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MULTIPLE POSITIVE SOLUTIONS FOR A SINGULARLY PERTURBED DIRICHLET PROBLEM IN “GEOMETRICALLY TRIVIAL” DOMAINS

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Abstract. In this paper we consider the singularly perturbed Dirichlet problem (P_ε), when the potential $a_\varepsilon(x)$, as ε goes to 0, is concentrating round a point $x_0 \in \Omega$. Under suitable growth assumptions on f, we prove that (P_ε) has at least three distinct solutions whatever Ω is and that at least one solution is not a one-peak solution.

1. Introduction

In this paper we consider the problem

$$(\mathrm{P}_\varepsilon) \qquad \begin{cases} -\varepsilon^2\Delta u + a_\varepsilon(x)u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, is a bounded domain having smooth boundary $\partial\Omega$, $\varepsilon \in \mathbb{R}^+ \setminus \{0\}$, $f\colon \mathbb{R}^+ \to \mathbb{R}$ is a $\mathcal{C}^{1,1}$ superlinear function and a_ε is a given nonnegative function of the form

$$a_\varepsilon(x) = a_\infty + \alpha\left(\frac{x - x_0}{\varepsilon}\right), \quad a_\infty \in \mathbb{R}^+, \; x_0 \in \Omega.$$

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During the last ten years the relations between the shape of Ω and the multiplicity of solutions to problems like (P_ε), when $\varepsilon \to 0$ have been intensively investigated. Most of the results are concerned with problems like

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

and we cannot mention here all of them. We recall only that at the beginning the effect of the domain topology was pointed out, by giving a lower bound to the number of solutions of (1.1) in terms of suitable topological invariants of Ω ([1]–[3]). Subsequently the role of the geometry of Ω and the importance of the distance function $d(x, \partial\Omega)$ have been stressed more and more. Starting from the fact (proven in [10]) that any least energy solution, when ε is suitably small, has a single spike layer, which converges to the point where the distance function admits its global maximum, the existence of single peaked solutions to (1.1) has been shown to be strictly linked to the existence of critical points of the distance function (see [11], [7], [4], [5] and references therein).

On the other hand, it is well known that if Ω is a ball (1.1) admits only one positive solution ([6]).

The aim of this paper is to show that multiplicity results can be obtained even if the domain is "geometrically trivial" (in the sense that the distance function admits only its global maximum as critical point) when the linear term contains a piece concentrating, as $\varepsilon \to 0$, around some point of Ω.

We make the following assumptions:

- (H_1) there exists $k \in \mathbb{R}$, $k > 0$ such that, for every $t > 0$,

$$|f(t)| < k + kt^p,$$
$$|f'(t)| < k + kt^{p-1},$$

where $p > 1$, and $p < (N+2)/(N-2)$ for $N \geq 3$.

- (H_2) there exists $\theta \in (0, 1/2)$ such that

$$F(t) \leq \theta t f(t) \quad \text{for all } t > 0,$$

where

$$F(t) = \int_0^t f(s)\, ds \quad \text{for all } t > 0,$$

- (H_3) $f(0) = f'(0) = 0$,
- (H_4) $\dfrac{d}{dt}\left(\dfrac{f(t)}{t}\right) > 0$ for all $t > 0$,
- (H_5) $a_\infty > 0$, $\alpha(x) \geq 0$, $\alpha \in L^{N/2}(\mathbb{R}^N)$, $|\alpha|_{L^{N/2}(\mathbb{R}^N)} \neq 0$.

The result we obtain is

Theorem 1.1. *Suppose the assumptions* (H_1)–(H_5) *are satisfied, then there exists* $\varepsilon^* > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon^*]$, *problem* (P_ε) *has at least three distinct solutions. Moreover, at least one solution is not a one-peak solution.*

The proof of the above theorem is contained in Section 3, while in Section 2 some useful facts are collected.

2. The functional analytic setting and some useful facts

Throughout the paper we make use of the following notations

- $L^p(\mathcal{D})$, $1 \le p < \infty$, $\mathcal{D} \subseteq \mathbb{R}^N$ denotes a Lebesgue space; the norm in $L^p(\mathcal{D})$ is denoted by $|\cdot|_{p,\mathcal{D}}$.
- $H_0^1(\mathcal{D})$, $\mathcal{D} \subseteq \mathbb{R}^N$ denotes the Sobolev space obtained as closure of $\mathcal{C}_0^\infty(\mathcal{D})$ under the norm
$$\|u\|_{\mathcal{D}} = \left[\int_{\mathcal{D}} (|\nabla u|^2 + a_\infty u^2)\,dx\right]^{1/2}.$$
If $u \in H_0^1(\mathcal{D}_1)$ and $\mathcal{D}_1 \subset \mathcal{D}_2 \subseteq \mathbb{R}^N$ we denote also by u its extension to $\mathcal{D}_2$ made setting $u \equiv 0$ outside of $\mathcal{D}_1$.
- $\mathcal{B}_\rho(y)$ denotes the open ball of radius ρ centered at y in $\mathbb{R}^N$:
$$\mathcal{B}_\rho(y) = \{x \in \mathbb{R}^N : |x - y|_{\mathbb{R}^N} < \rho\}.$$
- $\mathcal{D}_\varepsilon$ denotes the subset of $\mathbb{R}^N$ $\{y \in \mathbb{R}^N : \varepsilon y \in \mathcal{D}\}$, $\mathcal{D} \subset \mathbb{R}^N$.
- For what follows it is also useful to extend f and F to $\mathbb{R}^-$ in the following way
$$f(t) = 0 \quad \text{for } t < 0,$$
$$F(t) = 0 \quad \text{for } t < 0.$$

