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STRUCTURE OF LARGE POSITIVE SOLUTIONS OF SOME SEMILINEAR ELLIPTIC PROBLEMS WHERE THE NONLINEARITY CHANGES SIGN

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Abstract. Existence and uniqueness of large positive solutions are obtained for some semilinear elliptic Dirichlet problems in bounded smooth domains Ω with a large parameter λ. It is shown that the large positive solution has flat core. The distance of its flat core to the boundary $\partial\Omega$ is exactly measured as $\lambda \to \infty$.

1. Introduction

In this paper we study the following eigenvalue problem

$$-\Delta u = \lambda f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary $\partial\Omega$, $\lambda > 0$. We are interested in the structure of positive solutions of (1.1) for large positive λ in the case that $f(0) = 0$, $f'(0) = 0$, $f(a) = f(b) = 0$, $0 < a < b$, f changes sign on $[0,\infty)$. More precisely, we assume that $f \in C^1((0,\infty)\setminus\{b\}) \cap C^0([0,\infty))$ satisfies the following conditions:

(f_1) $f(0) = 0$, $f'(0) = 0$, f has two positive zeros a and b such that $a < b$; $f < 0$ in $(0,a)$, $f > 0$ in (a,b); there exists $0 < \delta < b - a$ such that $f'(s) < 0$ for $s \in (b-\delta, b)$ and f has no other positive zeros,

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(f$_2$) $\lim_{s\to b^-} f(s)/(b-s)^\omega = C_1$, $\lim_{s\to b^-} f'(s)/(b-s)^{\omega-1} = -C_2$ for some $0 < C_1, C_2 < \infty$ and $0 < \omega < 1$,

(f$_3$) $\int_0^b f(s)\,ds > 0$ and β is the unique number in (a,b) such that $\int_0^\beta f(s)\,ds = 0$.

Note that (f$_2$) implies $\lim_{s\to b^-} f'(s) = -\infty$.

Problem (1.1) has appeared in various models in applied mathematics, including population genetics and chemical reactor theory (see e.g. [16] and the references therein) and has been studied by many authors (see for example [1], [7]–[9], [19], [20], [5], [16]). Notice that if we set $\varepsilon^2 = 1/\lambda$, (1.1) can be viewed as a singularly perturbed problem. The case that $f'(0) = 0$ can be viewed as a border line case of singular perturbation problems (see [10]). Benci and Cerami [2] raised the question what happens for the structure of positive solutions in this borderline case, also called the *zero mass case*.

In paper [4], Clement and Sweers obtained that (1.1) has a unique positive solution u_λ with $\max u_\lambda \to b$ as $\lambda \to \infty$ and $u_\lambda \to b$ in compact sets of Ω as $\lambda \to \infty$ if f satisfies (f$_1$) and (f$_3$) with $f'(0) < 0$ and $-\infty < f'(b) < 0$. Notice that under such conditions on f, the fact that $\max u_\lambda < b$ can be obtained by the maximum principle. In a recent paper [6], Dancer studied (1.1) in a domain D of type R_N with $f'(0) = 0$ and $-\infty < f'(b) < 0$. He showed that when f satisfies (f$_1$), (f$_3$) and some extra conditions, (1.1) has exactly 2 positive solutions $\overline{u}_\lambda$, $\underline{u}_\lambda$ with $0 < \|u\|_\infty < b$ for all large positive λ: $\overline{u}_\lambda$ is a large solution, i.e. $\overline{u}_\lambda \to b$ uniformly on compact subsets of D as $\lambda \to \infty$; $\underline{u}_\lambda$ is a small solution, i.e. $\|\underline{u}_\lambda\|_\infty < b$ and $v_\lambda(y) := \underline{u}_\lambda(\lambda^{-1/2}y) \to V$ as $\lambda \to \infty$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$, where $V = V(y)$ is the unique positive (radial) solution of

$$\Delta V + f(V) = 0 \quad \text{in } \mathbb{R}^N, \quad V'(|y|) < 0, \quad V(y) \to 0 \quad \text{as } |y| \to \infty. \tag{1.2}$$

In this paper we shall show that when f satisfies (f$_1$)–(f$_3$), (1.1) has a unique large positive solution u_λ for λ sufficiently large. By a large solution u_λ of (1.1), we mean that $u_\lambda \in C^2(\Omega)$ and that there exists an open set $\Omega_0 \subset \Omega$ independent of λ with meas $(\Omega_0) > 0$ such that

$$\varliminf_{\lambda\to\infty} \inf_{x\in\Omega_0} u_\lambda(x) > a. \tag{1.3}$$

Since $\lim_{s\to b^-} f'(s) = -\infty$ (see (f$_2$)), the large positive solution u_λ of (1.1) may have *flat core*, i.e.

$$G_\lambda = \{x \in \Omega \mid u_\lambda(x) = b\} \neq \emptyset$$

(see [22]). We shall prove that under the assumptions (f$_1$)–(f$_3$), there exists flat core for the large positive solution u_λ of (1.1) when λ is sufficiently large. We also give the exact estimate of the flat core of u_λ.

The flat core properties of the positive solutions of elliptic equations similar to (1.1) have also been discussed by several authors, see for example [17], [18], [21]. In [21], Sweers obtained a positive solution of (1.1) which has flat core for λ sufficiently large. In a recent paper [18], Melian and Lis studied the flat core properties of the positive solutions of some elliptic problems involving p-Laplacian, but with simpler nonlinearity. It was known from [18] that under the conditions: $f \in C^2(0,b)$ with $f(s) > 0$ in $(0,b)$; $\lim_{s\to 0^+} f(s)/s = m > 0$; $f(b) = 0$; $f(s)/s$ is decreasing in $(0,b)$ and $\lim_{s\to b^-} f(s)/(b-s)^\omega = C > 0$ with $0 < \omega < 1$, (1.1) has a unique positive solution for λ sufficiently large and flat core of this solution exists.

Since the large positive solution u_λ of (1.1) has flat core, one will see that the problem studied in this paper becomes more difficult. For example, it is known from [11] that if f is a Lipschitz continuous function, the positive solutions of (1.1) with Ω being an N-ball is radially symmetric. We shall see in Section 5 below that such result is also true for the large positive solutions of (1.1), but $f(s)$ in our case is not Lipschitz for s near $s = b$. Moreover, we shall see later that it is difficult to establish the sweeping out results when we use sub- and supersolution argument because of the flat core of u_λ.

2. Existence of large positive solutions

In this Section we study the existence of large positive solutions of (1.1). The results in this section are strongly related to [4], but we need to overcome a difficulty arising from the singularity of $f'(s)$ at $s = b$. To deal with the case that $f'(b) = -\infty$, we modify f in the following way.

For any $\varepsilon > 0$ sufficiently small, define $f_\varepsilon(s) = f(s) - \varepsilon$. Then there exists $a(\varepsilon) > a$ and $b(\varepsilon) < b$ such that $f_\varepsilon(a_\varepsilon) = 0$, $f_\varepsilon(b_\varepsilon) = 0$ and $f_\varepsilon(0) = -\varepsilon$. (It is easy to see that $a(\varepsilon) \to a$ and $b(\varepsilon) \to b$ as $\varepsilon \to 0$ and $f_\varepsilon \in C^1([0, b(\varepsilon)])$.) We make an extension F_ε of f_ε:

$$\begin{cases} F_\varepsilon \text{ is bounded,} & \\ F_\varepsilon(s) \equiv 0 & \text{for } s \in (-\infty, -1], \\ F_\varepsilon \in C^1(-\infty, b(\varepsilon)) \text{ and } F_\varepsilon < 0 & \text{for } s \in (-1, 0), \\ F_\varepsilon \to 0 & \text{uniformly for } s \in [-1, 0] \text{ as } \varepsilon \to 0, \\ \lim_{s\to 0^-} F'_\varepsilon(s) = 0,\ \lim_{s\to(-1)^+} F'_\varepsilon(s) = 0, & \\ F_\varepsilon \equiv f_\varepsilon & \text{for } s \in [0, b(\varepsilon)], \\ F_\varepsilon(s) < 0 & \text{for } s \in [b(\varepsilon), \infty), \\ \int_{-1}^{b(\varepsilon)} F_\varepsilon(s)\, ds > 0. & \end{cases}$$

Lemma 2.1. *Let F_ε be defined as above. Then there exists $\mu_0 > 0$ independent of ε such that for $\mu > \mu_0$, there exists $v_{\varepsilon,\mu} \in C^1(\mathbb{R}^N)$, radially symmetric, which satisfies*

$$-\Delta v = \mu F_\varepsilon(v) \quad \text{in } \mathbb{R}^N, \quad v(1) = -1. \tag{2.1}$$

Moreover, $\max v_{\varepsilon,\mu} < b(\varepsilon)$ *and* $\max v_{\varepsilon,\mu} \to b(\varepsilon)$ *as* $\mu \to \infty$.

Proof. Define $\widetilde{f}_\varepsilon(s) = F_\varepsilon(s-1)$. We have that $\widetilde{f}_\varepsilon$ satisfies $\widetilde{f}_\varepsilon(0) = 0$ and $\widetilde{f}'_\varepsilon(0)=0$. Moreover, $\widetilde{f}_\varepsilon$ is bounded in $[0, b(\varepsilon)+1]$. Since $\widetilde{f}_\varepsilon(b(\varepsilon)+1) = F_\varepsilon(b(\varepsilon)) = 0$, without loss of generality, we assume $\widetilde{f}_\varepsilon(s) \equiv 0$ for $s \in [b(\varepsilon)+1, \infty)$. Now we consider the problem

$$-\Delta u = \mu \widetilde{f}_\varepsilon(u) \quad \text{in } B, \quad u = 0 \quad \text{on } \partial B, \tag{2.2}$$

where B is the unit ball in $\mathbb{R}^N$. By the arguments similar to that in [4], we can obtain a global minimizer $y_{\varepsilon,\mu} \in H_0^1(B)$ to the functional

$$I_\mu(u) = \frac{1}{2}\int_B |\nabla u|^2 - \mu \int_B \widetilde{F}_\varepsilon(u),$$

where $\widetilde{F}_\varepsilon(s) = \int_0^s \widetilde{f}_\varepsilon(\xi)\,d\xi$. It is known from the regularity of $-\Delta$ and the maximum principle that $y_{\varepsilon,\mu} \in C_0^2(B)$ which is a positive solution of (2.2). By [11], we know that $y_{\varepsilon,\mu}$ is radially symmetric and $y'_{\varepsilon,\mu} < 0$ for $r \in (0,1]$. Moreover, the fact that max $y_{\varepsilon,\mu} \to b(\varepsilon) + 1$ can also be obtained from the argument similar to that in [4].

