpha_j$ is given by (14), which in the one-dimensional case becomes

$$\alpha_j = \frac{y_0 + \sum_{l=1}^{M-1}(\lambda_j^l - \lambda_j^{l-1} - \frac{1}{N}\sum_{k=0}^{l-1} L_k \lambda_j^{l-1-k})y_{-l}}{p'(\lambda_j)} = \frac{K(Y_0,\lambda_j)}{p'(\lambda_j)}. \tag{41}$$

With Lemma l0 we have exactly $m_0$ eigenvalues in $S_\mu$, say $\lambda_1,\dots,\lambda_{m_0}$. Lemma 9 implies, that $P_\mu$ corresponds to $\mathrm{lin}\,\{v_1,\dots,v_{m_0}\} \subset {}^*\mathbb{C}^M$. $\alpha_1,\dots,\alpha_{m_0}$ don't have an interpretation as coefficients of functions, since there may be infinite, as we already mentioned before. So we change basis by Lemma 9 and

$$\sum_{m=1}^{m_0} \alpha_m v_m = \sum_{m=1}^{m_0} \beta_m \nu_{m,m},$$

where $\nu_{m,m} \stackrel{\wedge}{=} z_m$, $m = 1,\dots,m_0$. Define

$$\mathrm{pr}_\mu \Phi = {}^\circ\!\left(\sum_{m=1}^{m_0} \alpha_m v_m\right) = {}^\circ\!\left(\sum_{m=1}^{m_0} \beta_m \nu_{m,m}\right).$$

The so defined $\mathrm{pr}_\mu$ is the projection we are looking for: Assume $\mu_1,\dots,\mu_q \in \mathbb{C}$ are all the eigenvalues of the RFDE (26) satisfying $\mathrm{Re}\,\mu_j \ge \rho$, with corresponding eigenspaces $P_{\mu_j}$, $j = 1,\dots,q$. We define the projections $\mathrm{pr}_{\mu_j}$ accordingly. Then for $t \approx n/N$, $x(t)$ the solution of (26), with initial value $\Phi \stackrel{\wedge}{=} Y_0 = \sum_{j=1}^M \alpha_j v_j$, and $0 < \varepsilon \in \mathbb{R}$ small enough

$$\begin{aligned} \left| x(t) - \sum_{j=1}^q pr_{\mu_j}(\Phi)(t)\right| &\approx \left|\sum_{j=1}^M \alpha_j \lambda_j^n v_j - \sum_{\substack{j=1\\ N(|\lambda_j|-1)>\rho-\varepsilon}}^M \alpha_j \lambda_j^n v_j\right| \\ &= \left|\sum_{\substack{j=1\\ N(|\lambda_j|-1)<\rho-\varepsilon}}^M \alpha_j \lambda_j^n v_j\right| < C\left(1+\frac{\rho-\varepsilon}{N}\right)^n \|Y_0\|_\infty \approx Ce^{t(\rho-\varepsilon)}\|\Phi\|_\infty, \end{aligned}$$

using Lemma 8, for a constant $C \in \mathbb{R}$. Of course, we have to show, that $\sum_{j=1}^{m_0} \beta_j \mu_{j,j}$ represents a function, i.e. all $\beta_j$ are finite. For this and the proof of (39), it suffices to show ${}^\circ\beta_m = \Psi_m$, $m = 1,\dots,m_0$. $\beta_m$ is given by (see (30) and (41))

$$\beta_m = N^{m-1} \sum_{l=m}^{m_0} \frac{1}{\prod_{\substack{k=m\\ k\neq l}}^{m_0}(\lambda_l - \lambda_k)} \frac{K(Y_0,\lambda_l)}{\prod_{k=m_0+1}^M (\lambda_l - \lambda_k)}. \tag{42}$$

We have to compute the standard part of $\beta_m$. We start with $K(Y_0, \lambda_m)$:

$$K(Y_0,\lambda_m) = y_0 + \frac{1}{N}\sum_{l=1}^{M-1}\lambda_m^{l-1}N(\lambda_m-1)y_{-l} - \sum_{k=0}^{M-2}L_k\frac{1}{N}\sum_{l=k+1}^{M-1}\lambda_m^{l-1-k}y_{-l}$$
$$\approx \Phi(0) + \int_{-r}^{0}\mu e^{-\mu t}\Phi(t)\,dt - \int_{-r}^{0}\int_{-r}^{\theta}e^{\mu(\theta-t)}\Phi(t)\,dt\,d\eta(\theta) = N(\mu).$$