On $H_0^1(\mathcal{D})$ we consider the functionals E_ε and $\mathcal{G}_\varepsilon$ defined by

$$E_\varepsilon(u) = \frac{1}{2}\int_{\mathcal{D}} (\varepsilon^2|\nabla u|^2 + a_\varepsilon(x)u^2)\,dx - \int_{\mathcal{D}} F(u)\,dx, \tag{2.1}$$

$$\mathcal{G}_\varepsilon(u) = E'_\varepsilon(u)[u] = \int_{\mathcal{D}} (\varepsilon^2|\nabla u|^2 + a_\varepsilon(x)u^2)\,dx - \int_{\mathcal{D}} f(u)u\,dx. \tag{2.2}$$

Let us remark that by the assumptions, E_ε is a $\mathcal{C}^2$-functional on $H_0^1(\mathcal{D})$, so $\mathcal{G}_\varepsilon$ is well defined and $\mathcal{C}^1$.

Set

$$V_\varepsilon(\mathcal{D}) = \{u \in H_0^1(\mathcal{D}) : u \ne 0 \text{ and } \mathcal{G}_\varepsilon(u) = 0\} \tag{2.3}$$

and denote by

$$\mathcal{S}_{\mathcal{D}} = \{u \in H_0^1(\mathcal{D}) : \|u\|_{\mathcal{D}} = 1\} \setminus \{u \in H_0^1(\mathcal{D}) : u \le 0 \text{ a.e.}\}. \tag{2.4}$$

The following three lemmas collect the properties of E_ε and $V_\varepsilon(\mathcal{D})$.

Lemma 2.1. *For any $\mathcal{D} \subseteq \mathbb{R}^N$, $V_\varepsilon(\mathcal{D})$ is a smooth manifold of codimension* 1 *in $H_0^1(\mathcal{D})$. $V_\varepsilon(\mathcal{D})$ is diffeomorphic to $\mathcal{S}_\mathcal{D}$ by a $\mathcal{C}^{1,1}$-diffeomorphism $\psi_\varepsilon\colon \mathcal{S}_\mathcal{D} \to V_\varepsilon(\mathcal{D})$. Moreover, there exist $h_\varepsilon = h_\varepsilon(\mathcal{D}) > 0$ and $k_\varepsilon = k_\varepsilon(\mathcal{D}) > 0$ such that, for any $u \in V_\varepsilon(\mathcal{D})$,*

$$\begin{cases} \text{(i)} \quad \|u\|_\mathcal{D} \geq h_\varepsilon, \\ \text{(ii)} \quad E_\varepsilon(u) \geq k_\varepsilon. \end{cases} \tag{2.5}$$

Lemma 2.2. *For any $\mathcal{D} \subset \mathbb{R}^N$, bounded, the Palais–Smale condition holds for both the free functional E_ε and the functional E_ε constrained on $V_\varepsilon(\mathcal{D})$.*

Lemma 2.3. *For any $\mathcal{D} \subseteq \mathbb{R}^N$, $u \in H_0^1(\mathcal{D})$ is a free critical point of E_ε if and only if u is a critical point of E_ε constrained on $V_\varepsilon(\mathcal{D})$.*

The above listed properties can be proven by the same arguments used in [2] (Lemmas 2.1–2.4).

It is useful to remark that $V_\varepsilon(\mathcal{D})$ turn out to be the graph of a $\mathcal{C}^{1,1}$-function ψ_ε defined on $\mathcal{S}_\mathcal{D}$ by

$$\psi_\varepsilon(\overline{u}) = \xi_\varepsilon(\overline{u})\overline{u}, \quad \overline{u} \in \mathcal{S}_\mathcal{D}, \tag{2.6}$$

$\xi_\varepsilon(\overline{u})$ being the unique positive number which realizes the maximum of the function defined on $\mathbb{R}^+$ by

$$\lambda \to E_\varepsilon(\lambda \overline{u}).$$

Moreover, $\xi_\varepsilon\colon \mathcal{S}_\mathcal{D} \to \mathbb{R}$ is a $\mathcal{C}^{1,1}$-function.

We set

$$m_\varepsilon(\mathcal{D}) = \inf\{E_\varepsilon(u),\ u \in V_\varepsilon(\mathcal{D})\}. \tag{2.7}$$

By (2.5)(ii) $m_\varepsilon(\mathcal{D})$ is well defined and positive. Moreover, whenever $\mathcal{D} \subset \mathbb{R}^N$ is bounded the infimum is achieved since $E_\varepsilon(u)$ satisfies the (PS) condition on $V_\varepsilon(\mathcal{D})$.

It is clear that *to critical points $u \in H_0^1(\Omega)$ of E_ε, there correspond* (*weak*) *solutions of* (P_ε), *so the existence of at least one solution of* (P_ε), *having energy $m_\varepsilon(\Omega)$, easily follows using Lemma* 2.3.