Set $v_{\varepsilon,\mu}(r) = y_{\varepsilon,\mu}(r) - 1$ for $r \in [0,1]$ and

$$v_{\varepsilon,\mu}(r) = \begin{cases} -1 + \dfrac{1}{2-N}(r^{2-N} - 1) y'_{\varepsilon,\mu}(1) & \text{for } r \in (1,\infty) \text{ if } N > 2, \\ -1 + y'_{\varepsilon,\mu}(1) \log r & \text{for } r \in (1,\infty) \text{ if } N = 2. \end{cases}$$

Since $F_\varepsilon = 0$ on $(-\infty, -1]$, one verifies that $v_{\varepsilon,\mu}$ is the required function. This completes the proof. □

Remark. By the well-known result of [11], we know that all the positive solutions of (2.2) are radially symmetric for $\varepsilon > 0$. But we do not know whether such conclusion is true or not when $\varepsilon = 0$ since $\widetilde{f}_0$ is not Lipschitz continuous near $s = b + 1$.

Corollary 2.2. *Let* $(\mu, v_{\varepsilon,\mu})$ *be as in Lemma* 2.1, *and let* $\alpha_{\varepsilon,\mu} \in (0,1)$ *be the unique zero of* $v_{\varepsilon,\mu}$. *Then for* $y \in \Omega$ *and* $\lambda > \mu \cdot \alpha^2_{\varepsilon,\mu} \cdot d(y, \partial\Omega)^{-2}$,

$$w_{\mu,\varepsilon}(\lambda, y; x) := v_{\varepsilon,\mu}((\lambda/\mu)^{1/2} \cdot (x-y)), \quad x \in \Omega \tag{2.3}$$

is a subsolution of the problem

$$-\Delta u = \lambda F_\varepsilon(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{2.4}$$

Remark. We can show that for any $\varepsilon > 0$ sufficiently small, $\alpha_{\varepsilon,\mu} \to 1$ as $\mu \to \infty$. In fact, for any sequence $\{\mu_n\}$ with $\mu_n \to \infty$ as $n \to \infty$, by the arguments similar to that in the proof of Lemma 2.1, we have that $\tau_n :=$

$v_{\varepsilon,\mu_n}(0) = \max_B v_{\varepsilon,\mu_n} \to b(\varepsilon)$ as $n \to \infty$. Defining $y = \mu_n^{1/2} r$ and $\widetilde{v}_{\varepsilon,\mu_n}(y) = v_{\varepsilon,\mu_n}(r)$, we have that $\widetilde{v}_{\varepsilon,\mu_n}$ satisfies

$$\widetilde{v}''_{\varepsilon,\mu_n} + \frac{N-1}{y}\widetilde{v}'_{\varepsilon,\mu_n} + F_\varepsilon(\widetilde{v}_{\varepsilon,\mu_n}) = 0, \quad \widetilde{v}'_{\varepsilon,\mu_n}(0) = 0, \quad \widetilde{v}_{\varepsilon,\mu_n}(0) = \tau_n.$$

Since $\tau_n \to b(\varepsilon)$ as $n \to \infty$ and $b(\varepsilon)$ is the unique solution of the problem

$$u'' + \frac{N-1}{y}u' + F_\varepsilon(u) = 0 \quad \text{in } (0,\infty), \quad u'(0) = 0, \quad u(0) = b(\varepsilon),$$

one obtains from the theory of ordinary differential equations that

$$\widetilde{v}_{\varepsilon,\mu_n} \to b(\varepsilon) \quad \text{in } C^1_{\mathrm{loc}}(0,\infty) \text{ as } n \to \infty.$$

(We can choose subsequences if necessary.) This implies that

$$v_{\varepsilon,\mu_n} \to b(\varepsilon) \quad \text{in } C^1_{\mathrm{loc}}(B) \text{ as } n \to \infty.$$

Thus, $\alpha_{\varepsilon,\mu_n} \to 1$ as $n \to \infty$.

Let $x^* \in \Omega$. We define $\lambda^* := \mu d(x^*, \partial\Omega)^{-2} > \mu\alpha^2_{\mu,\varepsilon} d(x^*, \partial\Omega)^{-2}$ and $z_\lambda = w(\lambda, x^*)$, where μ, α are as defined in Corollary 2.2. Note that λ^* is independent of ε.

THEOREM 2.3. *Let f satisfy* (f$_1$)–(f$_3$). *Then there exists $\lambda_0 > 0$ such that for $\lambda > \lambda_0$,* (1.1) *has at least one large positive solution u_λ such that*

$$\max u_\lambda \to b \quad \text{as } \lambda \to \infty. \tag{2.5}$$

Moreover, $u_\lambda \to b$ on compact sets of Ω as $\lambda \to \infty$.

The proof of this theorem is similar to that in [4], but we need to overcome a difficulty arising from the singularity of $f'(s)$ at $s = b$. We first present the following lemmas.

LEMMA 2.4. *Let F_ε be as above. Then*

(i) *for $\lambda > \lambda^*$* (2.4) *has a solution $u_\lambda^{(\varepsilon)} \in [z_\lambda, b(\varepsilon))$,*

(ii) *there exist $\lambda^{**} > \lambda^*$, $c > 0$ and $\tau \in (a, b(\varepsilon))$, such that for $\lambda > \lambda^{**}$ every solution $u_\lambda^{(\varepsilon)} \in [z_\lambda, b(\varepsilon))$ of* (2.4) *satisfies*

$$u_\lambda^{(\varepsilon)}(x) > \min\{c\lambda^{1/2} d(x, \partial\Omega), \tau\} \quad \text{for all } x \in \Omega. \tag{2.6}$$

PROOF. By Corollary 2.2, for $\lambda > \lambda^*$ we have that z_λ is a subsolution of (2.4) and $z_\lambda < b(\varepsilon)$. Since $b(\varepsilon)$ is a supersolution of (2.4) and there exists $M_\varepsilon > 0$ such that $F_\varepsilon(s) + M_\varepsilon s$ is strictly increasing in $(\min_\Omega z_\lambda, b(\varepsilon))$, by a monotone method, there exists a minimal solution $u_\lambda^{(\varepsilon)} \in [z_\lambda, b(\varepsilon))$ of (2.4) for $\lambda > \lambda^*$. This completes the proof of the first assertion.

Since Ω satisfies a uniform interior sphere condition, there exists $\eta_0 > 0$ such that $\Omega = \bigcup\{B(x,\eta) \mid x \in \Omega_\eta\}$ for $\eta \in (0,\eta_0]$, where $\Omega_\eta = \{x \in \Omega \mid d(x,\partial\Omega) > \eta\}$. Set

$$
\begin{aligned}
\lambda^{**} &= \max(\lambda^*, \mu\eta_0^{-2}),\\
c &= \mu^{-1/2}\inf\{(\alpha - r)^{-1} v(r) \mid r \in [0,\alpha)\},\\
\tau &= v(0),
\end{aligned}
\tag{2.7}
$$

with μ, v and α as in Corollary 2.2. (Note that λ^{**} is independent of ε.)

Let $(\lambda, u_{\varepsilon,\lambda})$ be any solution of (2.4), $\lambda > \lambda^{**}$ and $u_{\varepsilon,\lambda} \in [z_\lambda, b(\varepsilon))$. Since for $\lambda > \lambda^{**}$, $\Omega_{\alpha(\mu/\lambda)^{1/2}}$ is arcwise connected and since $w(\lambda,y)$ is a subsolution for $y \in \Omega_{\alpha(\mu/\lambda)^{1/2}}$ with $w(\lambda,y) < 0$ on $\partial\Omega$, by the sweeping out result (see [5]) we obtain

$$u_{\varepsilon,\lambda} > w(\lambda,y) \quad \text{in } \Omega \text{ for all } y \in \Omega_{\alpha(\mu/\lambda)^{1/2}}.$$

Hence, a similar argument to that in [4] implies

$$u_{\varepsilon,\lambda} > c\lambda^{1/2} d(x,\partial\Omega) \quad \text{for all } x \in \Omega \setminus \Omega_{\alpha(\mu/\lambda)^{1/2}}, \tag{2.8}$$

$$u_{\varepsilon,\lambda}(x) > \tau \qquad \text{for all } x \in \Omega_{\alpha(\mu/\lambda)^{1/2}}, \tag{2.9}$$

which completes the proof. □

Remarks. (1) The sweeping out result as in [5] holds here since there exists $M_\varepsilon > 0$ such that $|F'_\varepsilon(s)| \le M_\varepsilon$ for $s \in [0, b(\varepsilon)]$.

(2) It follows from (2.8)–(2.9) that the minimal solution $u_\lambda^{(\varepsilon)} > 0$ for $\lambda > \lambda^{**}$, and $\max u_\lambda^{(\varepsilon)} \in (a, b(\varepsilon))$ for μ and λ sufficiently large. This implies that $u_\lambda^{(\varepsilon)}$ is a positive solution of

$$-\Delta u = \lambda(f(u) - \varepsilon) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{2.10}$$

(3) We know that the constant c in (2.7) depends upon ε. But we can show that $c \ge c_0/2 > 0$ for any ε sufficiently small, where c_0 is independent of ε. In fact, we know that for any fixed μ sufficiently large, and any sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ as $n \to \infty$, there exists a subsequence (still denoted by $\{\varepsilon_n\}$) such that $v_{\varepsilon_n,\mu} \to v_{0,\mu}$ in $C^1(B)$ and $v_{0,\mu}$ is a positive radial solution of the problem

$$-\Delta u = \mu F_0(u) \quad \text{in } B, \quad u = 0 \quad \text{on } \partial B,$$

where $F_0(s) = f(s)$ in $[0,b]$ and $F_0(s) \equiv 0$ in $(-\infty, 0]$. We can choose

$$c_0 = \mu^{-1/2}\inf\{(1-r)^{-1} v_{0,\mu}(r) \mid r \in [0,1)\}.$$

Note that for μ sufficiently large, $\max v_{0,\mu} = b$ may hold.