Before we continue to take standard parts to get $D(\mu)$, a few notations we will need later on. Let $\delta := \max\{|\lambda_j-\lambda_l| : 1 \le j \le l \le m_0\}$, $\Delta := \min\{|\lambda_j - \lambda_l| : 1 \le j \le m_0,\ m_0 < l \le M\}$. Since $\lambda_1, \dots, \lambda_{m_0}$ are all the eigenvalues in $S_\mu$, we have $N\Delta \not\approx 0$. And by Lemma l0 we can assume $\lambda_1,\dots,\lambda_{m_0}$ to be arbitrarily close to each other, in particular we assume at least $N^{m_0+1}\delta/\Delta \approx 0$.

Now we relate $D(\mu)$ to a derivative of $p(\lambda)$. For $1 \le m \le m_0$ we have

$$\begin{aligned}
(43)\qquad & N^{-m_0+1}p^{(m_0)}(\lambda_m)\\
&= \frac{M-1}{N}\frac{M-2}{N}\cdots\frac{M-m_0+1}{N}\lambda_m^{M-1-m_0}\Big(N(\lambda_m-1)\frac{M}{N}+m_0\Big)\\
&\quad -\sum_{k=0}^{M-1}L_k\frac{M-1-k}{N}\cdots\frac{M-k-m_0}{N}\lambda_m^{M-1-k-m_0}\\
&\approx r^{m_0-1}e^{\mu r}(\mu r+m_0) - \int_r^0(\theta+r)^{m_0}e^{\mu(\theta+r)}\,d\eta(\theta) = m_0!D(\mu),
\end{aligned}$$

and the maximality of $m_0$ implies, either directly or via Lemma 10, $D(\mu) \ne 0$. The link between the second part of the denominator in (42) and $p^{(m_0)}(\lambda)$ is:

$$\begin{aligned}
p^{(m_0)}(\lambda) &= m_0!\sum_{\substack{\gamma\in\{0,1\}^M\\|\gamma|=M-m_0}}(\lambda-\lambda_1)^{\gamma_1}\dots(\lambda-\lambda_M)^{\gamma_M}\\
&= m_0!\Big[\prod_{k=m_0+1}^{M}(\lambda-\lambda_k) + \sum_{l=1}^{m_0}\sum_{\substack{\gamma\in\{0,1\}^{m_0}\\|\gamma|=l}}(\lambda-\lambda_1)^{\gamma_1}\dots(\lambda-\lambda_{m_0})^{\gamma_{m_0}}\\
&\qquad\cdot\sum_{\substack{\gamma\in\{0,1\}^{M-m_0}\\|\gamma|=M-m_0-l}}(\lambda-\lambda_{m_0+1})^{\gamma_{m_0+1}}\dots(\lambda-\lambda_M)^{\gamma_M}\Big].
\end{aligned}$$

For $\lambda \in \{\lambda_1,\dots,\lambda_{m_0}\}$, each term in the sum can be estimated by $(\delta/\Delta)^l \cdot |\prod_{k=m_0+1}^{M}(\lambda-\lambda_k)|$, the number of these terms is bounded by $\binom{M}{m_0}$. Hence for these $\lambda$

$$p^{(m_0)}(\lambda) = m_0!\prod_{k=m_0+1}^{M}(\lambda-\lambda_k)[1+R],$$

where

$$|R| \le \sum_{l=1}^{m_0}\Big(\frac{\delta}{\Delta}\Big)^l\binom{M}{m_0} \le \frac{M^{m_0}}{m_0!}\frac{\delta}{\Delta}\cdot 2 \approx 0.$$