Setting, for all $u \in H_0^1(\mathcal{D})$

$$u_\varepsilon(x) := u(\varepsilon x) \tag{2.8}$$

we obtain a one to one map between $H_0^1(\mathcal{D})$ and $H_0^1(\mathcal{D}_\varepsilon)$ such that

$$\begin{aligned} &\frac{1}{2}\int_\mathcal{D} \bigg(\varepsilon^2|\nabla u|^2 + \bigg[a_\infty + \alpha\bigg(\frac{x - x_0}{\varepsilon}\bigg)\bigg]u^2\bigg)\,dx - \int_\mathcal{D} F(u)\,dx \\ &\qquad = \varepsilon^N\bigg[\frac{1}{2}\int_{\mathcal{D}_\varepsilon} (|\nabla u_\varepsilon|^2 + [a_\infty + \alpha(x - x_0)]u_\varepsilon^2)\,dx - \int_{\mathcal{D}_\varepsilon} F(u_\varepsilon)\,dx\bigg], \end{aligned} \tag{2.9}$$

and

$$\int_{\mathcal{D}}\left(\varepsilon^2|\nabla u|^2+\left[a_\infty+\alpha\left(\frac{x-x_0}{\varepsilon}\right)\right]u^2\right)dx-\int_{\mathcal{D}}f(u)u\,dx \tag{2.10}$$
$$=\varepsilon^N\left[\int_{\mathcal{D}_\varepsilon}(|\nabla u_\varepsilon|^2+[a_\infty+\alpha(x-x_0)]u_\varepsilon^2)\,dx-\int_{\mathcal{D}_\varepsilon}f(u_\varepsilon)u_\varepsilon\,dx\right].$$

Hence (2.8) maps in a one to one way critical points of E_ε constrained on $V_\varepsilon(\mathcal{D})$ in critical points of E_1 constrained in $V_1(\mathcal{D}_\varepsilon)$. In particular we have

$$m_\varepsilon(\mathcal{D})=\varepsilon^N m_1(\mathcal{D}_\varepsilon). \tag{2.11}$$

Let us denote by

$$M_\infty=\inf\left\{\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+a_\infty u^2)\,dx-\int_{\mathbb{R}^N}F(u)\,dx:\right. \tag{2.12}$$
$$\left.u\in H_0^1(\mathbb{R}^N),\ u\neq 0,\ \int_{\mathbb{R}^N}(|\nabla u|^2+a_\infty u^2)\,dx=\int_{\mathbb{R}^N}f(u)u\,dx\right\}$$

it is well known (see [2, Lemma 3.1]) that the following result holds:

LEMMA 2.4. *M_∞ is achieved. Any function ω that realizes M_∞ is positive, radially symmetric about some point in $\mathbb{R}^N$, decreasing when the radial coordinate increases and such that*

$$\lim_{\rho\to\infty}|D^\alpha\omega(\rho)|\rho^{(N-1)/2}\exp(\sqrt{a_\infty}\rho)=\text{const.}>0,\quad \alpha=0,1. \tag{2.13}$$

REMARK 2.5. By the definition of M_∞, it is clear that to any function ω that realizes M_∞ there corresponds a class of functions, obtained from ω by translations having the same properties. The uniqueness modulo translations of the function ω realizing M_∞ has been proven in [8] when $f(u)=u^p$.

LEMMA 2.6. *Let $\alpha\in L^{N/2}(\mathbb{R}^N)$ be such that $\alpha(x)\geq 0$ for all $x\in\mathbb{R}^N$. Let ω be a function, radially symmetric about $y_0\in\mathbb{R}^N$, that realizes M_∞. Then for all $x_0\in\mathbb{R}^N$ fixed*

$$\lim_{r\to\infty}\sup\left\{\int_{\mathbb{R}^N}\alpha(x-x_0)(\omega(x+y_0-y))^2\,dx:|y-y_o|=r\right\}=0. \tag{2.14}$$

PROOF. Without any loss of generality we can suppose $y_0=0$. Let us assume, by way of contradiction, that there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$, $y_n\in\mathbb{R}^N$, $\lim_{n\to\infty}|y_n|=+\infty$, such that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\alpha(x-x_0)(\omega(x-y_n))^2\,dx>0. \tag{2.15}$$

We have for any $r \in \mathbb{R},\ r > 0$,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \alpha(x-x_0)(\omega(x-y_n))^2 dx \\
&= \int_{B_r(y_n)} \alpha(x-x_0)(\omega(x-y_n))^2 dx + \int_{\mathbb{R}^N\setminus B_r(y_n)} \alpha(x-x_0)(\omega(x-y_n))^2 dx \\
&\leq |\omega|^2_{2^*,\mathbb{R}^N} \bigg(\int_{B_r(y_n)} |\alpha(x-x_0)|^{N/2}\, dx \bigg)^{2/N} \\
&\quad + |\alpha|_{N/2,\mathbb{R}^N} \bigg(\int_{\mathbb{R}^N\setminus B_r(y_n)} |\omega(x-y_n)|^{2^*} dx \bigg)^{2/2^*}.
\end{aligned}
$$

Now

$$\lim_{n\to\infty} \int_{B_r(y_n)} |\alpha(x-x_0)|^{N/2} dx = 0,$$

so we have, for all $r > 0$,

$$
\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{R}^N} \alpha(x-x_0)(\omega(x-y_n))^2\, dx \\
&\qquad\qquad \leq |\alpha|_{N/2,\mathbb{R}^N} \bigg(\int_{\mathbb{R}^N\setminus B_r(0)} |\omega(x)|^{2^*} dx \bigg)^{2/2^*} + o(1),
\end{aligned}
$$

hence the relation

$$\lim_{r\to\infty} \int_{\mathbb{R}^N\setminus B_r(0)} |\omega(x)|^{2^*}\, dx = 0$$

gives

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \alpha(x-x_0)(\omega(x-y_n))^2\, dx = 0$$

contradicting (2.15). □

LEMMA 2.7. *Let* $\alpha \in L^{N/2}(\mathbb{R}^N)$ *be such that* $\alpha(x) \geq 0$ *for all* $x \in \mathbb{R}^N$, $|\alpha|_{N/2,\mathbb{R}^N} \neq 0$. *Put*