By the conditions on f, we can choose a fixed $\widehat{b} \in (b-\delta, b)$ and $M > 0$ such that $f(s) + Ms$ is increasing for $s \in [0, \widehat{b}]$. Let $\gamma > 1$ be a fixed number. Now, setting

$$g(s) = \begin{cases} Ms & \text{for } s \in (0, \widehat{b}], \\ M\widehat{b} + \gamma f(\widehat{b}) - \gamma f(s) & \text{for } s \in (\widehat{b}, b], \end{cases}$$

we can easily show that g is continuous in $[0, b]$ and g is increasing in $[0, b]$. Moreover, we aslo know that $f(s) + g(s)$ is increasing in $[0, b]$. (Note that $f'(s) < 0$ for $s \in (\widehat{b}, b)$.)

LEMMA 2.5 (Maximum Principle). *If $u_1, u_2 \in H_0^1(\Omega) \cap C^0(\overline{\Omega})$ such that, for any $\phi \in H_0^1(\Omega)$, $\phi \geq 0$,*

$$\int_\Omega \nabla u_2 \cdot \nabla \phi + \int_\Omega g(u_2)\phi \geq \int_\Omega \nabla u_1 \cdot \nabla \phi + \int_\Omega g(u_1)\phi \tag{2.11}$$

and

$$u_2 \geq u_1 \quad \text{on } \partial\Omega,$$

then $u_2 \geq u_1$ in Ω.

PROOF. Let us choose $\phi = (u_1 - u_2)^+ \in H_0^1(\Omega) \cap C_0^0(\Omega)$. Then it follows from (2.11) that

$$0 \geq \int_\Omega |\nabla(u_1 - u_2)^+|^2 + \int_\Omega [g(u_1) - g(u_2)](u_1 - u_2)^+.$$

This implies that $(u_1 - u_2)^+ = 0$ in Ω and thus, $u_1 \leq u_2$ in Ω. □

PROOF OF THEOREM 2.3. Since $u_\lambda^{(\varepsilon)}$ is a positive subsolution of (1.1) with $\max u_\lambda^{(\varepsilon)} < b$, b is a supersolution of (1.1) and $f(s) + g(s)$ is increasing in $(0, b)$, by a monotone argument as in Theorem 2.4 in [3] that there exist a minimal positive solution $u_\lambda \in C_0^2(\Omega)$ of the problem (1.1) such that

$$u_\lambda^{(\varepsilon)} \leq u_\lambda(x) \leq b \quad \text{in } \Omega. \tag{2.12}$$

Here we use Lemma 2.5. It follows from Lemma 2.4 that for any $\varepsilon > 0$ and $\lambda > \lambda^*$,

$$u_\lambda > \min\{c\lambda^{1/2} d(x, \partial\Omega), \tau\} \quad \text{for all } x \in \Omega.$$

Since $\tau \to b$ as $\varepsilon \to 0$ and $\mu \to \infty$, we have from (2.9) (we may choose $\lambda = \mu^2$) that for λ sufficiently large that

$$u_\lambda > \tau \quad \text{for all } x \in \Omega_{\alpha(\mu/\lambda)^{1/2}}. \tag{2.13}$$

Since $\lim_{\lambda \to \infty} \mu/\lambda = 0$ (here we use $\lambda = \mu^2$) and $\tau \to b$ as $\lambda \to \infty$ and $\varepsilon \to 0$ (noticing that u_λ is independent of ε and $\alpha > \widetilde{\alpha} > 0$), we have from (2.13) that

$$u_\lambda \to b \text{ on compact sets of } \Omega \text{ as } \lambda \to \infty.$$

We easily know that u_λ is a large positive solution of (1.1) according to the definition of large positive solutions. This completes the proof. □

Remarks. (1) Note that the monotone argument was used to the weak solutions in [3], but the arguments can be applied for our case to obtain the solution in $C^2(\Omega)$ since g is bounded.

(2) We can obtain that if u_λ is a positive solution of (1.1) with $u_\lambda \in [z_\lambda, b]$, then $u_\lambda \geq u_\lambda^{(\varepsilon)}$ in Ω. In fact, $u_\lambda^{(\varepsilon)}$ can be obtained by the similar monotone argument to that in the proof of Theorem 2.3 with $f(s)$ replaced by F_ε. Note that by modifying $g(s) = Ms$ for $s \leq 0$ we can show that $F_\varepsilon(s)+g(s)$ is increasing in $(-\infty, b(\varepsilon))$. Thus, the monotone argument can be used for the problem (2.4). On the other hand, by modifying $f(s) \equiv 0$ for $s < 0$ and g as above for $s < 0$, we also know that $f(s)+g(s)$ is increasing for $s \in (-\infty, b)$. Since z_λ is a subsolution for both (2.4) and (1.1), we can use the monotone argument starting from z_λ, i.e.

$$-\Delta\zeta_n^{(1)} + \lambda g(\zeta_n^{(1)}) = \lambda(F_\varepsilon + g)(\zeta_{n-1}^{(1)}) \quad \text{in } \Omega, \quad \zeta_n^{(1)} = 0 \quad \text{on } \Omega$$

with $\zeta_0^{(1)} = z_\lambda$ and

$$-\Delta\zeta_n^{(2)} + \lambda g(\zeta_n^{(2)}) = \lambda(f + g)(\zeta_{n-1}^{(2)}) \quad \text{in } \Omega, \quad \zeta_n^{(2)} = 0 \quad \text{on } \Omega$$

with $\zeta_0^{(2)} = z_\lambda$. Since $f(s) > F_\varepsilon(s)$ on $(0, b]$, then it follows from the maximum principle in Lemma 2.5 that $\zeta_n^{(2)} \geq \zeta_n^{(1)}$ in Ω. Since $u_\lambda^{(\varepsilon)}$ is the minimal solution of (2.4) in $[z_\lambda, b(\varepsilon))$, thus, $u_\lambda \geq u_\lambda^{(\varepsilon)}$ in Ω.

3. Asymptotic behaviour of large positive solutions of (1.1) when λ is large

In this Section we shall study the asymptotic behaviour of the positive solutions of (1.1) when λ is large. We first consider the following ordinary differential equations

$$-y'' = f(y) - \varepsilon, \quad y(0) = 0, \quad y(\infty) = b(\varepsilon), \tag{3.1}$$

$$-y'' = f(y), \quad y(0) = 0, \quad y(\infty) = b. \tag{3.2}$$

By the first integrals of the equations, we have that each of (3.1) and (3.2) has a unique positive solution $y_\varepsilon(t)$ and $y(t)$ respectively which satisfies $(y_\varepsilon)'(t) > 0$ for $t \in [0, \infty)$ and $y'(t) \geq 0$ for $t \in [0, \infty)$ (see [5], [15]). To show $(y_\varepsilon)'(t) > 0$ for $t \in [0, \infty)$, we use the fact that $|f'(s)|$ is bounded for $s \in [0, b(\varepsilon)]$. Now we show that there exists $t_0 > 0$ such that $y'(t) > 0$ for $t \in (0, t_0)$ and $y'(t_0) = 0$ and $y(t) \equiv b$ for $t \in [t_0, \infty)$. In fact, from the first integral of (3.2), we have

$$|y'(t)|^2 + 2F(y(t)) \equiv C, \quad t \in (0, \infty),$$

where $F(s) = \int_0^s f(\xi)\, d\xi$. Therefore,

$$|y'(t)|^2 = 2(F(b) - F(y)).$$

Since $F(b) > F(s)$ for $0 < s < b$, we have

$$\int_0^{y(t)} (F(b) - F(s))^{-1/2}\, ds = 2^{1/2} t.$$

Since $f(s) \sim (b-s)^{\omega}$ for s near b, $0 < \omega < 1$, we know that $F(b) - F(s) \geq \rho(b-s)^{1+\omega}$, where $\rho > 0$. Thus, $\int_0^b (F(b) - F(s))^{-1/2}\, ds = A < \infty$. Let

$$t_0 = 2^{-1/2} A. \tag{3.3}$$

Then the first integral of (3.2) implies that $y'(t) > 0$ for $t \in (0, t_0)$, $y'(t_0) = 0$ and $y \equiv b$ in $[t_0, \infty)$. On the other hand, by the first integrals of the equations (3.1)–(3.2), we also know that

$$(y_\varepsilon)'(0) = \left(2 \int_0^{b(\varepsilon)} [f(s) - \varepsilon]\, ds\right)^{1/2},$$
$$y'(0) = \left(2 \int_0^{b} f(s)\, ds\right)^{1/2}.$$

Thus $(y_\varepsilon)'(0) \to y'(0)$ as $\varepsilon \to 0$. Therefore,

$$y_\varepsilon \to y \quad \text{in } C^1_{\text{loc}}(0, \infty).$$

If $x \in \Omega$ and x is near $\partial\Omega$, x can be uniquely written in the form $x = s + t\nu(s)$, where $s \in \partial\Omega$, $\nu(s)$ denotes the inward unit normal vector to $\partial\Omega$ at s, and t is small and positive. We will make frequent use of these coordinates. If $\lambda > 0$, define $\eta_\lambda(x) = y(\lambda^{1/2} t)$ if x is near $\partial\Omega$ and $\eta_\lambda(x) = b$ otherwise.

Proposition 3.1. *Let f satisfy* (f$_1$)–(f$_3$). *For any $\theta > 0$ sufficiently small, there is $\overline{\lambda} = \overline{\lambda}(\theta) > \lambda^{**}$ such that if $\lambda > \overline{\lambda}$ and $u_\lambda \in [z_\lambda, b]$ is a positive solution of* (1.1)*, then*

$$(1-\theta)\eta_\lambda \leq u_\lambda \leq (1+\theta)\eta_\lambda. \tag{3.4}$$

To prove this result, we first obtain the following sweeping out result.

Proposition 3.2 (Sweeping Out Result). *Let f satisfy* (f$_1$)–(f$_3$)*,*

$$u \in H_0^1(\Omega) \cap C^0(\overline{\Omega})$$

with $\max u \leq b$ *be a solution of* (1.1) *and let $A = \{v_t \mid t \in [0,1]\}$ be a family of subsolutions of* (1.1) *satisfying $v_t \in H_0^1(\Omega) \cap C^0(\overline{\Omega})$,* $\max v_t \leq \widetilde{b} < b$ *and $v_t \leq 0$ on $\partial\Omega$ for all $t \in [0,1]$. If*

(i) *$t \to (v_t - v_0) \in C^0(\overline{\Omega})$ is continuous with respect to the $\|\cdot\|_0$-norm,*
(ii) *$u \geq v_0$ in $\overline{\Omega}$, and*
(iii) *$u \not\equiv v_t$, for all $t \in [0,1]$ and x near $\partial\Omega$, then $u \geq v_t$ in $\overline{\Omega}$ for $t \in [0,1]$.*

Proof. Define $G = \{x \in \Omega \mid u(x) = b\}$. We know that G depends upon λ, we shall omit the subscript λ here and below for simplicity. In the following, we only consider the case $G \neq \emptyset$. The case $G = \emptyset$ can be studied similarly.