Plugging this in (43), we get $N^{-m_0+1}\prod_{k=m_0+1}^{M}(\lambda-\lambda_k)\approx D(\mu)$. That is, $N^{-m_0+1}$ times the second part of the denominator in (42) is infinitely close to $D(\mu)$, and $^\circ(\beta_{m_0})=\Psi_{m_0}$. In other words, (39) is true for $m=0$. To prove it for $m\geq 1$, note that if all $\lambda_j\neq\lambda_{m_0}$ are fixed, then $\beta_{m_0}=\beta_{m_0}(\lambda_{m_0})$ is an $*$-analytic function. There is a series expansion

$$\beta_{m_0}(\lambda)=N^{m_0-1}\frac{K(Y_0,\lambda)}{\prod_{k=m_0+1}^{M}(\lambda-\lambda_k)}=\sum_{j\in{}^*\mathbb{N}}\frac{\beta_{m_0}^{(j)}(\lambda_{m_0})}{j!}(\lambda-\lambda_{m_0})^j,$$

valid for at least $N|\lambda-\lambda_{m_0}|\approx 0$. Writing

$$\beta_j=N^{j-m_0}\sum_{l=j}^{m_0}\beta_{m_0}(\lambda_l)\frac{1}{\prod_{\substack{k=j\\k\neq l}}^{m_0}(\lambda_l-\lambda_k)},$$

we see, the sum in above expression is the leading coefficient of the polynomial interpolating $\beta_{m_0}(\lambda)$ at $\lambda_j,\ldots,\lambda_{m_0}$. Since for coinciding nodes, the normal interpolation becomes the Hermite interpolation, the coefficients of the interpolating polynomial of an analytic function are analytic functions themselves (of the nodes). Hence for $\lambda_1,\ldots,\lambda_{m_0}$ near enough to each other, we get

$$\beta_j=N^{j-m_0}\frac{\beta_{m_0}^{(m_0-j)}(\lambda_{m_0})}{(m_0-j)!}+\text{infinitesimal}.$$

We are interested in the standard part of $\beta_j$, so all we need is the derivative of $\beta_{m_0}$. Setting $D_\beta(\lambda)=N^{-m_0+1}\prod_{k=m_0+1}^{M}(\lambda-\lambda_K)$ we have

$$\begin{aligned}\beta_{m_0}(\lambda)&=\frac{K(Y_0,\lambda)}{D_\beta(\lambda)},\\ \beta_{m_0}^{(m)}(\lambda)&=\frac{1}{(D_\beta(\lambda))^{m+1}}\sum_{\substack{\gamma\in\mathbb{N}^{m+1}\\|\gamma|=m}}a_{\gamma,m}\frac{\partial^{\gamma_0}}{\partial\lambda^{\gamma_0}}K(Y_0,\lambda)D_\beta^{(\gamma_1)}(\lambda)\ldots D_\beta^{(\gamma_m)}(\lambda),\end{aligned}$$

where $a_{\gamma,m}$ has been defined in (40). The only thing still missing in the proof of (39) is to show for $1\leq m\leq m_0-1$ the last step in

$$\begin{aligned}^\circ\beta_{m_0-m}&\approx\frac{N^{-m}}{m!}\beta_{m_0}^{(m)}(\lambda)\\ &=\frac{N^{-m}}{m!(D_\beta(\lambda_{m_0}))^{m+1}}\\ &\quad\cdot\sum_{\substack{\gamma\in\mathbb{N}^{m+1}\\|\gamma|=m}}a_{\gamma,m}\frac{\partial^{\gamma_0}}{\partial\lambda^{\gamma_0}}K(Y_0,\lambda_{m_0})D_\beta^{(\gamma_1)}(\lambda_{m_0})\ldots D_\beta^{(\gamma_m)}(\lambda_{m_0})\\ &\approx\frac{1}{m!}\sum_{\substack{\gamma\in\mathbb{N}^{m+1}\\|\gamma|=m}}\frac{a_{\gamma,m}}{\prod_{k=1}^{m}\binom{m_0+\gamma_k}{\gamma_k}}\frac{N^{(\gamma_0)}(\mu)D^{(\gamma_1)}(\mu)\ldots D^{(\gamma_m)}(\mu)}{D^{m+1}(\mu)}.\end{aligned}$$

For this to be true, it is sufficient to show $N^{(m)}(\mu) \approx N^{-m}\partial^m K(Y_0,\lambda)/\partial\lambda^m$, and $D^{(m)}(\mu) \approx N^{-m}\binom{m_0+m}{m} D_\beta^{(m)}(\lambda)$, for all $\lambda \in S_\mu$, $0 \le m \le m_0$.