$$
\begin{aligned}
\text{(2.16)}\qquad M_a &:= \inf\{E_1(u) : u \in H^1_0(\mathbb{R}^N),\ u \in V_1(\mathbb{R}^N)\} \\
&= \inf\bigg\{ \frac{1}{2} \int_{\mathbb{R}^N} [|\nabla u|^2 + (a_\infty + \alpha(x-x_0))u^2]\, dx - \int_{\mathbb{R}^N} F(u)\, dx : \\
&\qquad u \in H^1_0(\mathbb{R}^N), \\
&\qquad \int_{\mathbb{R}^N} [|\nabla u|^2 + (a_\infty + \alpha(x-x_0))u^2]\, dx = \int_{\mathbb{R}^N} f(u)u\, dx \bigg\}
\end{aligned}
$$

then

$$M_a = M_\infty \tag{2.17}$$

and the minimization problem (2.16) *has no solution.*

PROOF. For every $\overline{u} \in \mathcal{S}_{\mathbb{R}^N}$, let $\xi_\infty(\overline{u})$ and $\xi_1(\overline{u})$ be, respectively, the unique positive numbers such that

$$\int_{\mathbb{R}^N} [|\nabla(\xi_\infty(\overline{u})\ \overline{u})|^2 + a_\infty(\xi_\infty(\overline{u})\ \overline{u})^2]\,dx = \int_{\mathbb{R}^N} f(\xi_\infty(\overline{u})\ \overline{u})\xi_\infty(\overline{u})\overline{u}\,dx$$

and $\xi_1(\overline{u})\ \overline{u} \in V_1(\mathbb{R}^N)$.

Since $\xi_1(\overline{u})$ is the positive number that realizes $\max\{E_1(\lambda\overline{u}) : \lambda \in \mathbb{R}^+\}$ we have

$$\begin{aligned}
(2.18)\qquad & \frac{1}{2}\int_{\mathbb{R}^N} [|\nabla(\xi_\infty(\overline{u})\overline{u})|^2 + a_\infty(\xi_\infty(\overline{u})\overline{u})^2]\,dx - \int_{\mathbb{R}^N} F(\xi_\infty(\overline{u})\overline{u})\,dx \\
&\leq \frac{1}{2}\int_{\mathbb{R}^N} [|\nabla(\xi_\infty(\overline{u})\overline{u})|^2 + (a_\infty + \alpha(x-x_0))(\xi_\infty(\overline{u})\overline{u})^2]\,dx \\
&\quad - \int_{\mathbb{R}^N} F(\xi_\infty(\overline{u})\overline{u})\,dx \leq E_1(\xi_1(\overline{u})\overline{u}).
\end{aligned}$$

Then $M_a \geq M_\infty$.

To see that the equality holds, let us denote by $\omega \in H_0^1(\mathbb{R}^N)$ a function radially symmetric about the origin that realizes M_∞, by $\{y_n\}$ a sequence of points in $\mathbb{R}^N$ such that $\lim_{n\to\infty} |y_n| = +\infty$, and consider

$$v_n = t_n\ \omega(x - y_n),$$

where

$$t_n = \xi_1\bigg(\frac{\omega(x-y_n)}{\|\omega\|_{\mathbb{R}^N}}\bigg)\|\omega\|_{\mathbb{R}^N}^{-1}.$$

Then $v_n \in V_1(\mathbb{R}^N)$ and, because of (2.14), $t_n \to 1$, $E_1(v_n) \to M_\infty$ as $n \to \infty$. Finally, assume that a function Ψ exists such that

$$\Psi \in V_1(\mathbb{R}^N), \quad E_1(\Psi) = M_\infty,$$

then $\Psi(x) \geq 0$ a.e. in $\mathbb{R}^N$ and applying (2.18) to $\Psi/\|\Psi\|_{\mathbb{R}^N} = \widehat{\Psi}$ we obtain

$$\frac{1}{2}\int_{\mathbb{R}^N} [|\nabla(\xi_\infty(\widehat{\Psi})\widehat{\Psi})|^2 + a_\infty(\xi_\infty(\widehat{\Psi})\widehat{\Psi})^2]\,dx - \int_{\mathbb{R}^N} F(\xi_\infty(\widehat{\Psi})\widehat{\Psi})\,dx = M_\infty$$

and

$$\int_{\mathbb{R}^N} \alpha(x - x_0)\widehat{\Psi}^2\,dx = 0,$$

contradicting Lemma 2.4 and the assumptions on α. □

3. Proof of the result

In what follows without any loss of generality we shall assume $0 \in \Omega$, $x_0 = 0$, $a_\infty = 1$. Moreover, we denote by ρ a positive real number such that $B_{4\rho}(0) \subset \Omega$ and by A_ρ the subset of Ω

$$A_\rho = \{x \in \mathbb{R}^N : 2\rho \leq |x| \leq 3\rho\}.$$

We define for all $\varepsilon > 0$ a map $\beta_\varepsilon : H_0^1(\Omega) \to \mathbb{R}^N$ by

$$\beta_\varepsilon(u) := \frac{1}{|u|_{p,\Omega}^p} \int_\Omega \chi_\varepsilon(x)|u(x)|^p\,dx \tag{3.1}$$

where

$$\chi_\varepsilon(x) = \begin{cases} x & \text{if } |x| \le \varepsilon, \\ \varepsilon \dfrac{x}{|x|} & \text{if } |x| \ge \varepsilon, \end{cases}$$

and we set

$$c_\varepsilon = \inf\,\{E_\varepsilon(u) : u \in V_\varepsilon(\Omega),\ |\beta_\varepsilon(u)| < \varepsilon/2\}. \tag{3.2}$$

Also, for all $u \in H_0^1(\mathbb{R}^N)$, we put

$$\beta(u) := \frac{1}{|u|_{p,\Omega}^p} \int_{\mathbb{R}^N} \chi(x)|u(x)|^p\,dx \tag{3.3}$$

where

$$\chi(x) = \begin{cases} x & \text{if } |x| \le 1, \\ \dfrac{x}{|x|} & \text{if } |x| \ge 1. \end{cases}$$