Set $E = \{t \in [0,1] \mid u \geq v_t \text{ in } \overline{\Omega}\}$. By (ii), E is nonempty. Moreover, E is closed. We easily know that $G \subset\subset \Omega$ is closed. Since $\max v_t \leq \widetilde{b} < b$ for any $t \in [0,1]$, then, for $0 < \tau < b - \widetilde{b}$, we can choose a neighbourhood O of G such that $G \subset O \subset\subset \Omega$ and $u \geq v_t + \tau$ in $\overline{O}$ for any $t \in [0,1]$. Moreover, there exists $M > 0$ sufficiently large such that for any $t \in E$,

$$f(u) + Mu \geq f(v_t) + Mv_t \quad \text{for } x \in \Omega \setminus \overline{O}.$$

Thus, for $t \in E$, we can easily show that $u > v_t$ in $\Omega \setminus \overline{O}$. In fact, suppose that there exists $x_0 \in \Omega \setminus \overline{O}$ and $u - v_t$ vanishes at $x = x_0$, then $u - v_t$ attends its minimum at $x = x_0$ in $\Omega \setminus \overline{O}$. On the other hand,

$$-\Delta(u - v_t) + \lambda M(u - v_t) \geq 0 \quad \text{in } \Omega \setminus \overline{O}.$$

The Hopf's maximum principle implies that $u \equiv v_t$ in $\Omega \setminus \overline{O}$. This contradicts the assumption (iii).

Now applying the strong maximum principle, we have

$$\frac{\partial(u - v_t)}{\partial\nu} < 0 \quad \text{on } \partial\Omega.$$

Thus, $u - v_t \geq c\phi$ on $\Omega \setminus \overline{O}$ (see [14]), where $c > 0$ and ϕ is the unique positive solution of the problem

$$-\Delta\phi = 1 \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \partial\Omega.$$

This and the arguments above imply that $u - v_t \geq c_1\phi$ for $x \in \Omega$, where $c_1 > 0$. By (i), we have that E is an open set. Thus, $E = [0,1]$. □

Proof of Proposition 3.1. To prove this proposition, we first construct sub- and supersolutions of (1.1), then obtain (3.4) by sweeping out results. We only consider the case that u_λ has flat core, i.e. $G_\lambda = \{x \in \Omega \mid u_\lambda(x) = b\} \neq \emptyset$ in the proof. If flat core of u_λ does not exist, the proof is similar but is simpler. (The key step in the proof below is to establish the sweeping out results. If $\max u_\lambda < b$ for all λ large, for a fixed λ large, we can choose $M_\lambda > 0$ such that $|f'(s)| \leq M_\lambda$ for $s \in (0, \max_\Omega u_\lambda]$. Assuming $G_\lambda = \emptyset$ in the proof below, we obtain our conclusion in this case by the similar arguments.)

Near $\partial\Omega$, we use the s, t coordinates. In these variables,

$$\Delta u_\lambda = \frac{\partial^2 u_\lambda}{\partial t^2} + b(s,t)\frac{\partial u_\lambda}{\partial t} + \text{terms involving } s \text{ derivatives}.$$

If $\overline{\alpha}_\varepsilon < (y_\varepsilon)'(0)$ but is close, using the first integral of (3.1) we easily prove that the solution $\widetilde{y}_\varepsilon$ of (3.1) with the initial conditions:

$$\widetilde{y}_\varepsilon(0) = 0, \quad \widetilde{y}_\varepsilon'(0) = \overline{\alpha}_\varepsilon,$$

first increases to a number near $b(\varepsilon)$ but less than $b(\varepsilon)$, and then decreases to zero (see [5]). Hence there is $\widetilde{l}_\varepsilon$ near $b(\varepsilon)$ and $\widetilde{t}_\varepsilon$ sufficiently large such that

$$\widetilde{y}_\varepsilon(\widetilde{t}_\varepsilon) = \widetilde{l}_\varepsilon, \quad \widetilde{y}_\varepsilon'(\widetilde{t}_\varepsilon) = 0.$$

We know that there exists $M_\varepsilon > 0$ sufficiently large such that $h_\varepsilon(s) := f(s) - \varepsilon + M_\varepsilon s$ is strictly increasing for $s \in (0, b(\varepsilon)]$ (since $b(\varepsilon) < b$). Hence if $\widetilde{\mu}$ is close to 1 and β is small, the solution $\overline{y}_\varepsilon$ of

$$-x'' - \beta x' + M_\varepsilon x = \widetilde{\mu} h_\varepsilon(x(t)), \quad x(0) = 0, \quad x'(0) = \overline{\alpha}_\varepsilon \tag{3.5}$$

increases until $\overline{t}_\varepsilon$, where $\overline{y}_\varepsilon(\overline{t}_\varepsilon)$ is close to $b(\varepsilon)$ but less than $b(\varepsilon)$. Moreover, $\overline{t}_\varepsilon$ is sufficiently large.

Let t_0 be the number such that $y'(t) > 0$ for $0 < t < t_0$ and $y(t) \equiv b$ for $t \geq t_0$, where y is the unique positive solution of (3.2). We know that $\overline{t}_\varepsilon > t_0$ for any $\varepsilon > 0$ sufficiently small and $\overline{y}_\varepsilon(t_0) < b(\varepsilon)$. Define

$$\widetilde{\eta}_\lambda^{(\varepsilon)}(x) = \begin{cases} \overline{y}_\varepsilon(\lambda^{1/2} t) & \text{if } x \text{ is close to } \partial\Omega \text{ and } 0 \leq t \leq \lambda^{-1/2} t_0, \\ \overline{y}_\varepsilon(t_0) & \text{otherwise,} \end{cases}$$

where $x = s + t\nu(s)$ if x is near $\partial\Omega$. (Thus $\widetilde{\eta}_\lambda^{(\varepsilon)}$ is constant except near $\partial\Omega$.) We know that $\widetilde{\eta}_\lambda^{(\varepsilon)}$ is in $C^1(\Omega)$ except the points $x = s + t\nu(s)$ with $t = \lambda^{-1/2} t_0$. Suppose we can show that, for λ large and u_λ is a positive solution of (1.1), then

$$u_\lambda \geq \widetilde{\eta}_\lambda^{(\varepsilon)} \quad \text{for all } \varepsilon > 0 \text{ sufficiently small.} \tag{3.6}$$

Since $\overline{y}_\varepsilon$ is close to y_ε on compact intervals if $\overline{\alpha}_\varepsilon$ is near $(y_\varepsilon)'(0)$; $\widetilde{\mu}$ is near 1 and β is small, $u_\lambda \geq (1 - \theta/2)\overline{\eta}_\lambda^{(\varepsilon)}$, where

$$\overline{\eta}_\lambda^{(\varepsilon)}(x) = \begin{cases} y_\varepsilon(\lambda^{1/2} t) & \text{if } x \text{ is close to } \partial\Omega \text{ and } 0 \leq t \leq \lambda^{-1/2} t_0, \\ y_\varepsilon(t_0) & \text{otherwise.} \end{cases}$$

Since $y_\varepsilon \to y$ in $C^0([0, t_0])$, as $\varepsilon \to 0$, we have $\overline{\eta}_\lambda^{(\varepsilon)} \to \eta_\lambda$ in $C^0(\overline{\Omega})$ as $\varepsilon \to 0$, where

$$\eta_\lambda(x) := \begin{cases} y(\lambda^{1/2} t) & \text{if } x \text{ is close to } \partial\Omega \text{ and } 0 \leq t \leq \lambda^{-1/2} t_0, \\ b & \text{otherwise.} \end{cases}$$

Thus $u_\lambda \geq (1 - \theta)\eta_\lambda$. This will prove half of Proposition 3.1.

Now we show (3.6). By choosing $\beta < 0$ and $\widetilde{\mu} < 1$, we have $\widetilde{\eta}_{\lambda}^{(\varepsilon)}$ is in $C^0(\overline{\Omega}) \cap H_0^1(\Omega)$. Moreover, we can find $e \in (0,1)$ such that $u_\lambda \geq \widetilde{\eta}_{e\lambda}^{(\varepsilon)}$ by Theorem 2.3 and Remark 2 after its proof. Now we deduce that

$$u_\lambda \geq \widetilde{\eta}_{j\lambda}^{(\varepsilon)} \quad \text{for } j \in [e,1]. \tag{3.7}$$

We first show that $\widetilde{\eta}_{j\lambda}^{(\varepsilon)}$ are subsolutions of (1.1) for $j \in [e,1]$. We only need to check this for $\Omega_{t_0} := \{x = s + t\nu(s) \in \Omega \mid 0 \leq t \leq (j\lambda)^{-1/2} t_0\}$. Since

$$\begin{aligned}
-\Delta \widetilde{\eta}_{j\lambda}^{(\varepsilon)} &= -(\widetilde{\eta}_{j\lambda}^{(\varepsilon)})'' - b(s,t)(\widetilde{\eta}_{j\lambda}^{(\varepsilon)})' \\
&\leq \lambda[j\widetilde{\mu} f(\overline{y}_\varepsilon((j\lambda)^{1/2}t)) + j(\widetilde{\mu}-1)M_\varepsilon \overline{y}_\varepsilon((j\lambda)^{1/2}t)] \\
&\quad + (j\lambda)^{1/2}(\beta(j\lambda)^{1/2} - b(s,t))\overline{y}_\varepsilon'((j\lambda)^{1/2}t) \\
&\leq \lambda f(\overline{y}_\varepsilon((j\lambda)^{1/2}t)) + \lambda(j\widetilde{\mu}-1)[f(\overline{y}_\varepsilon((j\lambda)^{1/2}t)) \\
&\quad + \frac{j(\widetilde{\mu}-1)}{j\widetilde{\mu}-1} M_\varepsilon \overline{y}_\varepsilon((j\lambda)^{1/2}t)] \\
&\quad + (j\lambda)^{1/2}(\beta(j\lambda)^{1/2} - b(s,t))\overline{y}_\varepsilon'((j\lambda)^{1/2}t).
\end{aligned}$$