For $m = 0$ we have shown it already, and for $m > 0$

$$\begin{aligned}
N^{-m}\frac{\partial^m}{\partial\lambda^m}K(Y_0,\lambda) &= \frac{1}{N}\sum_{l=1}^{M-1} \underbrace{N^{-m+1}(\lambda^{l-1}(\lambda-1))^{(m)}}_{\approx(^\circ(l/N))^{m-1}e^{\mu^\circ(l/N)}(^\circ(l/N)\mu+m)} y_{-l} \\
&\quad - \sum_{k=0}^{M-2}\frac{L_k}{N}\sum_{l=k+1}^{M-1} \underbrace{N^{-m}(\lambda^{l-1-k})^{(m)}}_{\approx^\circ((l-k)/N)^m e^{\mu^\circ((l-k)/N)}} y_{-l} \\
&\approx \int_{-r}^{0}(-t)^{m-1}e^{-\mu t}(-\mu t+m)\Phi(t)\,dt \\
&\quad - \int_{-r}^{0}\int_{-r}^{\theta}(\theta-t)^m e^{\mu(\theta-t)}\Phi(t)\,dt\,d\eta(\theta) = N^{(m)}(\mu),
\end{aligned}$$

and for $\lambda_1, \ldots, \lambda_{m_0}$ sufficiently near to each other

$$\begin{aligned}
&\frac{N^{-m_0-m+1}}{m_0!}p^{(m_0+m)}(\lambda) \\
&\quad = N^{-m_0-m+1}\frac{(m_0+m)!}{m_0!}\sum_{\substack{\gamma\in\{0,1\}^M \\ |\gamma|=M-m_0-m}} (\lambda-\lambda_1)^{\gamma_1}\ldots(\lambda-\lambda_M)^{\gamma_M} \\
&\quad = N^{-m_0-m+1}\frac{(m_0+m)!}{m_0!}\Bigg[\sum_{\substack{\gamma\in\{0,1\}^{M-m_0} \\ |\gamma|=M-m_0-m}} (\lambda-\lambda_{m_0+1})^{\gamma_1}\ldots(\lambda-\lambda_M)^{\gamma_{M-m_0}} \\
&\qquad + \sum_{l=1}^{m_0}\sum_{\substack{\gamma\in\{0,1\}^{m_0} \\ |\gamma|=l}} (\lambda-\lambda_1)^{\gamma_1}\ldots(\lambda-\lambda_{m_0})^{\gamma_{m_0}} \\
&\qquad \cdot \sum_{\substack{\gamma\in\{0,1\}^{M-m_0} \\ |\gamma|=M-m_0-m-l}} (\lambda-\lambda_{m_0+1})^{\gamma_{m_0+1}}\ldots(\lambda-\lambda_M)^{\gamma_M}\Bigg] \\
&\quad = N^{-m}\binom{m_0+m}{m} D_\beta^{(m)}(\lambda) + \text{infinitesimal},
\end{aligned}$$

by the definition of $D_\beta(\lambda)$. On the other hand

$$\begin{aligned}
&\frac{N^{-m_0-m+1}}{m_0!}p^{(m_0+m)}(\lambda) \\
&= \frac{1}{m_0!}\Bigg[\frac{M-1}{N}\cdots\frac{M-m_0-m+1}{N}\lambda^{M-m_0-m-1}\left(\frac{M}{N}N(\lambda-1)+m_0+m\right) \\
&\quad - \sum_{k=0}^{M-1} L_k\frac{M-1-k}{N}\cdots\frac{M-m_0-m-k}{N}\lambda^{M-1-m_0-m-k}\Bigg]
\end{aligned}$$

$$\approx \frac{1}{m_0!}\left[r^{m_0+m-1}e^{\mu r}(r\mu+m_0+m)-\int_{-r}^{0}(r+\theta)^{m_0+m}e^{\mu(r+\theta)}\,d\eta(\theta)\right]=D^{(m)}(\mu)$$

and the proof is complete. □

As a last application, we show how the linear operator $L$ changes if one exchanges one eigenvalue $\mu_0$ by an arbitrary (complex) number $\mu_1$, leaving all other eigenvalues unchanged. Of course, if $\mu_0$ or $\mu_1$ is complex one has to repeat the step with the conjugated number to get a real RFDE.