For any $\varepsilon > 0$ and $y \in A_\rho$ we consider the function

$$\varphi_y^\varepsilon(x) := \frac{\zeta(x)\omega((x-y)/\varepsilon)}{\|\zeta(x)\omega((x-y)/\varepsilon)\|_\Omega}, \tag{3.4}$$

where $\omega \in H_0^1(\mathbb{R}^N)$ is a positive function spherically symmetric about the origin that realizes M_∞ and $\zeta\colon \mathbb{R}^N \to [0,1]$ is defined by

$$\zeta(x) = \widehat{\zeta}\left(\frac{|x-y|}{\rho}\right),$$

$\widehat{\zeta}\colon \mathbb{R}^+ \to [0,1]$ being a decreasing $\mathcal{C}^\infty$-function such that

$$\widehat{\zeta}(t) = \begin{cases} 1 & 0 \le t \le 1/2, \\ 0 & t \ge 1. \end{cases}$$

Then for any $\varepsilon > 0$ and $y \in A_\rho$ we define the operator $\Phi^\varepsilon\colon A_\rho \to V_\varepsilon(\Omega)$ by

$$(\Phi^\varepsilon(y))(x) = \xi_\varepsilon(\varphi_y^\varepsilon) \cdot \varphi_y^\varepsilon(x), \tag{3.5}$$

where ξ_ε is the function defined in (2.6). Let us remark that Φ^ε is continuous in A_ρ.

We set

$$\mu_\varepsilon = \max\{E_\varepsilon(\Phi^\varepsilon(y)) :\ y \in A_\rho\}. \tag{3.6}$$

LEMMA 3.1. *Assume* (H_1)–(H_5) *are satisfied. Then the relation*

$$\lim_{\varepsilon\to 0}\frac{c_\varepsilon}{\varepsilon^N} > M_\infty \tag{3.7}$$

holds.

PROOF. By (2.9), (2.10) and (3.1), (3.3), we have that

$$\frac{c_\varepsilon}{\varepsilon^N} = \inf\{E_1(u) : u \in V_1(\Omega_\varepsilon),\ |\beta(u)| < 1/2\}.$$

Remark that $\beta(u)$ is well defined for all $u \in V_1(\Omega_\varepsilon)$. Clearly

$$\inf\{E_1(u) : u \in V_1(\Omega_\varepsilon),\ |\beta(u)| < 1/2\} \geq m_1(\Omega_\varepsilon) > M_\infty$$

so

$$\lim_{\varepsilon\to 0}\frac{c_\varepsilon}{\varepsilon^N} \geq M_\infty.$$

To prove the strict inequality we argue by contradiction and we suppose that the equality holds. In this case there exist a sequence $\{\varepsilon_n\}$, $\varepsilon_n \in \mathbb{R}$, $\varepsilon_n > 0$, $\lim_{n\to\infty}\varepsilon_n = 0$ and a sequence of functions $\{u_n\}$, such that

$$u_n \in H^1_0(\Omega_{\varepsilon_n}), \quad u_n \in V_1(\Omega_{\varepsilon_n}), \quad |\beta(u_n)| < 1/2$$

and

$$M_\infty \leq \frac{1}{2}\int_{\Omega_{\varepsilon_n}} (|\nabla u_n|^2 + (1+\alpha(x))u_n^2)\,dx - \int_{\Omega_{\varepsilon_n}} F(u_n)\,dx \leq M_\infty + \frac{1}{n}.$$

Setting $u_n(x) = 0$ in $\mathbb{R}^N \setminus \Omega_{\varepsilon_n}$, we have then $u_n \in H^1_0(\mathbb{R}^N)$, $u_n \in V_1(\mathbb{R}^N)$ $|\beta(u_n)| < 1/2$ and

$$\begin{aligned} M_\infty &\leq \frac{1}{2}\int_{\mathbb{R}^N}[|\nabla u_n|^2 + u_n^2]dx - \int_{\mathbb{R}^N} F(u_n)\,dx \\ &\leq \frac{1}{2}\int_{\mathbb{R}^N}[|\nabla u_n|^2 + (1+\alpha(x))u_n^2]\,dx - \int_{\mathbb{R}^N} F(u_n)dx \leq M_\infty + \frac{1}{n}. \end{aligned}$$

Hence u_n, up to a subsequence, by well known results [9] and by Lemma 2.7, must be of the form

$$u_n(x) = \omega(x - y_n) + w_n(x)$$

where $\omega \in H^1_0(\mathbb{R}^N)$ is a positive function, spherically symmetric about the origin, that realizes M_∞, $\{y_n\}_{n\in\mathbb{N}}$ is a sequence of points of $\mathbb{R}^N$ such that $\lim_{n\to\infty}|y_n| = +\infty$ and $\{w_n\}_{n\in\mathbb{N}}$ is a sequence of functions belonging to $H^1_0(\mathbb{R}^N)$ and going strongly to 0 in $H^1_0(\mathbb{R}^N)$.