Define $m_{\varepsilon,j}(s) = f(s) + [j(\widetilde{\mu}-1)/(j\widetilde{\mu}-1)]M_\varepsilon s$. Since $j(\widetilde{\mu}-1)/(j\widetilde{\mu}-1) \geq \widetilde{\theta} > 0$ for $j \in [e,1]$, if we choose M_ε sufficiently large, we have that $m_{\varepsilon,j}$ is also strictly increasing in $(0, b(\varepsilon))$ for all $j \in [e,1]$. Thus, $m_{\varepsilon,j}(\overline{y}_\varepsilon((j\lambda)^{1/2}t)) \geq 0$ in Ω and $\widetilde{\eta}_{j\lambda}^{(\varepsilon)}$ is a subsolution of (1.1) for each $j \in [e,1]$ provided $\widetilde{\mu} < 1$ and $\beta < 0$. On the other hand, we also know that $\max \widetilde{\eta}_{j\lambda}^{(\varepsilon)} < b(\varepsilon) < b$ in Ω for any $\varepsilon > 0$ sufficiently small. Then the sweeping out result (see Proposition 3.2) implies that

$$u_\lambda \geq \widetilde{\eta}_{j\lambda}^{(\varepsilon)} \quad \text{for all } j \in [e,1].$$

Now we construct supersolutions of (1.1) to prove the right hand side of (3.4). If $\overline{\alpha}_1 > y'(0)$ and close, it is easy to show from the first integral that the solution $\widetilde{y}_1$ of (3.2) such that $\widetilde{y}_1(0) = 0$, $\widetilde{y}_1'(0) = \overline{\alpha}_1$, increases till it hits $y = b$. Hence if $\widehat{\mu} > 1$ close to 1 and $\beta > 0$ is small, the solution $\overline{y}_1$ of

$$-x'' - \beta x' + M_a x = \widehat{\mu}[f(x(t)) + M_a x(t)], \quad x(0) = 0, \quad x'(0) = \overline{\alpha}_1 \tag{3.8}$$

increases until $\bar{t}_1$, where $\overline{y}_1(\bar{t}_1) = b$. Where $M_a > 0$ satisfies that $|f'(s)| \leq [\widehat{\mu}/(\widehat{\mu}-1)]M_a$ for $s \in [0,a]$. Clearly $\bar{t}_1 < t_0$ provided that $\widehat{\mu}$ is near 1 and β is small. We define

$$\overline{\eta}_\lambda(x) = \begin{cases} \overline{y}_1(\lambda^{1/2}t) & \text{if } 0 < t < \lambda^{-1/2}\bar{t}_1, \\ b & \text{otherwise.} \end{cases}$$

Choosing $\widehat{\mu} > 1$ and $\beta > 0$, we shall show that

$$u_\lambda \leq \overline{\eta}_{j\lambda} \quad \text{for } j \in [1,e]$$

provided it is possible to choose $e > 1$ such that $u_\lambda \leq \overline{\eta}_{e\lambda}$ for λ large.

Define $E = \{j \in [1, e] \mid u_\lambda \le \overline{\eta}_{j\lambda}\}$. We know that $e \in E$ and E is closed. Let

$$G = \{x \in \Omega \mid u_\lambda(x) = b\}, \quad F_j = \{x \in \Omega \mid \overline{\eta}_{j\lambda}(x) = b\},$$

and

$$\Omega_j = \{x = s + t\nu(s) \in \Omega \mid 0 < t < (j\lambda)^{-1/2}\overline{t}_1\}.$$

(Note that G, F_j and Ω_j depend upon λ, we omit the subscript λ here and below.) We shall prove that for each $j_0 \in E$, there is a neighbourhood J_0 of j_0 such that

$$G \subset F_j \quad \text{for all } j \in J_0. \tag{3.9}$$

Notice that G is closed, we first show that for a sufficiently small neighbourhood Q of G such that $G \subset Q \subset\subset \Omega$, there exists $\tau > 0$ (depending upon Q) such that

$$\overline{\eta}_{j_0\lambda} \ge u_\lambda + \tau \quad \text{on } \partial Q. \tag{3.10}$$

(Note that both Q and τ depend upon λ.) On the contrary, there exists $x_0 \in \partial Q$ such that

$$\overline{\eta}_{j_0\lambda}(x_0) = u_\lambda(x_0).$$

Since $x_0 \notin G$, it is clear that $x_0 \notin F_{j_0}$. Setting $\widetilde{\delta} = \mathrm{dist}(x_0, F_{j_0})/2$, we have $B_{\widetilde{\delta}}(x_0) \cap F_{j_0} = \emptyset$. Since $\max_{\overline{B_{\widetilde{\delta}}(x_0)}} \overline{\eta}_{j_0\lambda} < b$, we can find $M_{j_0} > 0$ such that $\overline{g}(s) := f(s) + M_{j_0}s$ is strictly increasing for $s \in [0, \max_{\overline{B_{\widetilde{\delta}}(x_0)}} \overline{\eta}_{j_0\lambda}]$. This also implies that $B_{\widetilde{\delta}}(x_0) \subset \Omega_{j_0}$.

On the other hand, for λ sufficiently large,

$$\begin{aligned} &-\Delta(\overline{\eta}_{j_0\lambda} - u_\lambda) + \lambda M_{j_0}(\overline{\eta}_{j_0\lambda} - u_\lambda) \\ &\quad = \lambda(\overline{g}(\overline{\eta}_{j_0\lambda}) - \overline{g}(u_\lambda)) + \lambda(j_0\widehat{\mu} - 1)\left[f(\overline{\eta}_{j_0\lambda}) + \frac{j_0(\widehat{\mu} - 1)}{(j_0\widehat{\mu} - 1)} M_a \overline{\eta}_{j_0\lambda}\right] \\ &\qquad + (\lambda j_0)^{1/2}[\beta(j_0\lambda)^{1/2} - b(s,t)]\overline{y}_1' > 0 \quad \text{in } B_{\widetilde{\delta}}(x_0) \end{aligned} \tag{3.11}$$

provided $\widehat{\mu} > 1$ and $\beta > 0$, where we use the facts that $j_0(\widehat{\mu} - 1)/(j_0\widehat{\mu} - 1) > (\widehat{\mu} - 1)/\widehat{\mu}$ and that $|f'(s)| \le [\widehat{\mu}/\widehat{\mu} - 1]M_a$ for $s \in [0, a]$ (we can easily see that the second term on the right hand side of (3.11) is positive). Therefore, the strong maximum principle implies $\overline{\eta}_{j_0\lambda} \equiv u$ in $B_{\widetilde{\delta}}(x_0)$. This contradicts (3.11). Since $\overline{\eta}_{j\lambda}$ is continuous in the norm $\|\cdot\|_0$ about j, we have from (3.10) that there exists $\widehat{\delta} > 0$ sufficiently small such that

$$\overline{\eta}_{j\lambda} - u_\lambda \ge 0 \quad \text{on } \partial Q \tag{3.12}$$

for $j \in J := (j_0 - \widehat{\delta}, j_0 + \widehat{\delta})$. (If $j_0 = e$, we choose $J = (j_0 - \widehat{\delta}, j_0)$.)

Now we show that (3.9) holds for a neighbourhood J_0 of j_0 with $J_0 \subset J$. On the contrary, we have sequences $\{j_n\} \subset J$ and $\{x_n\} \subset G$ with $j_n \to j_0$ as $n \to \infty$ such that $\overline{\eta}_{j_n\lambda}(x_n) < u_\lambda(x_n)$. Define $m_n = \inf_{x \in \overline{Q}}[\overline{\eta}_{j_n\lambda}(x) - u_\lambda(x)]$. We have that m_n can be achieved at $\xi_n \in Q$ and $m_n < 0$ for n sufficiently large. Now, setting

$$H_n = \{x \in \overline{Q} \mid \overline{\eta}_{j_n\lambda}(x) - u_\lambda(x) \geq 0\},$$

we know that H_n is closed and $\xi_n \notin H_n$. Let $\widehat{\omega}_n = \operatorname{dist}(\xi_n, H_n)$ and $B_{\widehat{\omega}_n}(\xi_n)$ be the ball with certer at ξ_n and radius $\widehat{\omega}_n$. One easily knows that $B_{\widehat{\omega}_n}(\xi_n) \subset Q$ and

$$\overline{\eta}_{j_n\lambda}(x) - u_\lambda(x) < 0 \quad \text{for } x \in B_{\widehat{\omega}_n}(\xi_n) \tag{3.13}$$

and there is at least one point $\eta_n \in \partial B_{\widehat{\omega}_n}(\xi_n)$, where $\overline{\eta}_{j_n\lambda} - u_\lambda$ vanishes.

On the other hand, choosing Q such that

$$u_\lambda \geq b - \delta/4 \quad \text{in } \overline{Q}, \tag{3.14}$$

where $\delta > 0$ is as in (f_1), we have

$$\overline{\eta}_{j_0\lambda} \geq u_\lambda \geq b - \delta/4 \quad \text{in } \overline{Q}. \tag{3.15}$$

The continuity on the C^0-norm of $\overline{\eta}_{j\lambda}$ about j implies that for n sufficiently large,

$$\overline{\eta}_{j_n\lambda} \geq b - \delta/2 \quad \text{on } \overline{Q}. \tag{3.16}$$

Thus, for n sufficiently large,

$$\overline{\eta}_{j_n\lambda} \geq b - \delta/2 \quad \text{on } \overline{B_{\widehat{\omega}_n}(\xi_n)}. \tag{3.17}$$

Therefore, for $x \in B_{\widehat{\omega}_n}(\xi_n)$,

$$\begin{aligned}-\Delta(\overline{\eta}_{j_n\lambda} - u_\lambda) = \lambda(&j_n\widehat{\mu} f(\overline{\eta}_{j_n\lambda}) - f(u_\lambda)) \\ &+ (\lambda j_n)^{1/2}[\beta(\lambda j_n)^{1/2} - b(s,t)]\overline{y}_1' + \lambda j_n(\widehat{\mu} - 1)M_a\overline{\eta}_{j_n\lambda}.\end{aligned}$$

Since $f'(s) < 0$ for $s \in (b - \delta, b)$, we have that $f(\overline{\eta}_{j_n\lambda}) \geq f(u_\lambda)$ in $B_{\widehat{\omega}_n}(\xi_n)$. Therefore,

$$-\Delta(\overline{\eta}_{j_n\lambda} - u_\lambda) \geq 0 \quad \text{on } B_{\widehat{\omega}_n}(\xi_n) \tag{3.18}$$

provided $\beta > 0$, $\widehat{\mu} > 1$ and λ sufficiently large. It follows from (3.18) and the Hopf's maximum principle that

$$\overline{\eta}_{j_n\lambda} - u_\lambda \equiv m_n < 0 \quad \text{in } \overline{B_{\widehat{\omega}_n}(\xi_n)}. \tag{3.19}$$

But (3.19) contradicts the fact that $\overline{\eta}_{j_n\lambda} - u_\lambda$ has a zero point on $\partial B_{\widehat{\omega}_n}(\xi_n)$. This shows (3.9).