Pandolfi showed in [10], how one can change a finite number of eigenvalues. He gives a method to construct the resulting linear operator (in various dimensions). Here we only change one eigenvalue at a time, and have a one-dimensional RFDE, but we give an explicit formula for the resulting operator, not only a way to construct it.

Lemma 11. *Let $\mu_0$ be an eigenvalue of the* RFDE

$$x'(t)=\int_{-r}^{0}x(t+\theta)\,d\eta_0(\theta) \tag{44}$$

*and $\mu_1\in\mathbb{C}$ be an arbitrary number. Then the* RFDE

$$x'(t)=\int_{-r}^{0}x(t+\theta)\,d\eta_1(\theta),$$

*where $\eta_1(\theta)$ is defined by $\eta_1(0)=0$, and for $-r\le\theta<0$ by* (46) *below, has exactly the same eigenvalues* (*including multiplicities*) *as in the* RFDE (44)*, with the only exception of one eigenvalue $\mu_0$ having become $\mu_1$.*

Proof. Let $p_m(\lambda)=\lambda^M-\lambda^{M-1}-q_m(\lambda)/N$, $q_m(\lambda)=\sum_{j=0}^{M-1}L_j^{(m)}\lambda^{M-1-j}$, $m=0,1$, where $p_0(\lambda)$ is the characteristic polynomial of equation (44). We have to find $L_j^{(1)}$, so that the conclusion holds.

We can choose $L_j^{(0)}={}^*\eta_0(-j/N)-{}^*\eta_0((-j-1)/N)$ (without loss of generality assume $M/N<r$). Let $\lambda_0,\ \lambda_1\in{}^*\mathbb{C}$, such that $N(\lambda_m-1)\approx\mu_m$, $m=0,1$, and assume $p_0(\lambda_0)=0$.

Define $L_j^{(1)}$ by $p_1(\lambda)\equiv p_0(\lambda)(\lambda-\lambda_1)/(\lambda-\lambda_0)$. We will show $\sum_{j=0}^{M-1}|L_j^{(1)}|$ to be finite, and by Lemma 6 we get $\eta_1(\theta)$, which by Lemma 10 satisfies the requirements of this lemma.

Assume $\lambda_0,\lambda_2,\dots,\lambda_M$ to be the roots of $p_0(\lambda)$. Then

$$q_1(\lambda)-q_0(\lambda)=N(p_0(\lambda)-p_1(\lambda))=N\prod_{j=2}^{M}(\lambda-\lambda_j)(\lambda_1-\lambda_0) \tag{45}$$

and

$$\prod_{j=2}^{M}(\lambda-\lambda_j)=\frac{p_0(\lambda)}{\lambda-\lambda_0}=\sum_{j=0}^{M-1}\lambda^{M-1-j}\underbrace{\sum_{l=0}^{j}\lambda_0^l a_{M-j+l}}_{=:b_j},$$

where $a_M = 1$, $a_{M-1} = -1 - L_0^{(0)}/N$, $a_j = -L_{M-j-1}^{(0)}/N$, $j = 0, \dots, M-2$. $\sum_{j=0}^{M-1} |L_j^{(0)}|$ is bounded, by say $K_0 \in \mathbb{R}$, so we get $b_0 = a_M = 1$ and, for $j = 1, \dots, M-1$,

$$\begin{aligned} |b_j| &= \left| \lambda_0^j - \lambda_0^{j-1} - \frac{1}{N} \sum_{l=0}^{j-1} \lambda_0^{j-1-l} L_l^{(0)} \right| \\ &\leq \frac{1}{N} [|\lambda_0|^{j-1} N |\lambda_0 - 1| + (1 + |\lambda_0|^M) K_0] = \frac{\text{finite}}{N}. \end{aligned}$$

Hence $\sum_{j=0}^{M-1} |b_j|$ is finite. By (45) $L_j^{(1)} = L_j^{(0)} + N(\lambda_1 - \lambda_0) b_j$, and together with $N(\lambda_1 - \lambda_0)$ finite, $\sum_{j=0}^{M-1} |L_j^{(1)}|$ is finite too. We conclude the proof by giving an explicit formula for $\eta_1(\theta)$, as defined in (16) using $L_j^{(1)}$.