Thus, by the continuity of β,

$$\frac{1}{2} > \ |\beta(u_n)|\ \geq |\beta(\omega(x-y_n))| - o(1). \tag{3.8}$$

On the other hand, since $\lim_{n\to\infty}|y_n| = \infty$, for all $\eta > 0$ and for all $R > 0$, $\overline{n}$ exists such that for all $n > \overline{n}$

$$|x - y_n| < R \Rightarrow \left|\frac{x}{|x|} - \frac{y_n}{|y_n|}\right| < \eta$$

and the asymptotic decay of ω implies that, for all $\eta > 0$, $\overline{R} > 0$ exists so that, for all $R > \overline{R}$ and for all n,

$$\int_{\mathbb{R}^N\setminus B_R(y_n)} |\omega(x - y_n)|^p\,dx = \int_{\mathbb{R}^N\setminus B_R(0)} |\omega(x)|^p\,dx < \eta. \tag{3.9}$$

Hence, choosing $\eta > 0$ arbitrarily and fixing R so that (3.9) is verified, for n large enough we get

$$\begin{aligned}\left|\beta(\omega(x - y_n)) - \frac{y_n}{|y_n|}\right| &\le \frac{1}{|\omega|^p_{p,\mathbb{R}^N}} \int_{\mathbb{R}^N} \left|\chi(x) - \frac{y_n}{|y_n|}\right| |\omega(x - y_n)|^p\,dx \\ &= \frac{1}{|\omega|^p_{p,\mathbb{R}^N}}\bigg[\int_{\mathbb{R}^N\setminus B_R(y_n)} \left|\chi(x) - \frac{y_n}{|y_n|}\right| |\omega(x - y_n)|^p dx \\ &\qquad + \int_{B_R(y_n)} \left|\frac{x}{|x|} - \frac{y_n}{|y_n|}\right| |\omega(x - y_n)|^p\,dx\bigg] \le 2\eta + \eta = 3\eta.\end{aligned} \tag{3.10}$$

Thus $|\beta(\omega(x - y_n))| \ge 1 - o(1)$ contradicting (3.8). □

Lemma 3.2. *Assume* (H_1)–(H_5) *are satisfied. Then there exists* $\widehat{\varepsilon}$ *such that for all* $\varepsilon \in (0, \widehat{\varepsilon}]$ *the inequality*

$$\mu_\varepsilon < c_\varepsilon \tag{3.11}$$

holds.

Proof. Since $\Phi^\varepsilon(y)(x) = 0$ for all $x \notin B_\rho(y)$ we have

$$\begin{aligned}&E_\varepsilon(\Phi^\varepsilon(y)(x)) \\ &=\frac{1}{2}\int_{B_\rho(y)} (\xi_\varepsilon(\varphi^\varepsilon_y))^2[\varepsilon^2|\nabla\varphi^\varepsilon_y(x)|^2 + a_\varepsilon(x)(\varphi^\varepsilon_y(x))^2]\,dx \\ &\quad - \int_{B_\rho(y)} F(\xi_\varepsilon(\varphi^\varepsilon_y)\varphi^\varepsilon_y(x))\,dx = \varepsilon^N\bigg[\frac{(\xi_\varepsilon(\varphi^\varepsilon_y))^2/2}{\|\zeta(x)\omega((x-y)/\varepsilon)\|^2_{B_\rho(y)}} \\ &\quad \cdot \int_{B_{\rho/\varepsilon}(y/\varepsilon)} \left(\left|\nabla\zeta(\varepsilon x)\omega\left(x - \frac{y}{\varepsilon}\right)\right|^2 + (1 + \alpha(x))\left(\zeta(\varepsilon x)\omega\left(x - \frac{y}{\varepsilon}\right)\right)^2\right) dx \\ &\quad - \int_{B_{\rho/\varepsilon}(y/\varepsilon)} F\left(\xi_\varepsilon(\varphi^\varepsilon_y)\frac{\zeta(\varepsilon x)\omega(x - y/\varepsilon)}{\|\zeta(x)\omega((x-y)/\varepsilon)\|_{B_\rho(y)}}\right) dx\bigg].\end{aligned}$$

Using (H_1) and (2.13) we easily obtain for all $y \in A_\rho$

$$\left\|\omega\left(x - \frac{y}{\varepsilon}\right) - \zeta(\varepsilon x)\omega\left(x - \frac{y}{\varepsilon}\right)\right\|^2_{\mathbb{R}^N} \le c_1\int_{\mathbb{R}^N\setminus B_{\rho/2\varepsilon}(0)} (|\nabla\omega|^2 + a_\infty\omega^2)\,dx = o(\varepsilon), \tag{3.12a}$$

$$\text{(3.12b)}\quad \left|\omega\left(x-\frac{y}{\varepsilon}\right)-\zeta(\varepsilon x)\omega\left(x-\frac{y}{\varepsilon}\right)\right|^{p+1}_{p+1,\mathbb{R}^N} \le c_2\int_{\mathbb{R}^N\setminus B_{\rho/2\varepsilon}(0)}|\omega(x)|^{p+1}\,dx=o(\varepsilon),$$

$$\text{(3.12c)}\quad 0\le\int_{\mathbb{R}^N\setminus B_{\rho/2\varepsilon}(y/\varepsilon)} f\left(\omega\left(x-\frac{y}{\varepsilon}\right)\right)\omega\left(x-\frac{y}{\varepsilon}\right)dx \le k\int_{\mathbb{R}^N\setminus B_{\rho/2\varepsilon}(0)}[\omega(x)+(\omega(x))^{p+1}]\,dx=o(\varepsilon),$$

$$\text{(3.12d)}\quad 0\le\int_{\mathbb{R}^N\setminus B_{\rho/2\varepsilon}(y/\varepsilon)} F\left(\omega\left(x-\frac{y}{\varepsilon}\right)\right)dx \le \widehat{k}\int_{\mathbb{R}^N\setminus B_{\rho/2\varepsilon}(0)}[\omega(x)+(\omega(x))^{p+1}]\,dx=o(\varepsilon),$$

$$\text{(3.12e)}\quad \int_{\mathbb{R}^N\setminus B_{\rho/2\varepsilon}(y/\varepsilon)}\alpha(x)\left(\omega\left(x-\frac{y}{\varepsilon}\right)\right)^2dx \le |\alpha|_{N/2,\mathbb{R}^N}\left(\int_{\mathbb{R}^N\setminus B_{\rho/2\varepsilon}(0)}|\omega(x)|^{2^*}\,dx\right)^{2/2^*}=o(\varepsilon).$$