Now we show that there exists a neighbourhood J_1 of j_0 in J_0 such that $J_1 \subset E$. Since $j_0 \in E$ and $u_\lambda \leq \overline{\eta}_{j_0\lambda}$ in Ω, we can choose a small neighbourhood Q_1 of F_{j_0} such that $G \subset F_{j_0} \subset\subset Q_1$ and

$$u_\lambda \leq \overline{\eta}_{j_0\lambda} \quad \text{in } Q_1, \tag{3.20}$$

$$u_\lambda < \overline{\eta}_{j_0\lambda} \quad \text{on } \partial Q_1. \tag{3.21}$$

By the property of F_j; the fact that $G \subset F_j$ for all $j \in J_0$ and the continuity of $\overline{\eta}_{j\lambda}$ in the C^0-norm about j, we have that there exists a neighbourhood J_1 of j_0 in J_0 such that $G \subset F_j \subset\subset Q_1$ for all $j \in J_1$ and (3.20)–(3.21) hold for all $j \in J_1$. The existence of J_1 can be obtained by the arguments similar to that in the proof of (3.9). Without loss of generality, we assume $Q \subset Q_1$.

Now we consider the domain $\Omega^1 := \Omega \setminus \overline{Q_1}$. Since $\Omega^1 \subset \Omega_{j_0}$, we know that $\max u_\lambda \leq \max \overline{\eta}_{j_0\lambda} < b$ in $\overline{\Omega^1}$. Thus, assuming that for $M_{j_0} > 0$ and $\overline{g}(s)$ as above, we have

$$\overline{g}(\overline{\eta}_{j_0\lambda}) - \overline{g}(u_\lambda) \geq 0 \quad \text{in } \Omega^1.$$

Therefore,

$$\begin{aligned}
-\Delta(\overline{\eta}_{j_0\lambda} - u_\lambda) &+ \lambda M_{j_0}(\overline{\eta}_{j_0\lambda} - u_\lambda) \\
&= \lambda(\overline{g}(\overline{\eta}_{j_0\lambda}) - \overline{g}(u_\lambda)) + \lambda(j_0\widehat{\mu} - 1)\left[f(\overline{\eta}_{j_0\lambda}) + \frac{j_0(\widehat{\mu} - 1)}{j_0\widehat{\mu} - 1} M_a \overline{\eta}_{j_0\lambda}\right] \\
&\quad + (\lambda j_0)^{1/2}[\beta(j_0\lambda)^{1/2} - b(s,t)]\overline{y}_1' \geq 0 \quad \text{in } \Omega^1.
\end{aligned}$$

In fact, since $j(\widehat{\mu} - 1)/(j\widehat{\mu} - 1) \geq (\widehat{\mu} - 1)/\widehat{\mu}$ for $j \geq 1$, then

$$f(\overline{\eta}_{j_0\lambda}) + \frac{j_0(\widehat{\mu} - 1)}{j_0\widehat{\mu} - 1} M_a \overline{\eta}_{j_0\lambda} \geq 0$$

in Ω^1. The arguments similar to that in the proof of Proposition 3.2 imply that there exists $c > 0$ such that

$$\overline{\eta}_{j_0\lambda} - u_\lambda \geq c\phi \quad \text{in } \overline{\Omega^1}, \tag{3.22}$$

where ϕ is as that in the proof of Proposition 3.2. The continuity of $\overline{\eta}_{j\lambda}$ in the C^0-norm about j implies that

$$u_\lambda \leq \overline{\eta}_{j\lambda} \quad \text{in } \Omega^1 \tag{3.23}$$

for all $j \in J_2$, where J_2 is a neighbourhood of j_0 in J_1. (3.23) and the claim immediately after (3.20)–(3.21) above give the fact that $J_2 \subset E$. This implies that $E = [1, e]$.

Now we show that it is possible to choose $e > 1$ such that $u_\lambda \leq \overline{\eta}_{e\lambda}$ for λ large and all positive solutions $u_\lambda \in [z_\lambda, b]$ of (1.1). It is easy to see that this reduces to showing that there is $K > 0$ such that $u_\lambda(x) \leq K\lambda^{1/2}t$ if u_λ is a positive solution of (1.1), x is near $\partial\Omega$ and λ is large. Obviously, it suffices to prove the result for

$t \le K_1\lambda^{-1/2}$ ($K_1 > 0$). Now for arbitrary $x_0 \in \partial\Omega$, letting $X = \lambda^{1/2}(x - x_0)$ and $\widetilde{u}_\lambda(X) = u_\lambda(x)$, then

$$-\Delta\widetilde{u}_\lambda = f(\widetilde{u}_\lambda) \quad \text{in } \Omega_\lambda, \quad \widetilde{u}_\lambda = 0 \quad \text{on } \partial\Omega_\lambda,$$

where $\Omega_\lambda = \{X \mid \lambda^{-1/2}X + x_0 \in \Omega\}$. By a blow up argument as in [5], the stretching only flattens the boundary as $\lambda \to \infty$. Since $0 \in \partial\Omega_\lambda$ and $\|\widetilde{u}\|_\infty \le b$, we apply the regularity result of $-\Delta$ to obtain that $\nabla\widetilde{u}_\lambda$ is bounded on the bounded subsets of $\overline{\Omega_\lambda}$ which contain neighbourhoods of 0 on $\partial\Omega_\lambda$. Hence, in the original variables, $\|\nabla u_\lambda\|_\infty \le K\lambda^{1/2}$ on the subsets of $\overline{\Omega}$ which contain neighbourhoods of x_0 on $\partial\Omega$. The required estimate for u_λ near $\partial\Omega$ now follows since $\partial\Omega$ is compact. This completes the proof. □

4. Uniqueness results

In this Section we show that (1.1) has only one large positive solution u_λ when λ is sufficiently large.

First note that from the definition of the large positive solution of (1.1), there exists $\xi \in (a, b)$ and a ball $B(x_0, r) \subset \Omega$ which is independent of λ such that $u_\lambda \ge \xi$ in $B(x_0, r)$ for all λ sufficiently large. Let $w(\lambda, x_0)$ be as in (2.3). We know that $w(\lambda, x_0)$ is a subsolution of (1.1) with f replaced by F_ε for $\lambda > \mu\alpha^2 d(x_0, \partial\Omega)^{-2}$. Therefore, it follows from the monotone arguments as in Lemma 2.4 and Theorem 2.3 that for $\lambda > \lambda^{**}_{x_0}$ (with x^* replaced by x_0), (2.4); (1.1) has a positive solution $u^{(\varepsilon)}_\lambda$; $\widetilde{u}_\lambda$ in $[w(\lambda, x_0), b]$ respectively (both of them are minimal solutions) such that $\widetilde{u}_\lambda \ge u^{(\varepsilon)}_\lambda$ in Ω and

$$\widetilde{u}_\lambda \to b \quad \text{on compact sets of } \Omega \text{ as } \lambda \to \infty. \tag{4.1}$$

Therefore, for $\lambda \ge \overline{\lambda} > \max\{\lambda^{**}_{x_0}, \lambda^{**}_{x^*}\}$ and $\overline{\lambda}$ being a fixed sufficiently large number,

$$\widetilde{u}_\lambda \ge w(\lambda, x^*) \quad \text{in } \Omega \tag{4.2}$$

and

$$\widetilde{u}_\lambda \ge u_{\lambda,x^*} \quad \text{in } \Omega,$$

where $u_{\lambda,x^*} \in [w(\lambda, x^*), b]$ is the minimal positive solution of (1.1). (This is known from Remark 2 after the proof of Theorem 2.3.)

Now we show that $u_\lambda \ge \widetilde{u}_\lambda$ and hence

$$u_\lambda \ge u_{\lambda,x^*} \quad \text{in } \Omega$$

for $\lambda \ge \overline{\lambda}$. Therefore, the eatimates in Proposition 3.1 are true for u_λ.

It is enough to prove $u_\lambda \ge w(\lambda, x_0)$ in Ω. It is known from the above that $w(\lambda, x_0)$ is a subsolution of (2.4) for ε sufficiently small and

$$\tau = \max w(\lambda, x_0) < b(\varepsilon) < b.$$

Moreover, there exists $\sigma_\varepsilon > 0$ such that

$$f(s) \geq \sigma_\varepsilon (s - \xi) \quad \text{for } s \in [\xi, b(\varepsilon)] \tag{4.3}$$

since $\xi > a$ and $f(s) > 0$ for $s \in [\xi, b(\varepsilon)]$. (We know that $\xi < b(\varepsilon)$ when ε is sufficiently small).