$\eta_1(0) := 0$, and for $-r \leq \theta < 0$, $n/N \leq -\theta$ maximal, define $\eta_1(\theta)$ by

$$\begin{aligned} \eta_1(\theta) &\approx -\sum_{j=0}^{n} L_j^{(1)} \\ &= -\sum_{j=0}^{n} L_j^{(0)} + N(\lambda_0 - \lambda_1) \Bigg( \sum_{j=0}^{n} \lambda_0^j - \sum_{j=1}^{n} \lambda_0^{j-1} - \frac{1}{N} \sum_{j=1}^{n} \sum_{l=0}^{j-1} \lambda_0^{j-1-l} L_l^{(0)} \Bigg) \\ &\approx -\int_\theta^0 d\eta_0(t) \\ &\quad + (\mu_0 - \mu_1) \left( 1 + \mu_0 \int_\theta^0 e^{-\mu_0 t} dt - \int_\theta^0 \int_\theta^t e^{\mu_0(t-s)} \, ds \, d\eta_0(t) \right). \end{aligned}$$

On both sides there are standard quantities, which therefore have to be equal, and we get finally: $\eta_1(0) = 0$, and for $-r \leq \theta < 0$

$$\eta_1(\theta) = \begin{cases} \dfrac{\mu_1}{\mu_0}(\eta_0(\theta) - \eta_0(0)) + (\mu_0 - \mu_1) e^{-\mu_0 \theta} & \\ \qquad + \left( \dfrac{\mu_1}{\mu_0} - 1 \right) \displaystyle\int_\theta^0 e^{\mu_0(t-\theta)} \, d\eta_0(t) & \text{for } \mu_0 \neq 0, \\ \eta_0(\theta) - \eta_0(0) - \mu_1 + \mu_1 \displaystyle\int_\theta^0 (t - \theta) \, d\eta_0(t) & \text{for } \mu_0 = 0. \end{cases} \tag{46}$$

This $\eta_1(\theta)$ satisfies the conclusion of Lemma 11. □

## 6. Conclusion

Nonstandard Analysis has been applied fruitfully to various areas of the theory of ODE's (see e.g. Benoit's article in [1], [3], [6]), but to our knowledge not to RFDE's so far. The main advantage of Nonstandard Analysis in the theory of ODE's, in our mind, is that it offers alternative descriptions, together with the necessary tools. These are often very intuitive, making an understanding and subsequently an investigation easier.

This paper is a first presentation of above mentioned nonstandard description of RFDE's. It treats only the linear autonomous case in more detail. The whole field of non-linear and non-autonomous equations, while the description is applicable in these cases too, has still to be looked at more closely. Even in the case of autonomous linear equations of various dimensions, the counterparts of some results in one dimension remain to be done (see Lemma 11, part of Lemma 10, and Proposition 3).

The framework presented here is very new, and thus applications are few. Still, two examples of new standard results we got with this approach, have been included. In both cases we use simple representations in the nonstandard framework to get explicit formulas for standard quantities (see Proposition 3 and Lemma 11). In our opinion these applications show, that it is worthwhile to pursue our approach further. In particular, the possibility of relating the characteristic equation of a linear autonomous RFDE to a polynomial seems promising for further exploitation.

In the case of non-linear RFDE, the finite dimensionality (within Nonstandard Analysis) gives advantages too, but before one can think seriously of exploiting this to advance the (standard) theory of RFDE, there has to be a closer look into the features of our description in this more general case.

If one chooses $M/N$ infinite instead of near to the finite delay $r$, then one would have infinite delay. Proposition 1, which proves the applicability of our method, does not apply to this case. But if it were applicable, within the nonstandard description there would be no change at all. Obviously, one has to think about how far in this case the nonstandard features of the description remain interpretable in the real world. A new description cannot get rid of the differences between RFDE with finite, respectively infinite delay. Still, this is an other interesting problem to look into.

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