Hence when $\varepsilon\to0$, taking account of (2.17), we obtain for all $y\in A_\rho$

$$\frac{\xi_\varepsilon(\varphi^\varepsilon_y)}{\|\zeta(x)\omega((x-y)/\varepsilon)\|_\Omega}\to1,$$

$$\varepsilon^N M_\infty<m_\varepsilon(B_\rho(y))\le E_\varepsilon(\Phi^\varepsilon(y))\le\varepsilon^N[M_\infty+o(\varepsilon)].$$

So, because of the compactness of A_ρ,

$$\text{(3.13)}\qquad \lim_{\varepsilon\to o}\frac{\mu_\varepsilon}{\varepsilon^N}=M_\infty$$

that with (3.7) gives the claim. □

Lemma 3.3. *The relation*

$$\text{(3.14)}\qquad \lim_{\varepsilon\to0}\sup_{y\in A_\rho}\left|\frac{1}{\varepsilon}\beta_\varepsilon(\Phi^\varepsilon(y))-\frac{y}{|y|}\right|=0$$

holds true.

Proof. For any $y\in A_\rho$ and for any $\varepsilon>0$ small enough we have

$$\begin{aligned}&\left|\frac{1}{\varepsilon}\beta_\varepsilon(\Phi^\varepsilon(y))-\frac{y}{|y|}\right|=\left|\frac{1}{\varepsilon|\Phi^\varepsilon(y)|^p_{p,B_\rho(y)}}\int_{B_\rho(y)}\chi_\varepsilon(x)|\Phi^\varepsilon(y)(x)|^p dx-\frac{y}{|y|}\right|\\&=\left|\frac{1}{|\zeta(\varepsilon x)\omega(x-y/\varepsilon)|^p_{p,B_{\rho/\varepsilon}(y/\varepsilon)}}\int_{B_{\rho/\varepsilon}(y/\varepsilon)}\chi(x)\left|\zeta(\varepsilon x)\omega\left(x-\frac{y}{\varepsilon}\right)\right|^p dx-\frac{y}{|y|}\right|\\&\leq\frac{1}{|\zeta(\varepsilon x)\omega(x-y/\varepsilon)|^p_{p,B_{\rho/\varepsilon}(y/\varepsilon)}}\int_{B_{\rho/\varepsilon}(y/\varepsilon)}\left|\chi(x)-\frac{y}{|y|}\right|\left|\zeta\left(\varepsilon x)\omega(x-\frac{y}{\varepsilon}\right)\right|^p dx.\end{aligned}$$

Then, since $|y|/\varepsilon\to+\infty$ and $\rho/\varepsilon\to\infty$ as $\varepsilon\to 0$, (3.12.b), an argument similar to that used to prove (3.10) and the compactness of A_ρ give (3.14). $\square$

In what follows, for all $\sigma\in\mathbb{R}$ we set

$$V_\varepsilon^\sigma=\{u\in V_\varepsilon(\Omega):E_\varepsilon(u)\leq\sigma\}.$$

Proof of Theorem 1.1. By (3.14) there exists $\varepsilon^*\in\mathbb{R}$, $0<\varepsilon^*\leq\widehat{\varepsilon}$ such that for any $\varepsilon\in(0,\varepsilon^*]$ and for any $y\in A_\rho$

$$\left|\beta_\varepsilon\circ\Phi^\varepsilon(y)-\varepsilon\frac{y}{|y|}\right|\leq\frac{\varepsilon}{4}. \tag{3.16}$$

Fix now $\varepsilon\in(0,\varepsilon^*]$ and choose τ_ε such that $\mu_\varepsilon\leq\tau_\varepsilon<c_\varepsilon$ and

$$\{u\in V_\varepsilon(\Omega):E_\varepsilon(u)=\tau_\varepsilon,\ \nabla E_{\varepsilon|V_\varepsilon(\Omega)}(u)=0\}=\emptyset.$$

Indeed, if this choice were not possible, we would have infinitely many critical levels between μ_ε and c_ε.

Our aim is to show that E_ε has at least two solutions belonging to $V_\varepsilon^{\tau_\varepsilon}$ and at least another solution in $V_\varepsilon(\Omega)\setminus V_\varepsilon^{\tau_\varepsilon}$.

The functional E_ε satisfies the (PS) condition on the set $V_\varepsilon^{\tau_\varepsilon}$, hence applying a classical result of the Lusternik–Schnirelman theory, we deduce

$$\#\ \{u\in V_\varepsilon^{\tau_\varepsilon}:\nabla E_{\varepsilon|V_\varepsilon(\Omega)}(u)=0\}\geq\operatorname{cat}V_\varepsilon^{\tau_\varepsilon}.$$

Let us show that $\operatorname{cat}V_\varepsilon^{\tau_\varepsilon}\geq 2$. Assume, by contradiction that $V_\varepsilon^{\tau_\varepsilon}$ is a contractible set, then there exists $h\in\mathcal{C}([0,1]\times V_\varepsilon^{\tau_\varepsilon},V_\varepsilon^{\tau_\varepsilon})$ such that

$$\begin{aligned}h(0,u)&=u &&\text{for all } u\in V_\varepsilon^{\tau_\varepsilon},\\ h(1,u)&=w,\ w\in V_\varepsilon^{\tau_\varepsilon} &&\text{for all } u\in V_\varepsilon^{\tau_\varepsilon}.\end{aligned}$$