Lemma 4.1. *Let f satisfy* (4.3). *Then, for $\lambda > \overline{\lambda}$,*

$$u_\lambda \geq b(\varepsilon) \quad \text{in } B(x_0, r/2). \tag{4.4}$$

Proof. We know that $u_\lambda \geq \xi$ in $B(x_0, r)$. Now for any $x_1 \in B(x_0, r/2)$, we set

$$\theta(x_1, \lambda, t; x) = \xi + t\phi_1((\sigma_\varepsilon \lambda/\lambda_1)^{1/2}(x - x_1)) \quad \text{for } x \in \widetilde{B} \text{ and } t \in [0, b(\varepsilon) - \xi],$$

where λ_1 and ϕ_1 with $\|\phi_1\|_\infty = 1$ are the first eigenvalue and the corresponding eigenfunction of the eigenvalue problem of $-\Delta$ in the unit ball of $\mathbb{R}^N$ with the Dirichlet boundary condition; $\widetilde{B} = B(x_1, \lambda_1(\sigma_\varepsilon \lambda)^{-1})$. It is well-known that ϕ_1 is radially symmetric and $\phi_1(0) = 1$. Note that for $\lambda > \lambda_1(\sigma_\varepsilon r/2)^{-1}$, $\widetilde{B} \subset B(x_0, r)$. We assume that $\lambda > \max\{\lambda_1(\sigma_\varepsilon r/2)^{-1}, \overline{\lambda}\}$. We claim that the set $\{\theta(x_0, \lambda, t) \mid t \in [0, b(\varepsilon) - \xi]\}$ is a family of subsolutions of the problem

$$-\Delta v = \lambda f(v) \quad \text{in } \widetilde{B}, \quad v = u_\lambda \quad \text{on } \partial\widetilde{B}, \tag{4.5}$$

with the closure of $\widetilde{B}$ is contained in $B(x_0, r)$. It is clear that $u_\lambda \geq \theta(x_0, \lambda, 0)$ in $\widetilde{B}$ and $|f'(s)| \leq M_\varepsilon$ for $s \in [0, b(\varepsilon)]$. Thus, by the similar argument to that in the proof of Proposition 3.2, we obtain that

$$u_\lambda \geq \theta(x_0, \lambda, b(\varepsilon) - \xi) \quad \text{in } \widetilde{B} \tag{4.6}$$

and thus

$$u_\lambda(x_1) \geq b(\varepsilon) \quad \text{for all } x_1 \in B(x_0, r/2). \tag{4.7}$$

This completes the proof of Lemma 4.1. □

It is easily seen that when $\lambda > (r/2)^{-2}\mu$, $w(\lambda, x_0; x) \leq 0$ for $x \in \Omega \setminus B(x_0, r/2)$. We assume $\lambda > \widetilde{\lambda} := \max\{\lambda_1(\sigma_\varepsilon r/2)^{-1}, \overline{\lambda}, (r/2)^{-2}\mu\}$ in the follows. Then we obtain

$$u_\lambda \geq w(x_0, \lambda) \quad \text{in } \Omega, \tag{4.8}$$

since $\tau < b(\varepsilon)$. By the fact that $\widetilde{u}_\lambda$ is the minimal positive solution of (1.1) in $[w(x_0, \lambda), b]$, we have

$$u_\lambda \geq \widetilde{u}_\lambda \quad \text{in } \Omega. \tag{4.9}$$

This is our claim.

Theorem 4.2. *Assume that f satisfies* (f$_1$)–(f$_3$). *Then* (1.1) *has only one large positive solution u_λ for λ sufficiently large satisfying* $\max_\Omega u_\lambda \le b$ *and*

$$u_\lambda \to b \text{ in compact sets of } \Omega \text{ as } \lambda \to \infty. \tag{4.10}$$

Proof. The existence of at least one large positive solution u_λ of (1.1) for λ sufficiently large has been obtained in Theorem 2.3. We only need to study the uniqueness of u_λ.

By the argument above, we know that if u_λ and u_λ^* are two large positive solutions of (1.1) for λ sufficiently large, then $u_\lambda \le u_\lambda^*$ or $u_\lambda^* \le u_\lambda$ holds and the asymptotic behaviour in Proposition 3.1 holds for both u_λ and u_λ^* and λ large. Without loss of generality, we assume $u_\lambda^* \le u_\lambda$ in Ω in the follows.

Now we show that for λ sufficiently large,

$$u_\lambda \equiv u_\lambda^* \quad \text{in } \Omega. \tag{4.11}$$

On the contrary, there exist sequences $\{\lambda_n\}$ with $\lambda_n \to \infty$ as $n \to \infty$ and $\{u_n\} \equiv \{u_{\lambda_n}\}$, $\{u_n^*\} \equiv \{u_{\lambda_n}^*\}$ such that $u_n \not\equiv u_n^*$ for all n.

Define $v_n = (u_n - u_n^*)/\|u_n - u_n^*\|_\infty$. Then $v_n \ge 0$, $v_n \not\equiv 0$ in Ω and $\max_\Omega v_n = 1$ for all n. Setting

$$H_n = \{x \in \Omega \mid u_n(x) = b\},$$
$$H_n^* = \{x \in \Omega \mid u_n^*(x) = b\},$$

we easily know that $H_n^* \subset H_n \subset\subset \Omega$ and that v_n satisfies the problem

$$-\Delta v_n = \lambda_n f'(\xi_n) v_n \quad \text{in } \Omega \setminus H_n, \quad v_n = 0 \quad \text{on } \partial\Omega, \tag{4.12}$$

where $\xi_n \in (u_n^*, u_n)$. Now we show if $\eta_n \in \Omega$ such that $v_n(\eta_n) = 1$, then

$$\operatorname{dist}(\eta_n, \partial\Omega) \to 0 \quad \text{as } n \to \infty. \tag{4.13}$$

(Note that $v_n(x) = 0$ for $x \in H_n^*$.) In fact, it is known from Proposition 3.1 that if $K \subset\subset \Omega$ is a compact set, then $u_n^* \to b$, $u_n \to b$ in K as $n \to \infty$. If $\eta_n \in K$ for all n large, we know that $\eta_n \in K \setminus H_n^*$. There are two cases here: (i) there exists a subsequence of $\{\eta_n\}$ (still denoted by $\{\eta_n\}$) such that $\eta_n \in K \setminus H_n$, (ii) there exists a subsequence of $\{\eta_n\}$ (still denoted by $\{\eta_n\}$) such that $\eta_n \in H_n \setminus H_n^*$. Since H_n and H_n^* are closed sets in Ω, for the first case, there exists a small neighbourhood B_{η_n} of η_n in Ω such that $B_{\eta_n} \cap H_n = \emptyset$ and $f'(\xi_n) < 0$ in B_{η_n} for all n large. (We use the continuity of ξ_n in B_{η_n} here.) This is a contradiction since v_n attains its maximum on Ω at η_n. For the second case, we also can choose a small neighbourhood B_{η_n} of η_n in Ω such that $B_{\eta_n} \cap H_n^* = \emptyset$. On the other hand, we write (4.12) in the form

$$-\Delta v_n = \lambda_n \frac{f(u_n) - f(u_n^*)}{u_n - u_n^*} v_n$$

and easily know that $-\Delta v_n < 0$ in B_{η_n}. This is also a contradiction. Thus, (4.13) holds.

Now we use the blow up argument as in [12], [5] to show that (4.13) does not hold. We consider two cases here: (we can choose subsequences if necessary)

(i) $\lambda_n^{1/2}\mathrm{dist}(\eta_n, \partial\Omega) \to Z \ge t_0$ (Z can be ∞), as $n \to \infty$,

(ii) $\lambda_n^{1/2}\mathrm{dist}(\eta_n, \partial\Omega) \le Z < t_0$ for all n sufficiently large, where $t_0 > 0$ is the number defined in (3.3).

For the first case, we have from Proposition 3.1 that $u_n(\eta_n) \to b$, $u_n^*(\eta_n) \to b$ as $n \to \infty$. Thus we derive contradictions by the arguments similar to that in the proof of (4.13).

For the second case, we make a change of variables, $X^n = \lambda_n^{1/2}(x - \widetilde{\eta}_n)$, where $\widetilde{\eta}_n$ is the point on $\partial\Omega$ closest to η_n. Let $\widetilde{u}_n(X^n) = u_n(x)$, $\widetilde{u}_n^*(X^n) = u_n^*(x)$, $\widetilde{\xi}^n(X_n) = \xi_n(x)$ and $\widetilde{v}_n(X^n) = v_n(x)$. We have that $\widetilde{v}_n$ satisfies the problem

$$-\Delta\widetilde{v}_n = f'(\widetilde{\xi}_n)\widetilde{v}_n \quad \text{in } \widetilde{\Omega}_n \setminus \widetilde{H}_n, \quad \widetilde{v}_n = 0 \quad \text{on } \partial\widetilde{\Omega}_n, \tag{4.14}$$

where

$$\widetilde{\Omega}_n \equiv \{X^n = \lambda_n^{1/2}(x - \widetilde{\eta}_n) \mid x \in \Omega\}, \quad \widetilde{H}_n \equiv \{X^n = \lambda_n^{1/2}(x - \widetilde{\eta}_n) \mid x \in H_n\}.$$

Note that $\widetilde{v}_n(Z_n) = 1$, where $Z_n = \lambda_n^{1/2}(\eta_n - \widetilde{\eta}_n)$ is at distance at most Z from 0 and $Z < t_0$. By the argument similar to that in the proof of Theorem 2 of [5], we have that $\widetilde{u}_n \to y(x_1)$ in $C^1_{\mathrm{loc}}(T_1)$, $\widetilde{u}_n^* \to y(x_1)$ in $C^1_{\mathrm{loc}}(T_1)$ as $n \to \infty$. Where $T_1 = \{x \in \mathbb{R}^N \mid x_1 \ge 0\}$ and y is the unique solution of (3.2). Defining $H = \{x \in T_1 \mid x_1 \ge t_0\}$, we easily know that $\widetilde{H}_n \to H$ and $\widetilde{\xi}_n \to y(x_1)$ in $C^0_{\mathrm{loc}}(T_1 \setminus H)$ as $n \to \infty$. Moreover, $\widetilde{v}_n$ converges in $C^1_{\mathrm{loc}}(T_1 \setminus H)$ to a non-trivial non-negative bounded solution $\widetilde{v}$ of

$$-\Delta\widetilde{v} = f'(y(x_1))\widetilde{v} \quad \text{in } T_1 \setminus H, \quad \widetilde{v} = 0 \quad \text{on } \partial T_1. \tag{4.15}$$

Here $\widetilde{v}$ is non-trivial because $\widetilde{v}_n(Z_n) = 1$ and $\mathrm{dist}(0, Z_n) \le Z < t_0$.

Now we show that such $\widetilde{v}$ can not exist by the three steps as that in the proof of Proposition 2 of [5], but with a different definition domain of $y(x_1)$.

Step 1. We find a solution q of

$$-u'' = f'(y)u, \tag{4.16}$$

which is positive on $[0, t_0)$ and is not bounded as $x_1 \to t_0^-$.

By differentiating the equation satisfied by y ((3.2)) with respect to x_1, we see that $y'(x_1)$ is a solution of (4.16). Let Y be the solution of the initial value problem

$$-Y'' = f'(y)Y \quad \text{in } (0, t_0), \quad Y(0) = 0, \quad Y'(0) = 1. \tag{4.17}$$

We claim that $Y(x_1) \to \infty$ as $x_1 \to t_0^-$. In fact, we know from a simple computation

$$(y'Y' - Yy'')' \equiv 0 \quad \text{in } (0, t_0). \tag{4.18}$$

This implies that

$$y'Y' - Yy'' \equiv C \quad \text{in } [0, t_0], \tag{4.19}$$

where $C = y'(0)$. Our claim can be obtained from (4.19) and the facts that $y'(x_1) \to 0$ and $y''(x_1) \to 0$ as $x_1 \to t_0^-$. Define $q(x_1) = y'(x_1) + Y(x_1)$. We easily know that q satisfies our requirement.