Put $A_\varepsilon=\{x\in\mathbb{R}^N:\varepsilon/2\leq|x|\leq 3\rho\}$ and consider the map $g\in\mathcal{C}\ ([0,1]\times A_\rho,A_\varepsilon)$ defined by

$$g(t,y)=\begin{cases}(1-2t)y+2t(\beta_\varepsilon\circ\Phi^\varepsilon(y)) & 0\leq t\leq 1/2,\\ \beta_\varepsilon\circ h(2t-1,\Phi^\varepsilon(y)) & 1/2\leq t\leq 1,\end{cases}$$

g is well defined because (3.16), (3.2) and the choice of τ_ε and

$$g(0, y) = y \quad \text{for all } y \in A_\rho,$$
$$g(1, y) = \beta_\varepsilon(w) \in A_\varepsilon$$

so A_ρ turns out to be contractible to a point in A_ε, and this is clearly a contradiction.

Let us consider, now the set $\Phi^\varepsilon(A_\rho) = \Gamma_\varepsilon \subset V_\varepsilon^{\tau_\varepsilon}$, the same argument as before shows that Γ_ε is not contractible in $V_\varepsilon^{\tau_\varepsilon}$. Then, in order to prove the existence of another critical point, it is sufficient to construct an energy level $\sigma_\varepsilon > \tau_\varepsilon$ such that Γ_ε is contractible in $V_\varepsilon^{\sigma_\varepsilon}$.

Take $u^* \in V_\varepsilon(\Omega)$, $u^* \geq 0$, $u^* \notin \Gamma_\varepsilon$ (remark this choice is possible because Γ_ε is compact). Define the set

$$\Theta_\varepsilon = \{\theta u^* + (1-\theta)u : \theta \in [0,1],\ u \in \Gamma_\varepsilon\}.$$

Θ_ε is compact and contractible, moreover $0 \notin \Theta_\varepsilon$ (since any $u \in \Gamma_\varepsilon$ is positive on a set of positive measure). Hence the set

$$\Lambda_\varepsilon = \left\{ t_\varepsilon(v)v : v \in \Theta_\varepsilon,\ t_\varepsilon(v) = \xi_\varepsilon\left(\frac{v}{\|v\|_\Omega}\right)\|v\|_\Omega^{-1} \right\}$$

is well defined and $\Gamma_\varepsilon \subseteq \Lambda_\varepsilon \subseteq V_\varepsilon(\Omega)$. Then, setting

$$\sigma_\varepsilon = \max\{E_\varepsilon(z),\ z \in \Lambda_\varepsilon\}$$

we have that Γ_ε is contractible in $V_\varepsilon^{\sigma_\varepsilon}$. Finally let us show that for all $\varepsilon \in (0, \varepsilon^*]$ there exists a function $v_\varepsilon \in V_\varepsilon(\Omega)$ so that

$$E_\varepsilon(v_\varepsilon) \geq c_\varepsilon, \quad \nabla E_{\varepsilon|V_\varepsilon(\Omega)}(v_\varepsilon) = 0,$$

that, because of (3.7), cannot be a one-peak solution. We prove that denoted by w_ε the critical point such that $\tau_\varepsilon < E_\varepsilon(w_\varepsilon) \leq \sigma_\varepsilon$ either $E_\varepsilon(w_\varepsilon) \geq c_\varepsilon$ or exists $v_\varepsilon \neq w_\varepsilon$ such that $E_\varepsilon(v_\varepsilon) \geq c_\varepsilon$ and $\nabla E_{\varepsilon|V_\varepsilon(\Omega)}(v_\varepsilon) = 0$.

Assume in fact $E_\varepsilon(w_\varepsilon) = \gamma_\varepsilon < c_\varepsilon$, then we can find a level $\widehat{\tau}_\varepsilon$:

$$\frac{c_\varepsilon + \gamma_\varepsilon}{2} < \widehat{\tau}_\varepsilon < c_\varepsilon$$

such that $\{u \in V_\varepsilon(\Omega) : E_\varepsilon(u) = \widehat{\tau}_\varepsilon \text{ and } \nabla E_{\varepsilon|V_\varepsilon(\Omega)}(u) = 0\} = \emptyset$.

Otherwise any level $((c_\varepsilon + \gamma_\varepsilon)/2, c_\varepsilon)$ would be critical and, since (PS) condition holds, c_ε too would be critical. Then we can argue exactly as we have done before and prove that there exist $\widehat{\sigma}_\varepsilon > \widehat{\tau}_\varepsilon$ and $\widehat{w}_\varepsilon \in V_\varepsilon(\Omega)$ such that

$$\frac{c_\varepsilon + \gamma_\varepsilon}{2} < \widehat{\tau}_\varepsilon < E_\varepsilon(\widehat{w}_\varepsilon) < \widehat{\sigma}_\varepsilon, \quad \nabla E_{\varepsilon|V_\varepsilon(\Omega)}(\widehat{w}_\varepsilon) = 0$$

so, obviously $\widehat{w}_\varepsilon \neq w_\varepsilon$.

Iterating this argument we find either a function v_ε such that $E_\varepsilon(v_\varepsilon) \geq c_\varepsilon$ or a sequence of functions v_n such that

$$\lim_{n\to\infty} E_\varepsilon(v_n) = c_\varepsilon \quad \text{and} \quad \nabla E_{\varepsilon|V_\varepsilon(\Omega)}(v_n) = 0$$

and this, since the Palais–Smale condition holds, implies the existence of a critical point of E_ε at the level c_ε. □

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