Step 2. If (4.15) has a non-trivial bounded non-negative solution $\widetilde{v}$, the $\widetilde{v}$ can be chosen so that $T(x_1) = \sup_{y \in \mathbb{R}^{N-1}} \widetilde{v}(x_1, y)$ is continuous for $x_1 > 0$.

The proof of Step 2 is similar to that of Proposition 2 in [5].

Step 3. We show that $\widetilde{v}$ can not exist. If $\widetilde{v}$ exists, using the notation of Step 2, we consider $r(x) = \widetilde{v}(x)/q(x_1)$. By Steps 1 and 2 and the boundedness of $\widetilde{v}$, it follows that $\lim_{x_1 \to t_0^-} T(x_1)/q(x_1) = 0$. Thus, since $T(0) = 0$, we can find $0 < \widetilde{x}_1 < t_0$ such that

$$\sup\{T(x_1)/q(x_1) \mid 0 \le x_1 < t_0\} = T(\widetilde{x}_1)/q(\widetilde{x}_1).$$

By Step 2, $\widetilde{v}$ can be chosen so that $\widetilde{v}(\widetilde{x}_1, y)$ achieves its maximum on $\mathbb{R}^{N-1}$ at 0. By our construction, $r(x)$ achieves its maximum on $\{(x_1, y) \mid 0 \le x_1 < t_0, y \in \mathbb{R}^{N-1}\}$ at the *interior point* $(\widetilde{x}_1, 0)$. However, since q satisfies (4.16), a simple calculation shows that r satisfies an elliptic equation

$$r''_{x_1x_1} + 2(q'/q)r'_{x_1} + \Delta_{N-1} r = 0,$$

where Δ_{N-1} denotes the Laplacian in the y variables. Hence, by applying the maximum principle on compact sets, we see that $r(x_1, y)$ is constant of $0 \le x_1 < t_0$, $y \in \mathbb{R}^{N-1}$. This is impossible since $r = 0$ when $x_1 = 0$. □

We easily obtain the following corollary from Theorem 4.2.

Corollary 4.3. *Let f satisfy* (f_1)–(f_3) *and Ω be an N-ball or an annulus. Then* (1.1) *has exactly one large positive solution u_λ which is radially symmetric for λ sufficiently large. Moreover, $u_\lambda \to b$ in compact subsets of Ω as $\lambda \to \infty$.*

Remark. Corollary 4.3 implies that (1.1) has no non-radial large positive solutions for λ sufficiently large.

5. Flat core of the large positive solution

In this section we shall give the asymptotic behaviour of the flat core G_λ of the unique large positive solution u_λ as $\lambda \to \infty$. The existence of G_λ for u_λ was obtained in [21]. Our main result of this section is

THEOREM 5.1. *Let f satisfy* (f_1)–(f_3). *Then for λ sufficiently large, G_λ satisfies that if $d^*(\lambda) = \mathrm{dist}(G_\lambda, \partial\Omega)$, then*

$$\lim_{\lambda\to\infty} \lambda^{1/2} d^*(\lambda) = \frac{C(F)^{1/2}}{2},$$

where

$$C(F) = \frac{1}{2}\bigg(\int_0^b \frac{2ds}{(F(b)-F(s))^{1/2}}\bigg)^2 \quad \text{and} \quad F(s) = \int_0^s f(\xi)\,d\xi.$$

Moreover,

$$\lim_{\lambda\to\infty} \lambda^{1/2}\mathrm{dist}(x, G_\lambda) = \frac{C(F)^{1/2}}{2} \quad \textit{for any } x \in \partial\Omega.$$

To prove this theorem, we start the study from the simple case $N = 1$, i.e. the problem

$$-v'' = \lambda f(v) \quad \text{in } (0,\ell), \quad v(0) = 0, \quad v(\ell) = 0, \tag{5.1}$$

where $\ell > 0$ is independent of λ. The main idea of this section is similar to that in [18] but with many modifications.

LEMMA 5.2. *Assume that f satisfies* (f_1)–(f_3). *Then there exists a unique positive solution $v_\lambda(x)$ of* (5.1) *satisfying*

$$v_\lambda \to b \quad \textit{uniformly on compact sets of } (0,\ell) \textit{ as } \lambda \to \infty. \tag{5.2}$$

Moreover, for $\lambda \geq \widehat{\lambda} := (1/\ell^2)C(F)$,

$$E_\lambda = \{x \in (0,\ell) \mid v_\lambda(x) = b\} = [d^*(\lambda), \ell - d^*(\lambda)], \tag{5.3}$$

where

$$C(F) = \frac{1}{2}\bigg(\int_0^b \frac{2\,ds}{(F(b)-F(s))^{1/2}}\bigg)^2, \tag{5.4}$$

$$F(s) = \int_0^s f(\xi)\,d\xi,$$

$$d^*(\lambda) = \frac{1}{2}C(F)^{1/2}\lambda^{-1/2}. \tag{5.5}$$

PROOF. We claim that if $v_\lambda \in C^1([0,\ell])$ is a positive solution of (5.1) with $\|v_\lambda\|_\infty \leq b$, then v_λ is symmetric about $x = \ell/2$. In fact, the first integral of (5.1) implies that

$$|v_\lambda'|^2 + 2\lambda F(v_\lambda) = C, \quad x \in (0,\ell). \tag{5.6}$$

Let $\overline{v}_\lambda = \sup_{0<x<\ell} v_\lambda(x)$. Then it follows from (5.6) that

$$|v_\lambda'|^2 = 2\lambda(F(\overline{v}_\lambda) - F(v_\lambda)). \tag{5.7}$$

On the other hand, we easily know from (5.6) that $\overline{v}_\lambda$ is the only critical value of $v_\lambda = v_\lambda(x)$. Therefore, if $x_1^\lambda = \min\{x \mid v_\lambda = \overline{v}_\lambda\}$, $x_2^\lambda = \max\{x \mid v_\lambda = \overline{v}_\lambda\}$, then v_λ increases before x_1^λ, decreases after x_2^λ, while $v_\lambda \equiv \overline{v}_\lambda$ in $x_1^\lambda \le x \le x_2^\lambda$. Thus, it follows from (5.7) that

$$\int_0^{v_\lambda(x)} \frac{ds}{(F(\overline{v}_\lambda) - F(s))^{1/2}} = (2\lambda)^{1/2} x, \quad 0 < x < x_1^\lambda, \tag{5.8}$$

and

$$\int_0^{v_\lambda(x)} \frac{ds}{(F(\overline{v}_\lambda) - F(s))^{1/2}} = (2\lambda)^{1/2} (\ell - x), \quad x_2^\lambda < x < \ell. \tag{5.9}$$

(5.8) and (5.9) imply that v_λ is symmetric with respect to $\ell/2$ and

$$\int_0^{\overline{v}_\lambda} \frac{ds}{(F(\overline{v}_\lambda) - F(s))^{1/2}} = (2\lambda)^{1/2} x_1^\lambda. \tag{5.10}$$

This implies our claim.

To prove the existence, we first notice that it follows from (f_2) that

$$\int_0^b \frac{ds}{(F(b) - F(s))^{1/2}} < \infty.$$

Defining $C(F)$ and $d^*(\lambda)$ as in (5.4) and (5.5) and

$$\widehat{\lambda} = \frac{1}{\ell^2} C(F),$$

we have that if $\lambda > \widehat{\lambda}$, then $d^*(\lambda) < \ell/2$. Now we define $v_\lambda(x)$ by

$$\int_0^{v_\lambda(x)} \frac{ds}{(F(b) - F(s))^{1/2}} = (2\lambda)^{1/2} x, \quad 0 < x < d^*(\lambda) \tag{5.11}$$

and

$$v_\lambda(x) \equiv b \quad \text{for } x \in [d^*(\lambda), \ell/2]. \tag{5.12}$$

We can define v_λ on $[\ell/2, \ell]$ such that v_λ is symmetric about $x = \ell/2$. It is clear that v_λ is the required positive solution of (5.1).

Now we show that v_λ is the unique positive solution of (5.1) such that $\max v_\lambda \to b$ as $\lambda \to \infty$. In fact, suppose w_λ is a positive solution of (5.1) such that $\max w_\lambda \to b$ as $\lambda \to \infty$, we can show that $w_\lambda(\ell/2) = b$ for λ sufficiently large. On the contrary, we know that $w_\lambda(\ell/2) := \overline{w}_\lambda < b$ for all λ large. Since $F(s) = \int_0^s f(\xi)\, d\xi$, we know that for $s < \overline{w}_\lambda$ and near $\overline{w}_\lambda$,

$$F(s) = F(\overline{w}_\lambda) + f(\overline{w}_\lambda)(s - \overline{w}_\lambda) + \frac{1}{2} f'(\overline{w}_\lambda)(s - \overline{w}_\lambda)^2 + o((s - \overline{w}_\lambda)^2).$$

We know that $f(\overline{w}_\lambda) > 0$ and $f'(\overline{w}_\lambda) < 0$ for λ sufficiently large (since $\overline{w}_\lambda \to b$ as $\lambda \to \infty$). Thus,

$$F(\overline{w}_\lambda) - F(s) \ge \frac{1}{2} f(\overline{w}_\lambda)(\overline{w}_\lambda - s) \quad \text{for } s \text{ near } \overline{w}_\lambda.$$

Therefore,

$$\int_{s_0}^{\overline{w}_\lambda} (F(\overline{w}_\lambda) - F(s))^{-1/2} \le 2(f(\overline{w}_\lambda))^{-1/2} \int_{s_0}^{\overline{w}_\lambda} (\overline{w}_\lambda - s)^{-1/2}\, ds < \infty \tag{5.13}$$

for s_0 near $\overline{w}_\lambda$ and λ sufficiently large. On the other hand, we know from a similar identity to (5.10) that

$$\int_0^{\overline{w}_\lambda} \frac{ds}{(F(\overline{w}_\lambda) - F(s))^{1/2}} = (2\lambda)^{1/2} \frac{\ell}{2}. \tag{5.14}$$

(Since $\overline{w}_\lambda < b$, $\overline{w}_\lambda$ can only attain at $x = \ell/2$.) We easily derive a contradiction from (5.13) and (5.14). Since w_λ can also be written to the forms same as (5.11) and (5.12), we have that $w_\lambda \equiv v_\lambda$ in $(0, \ell)$. □

Now we are dealing with the case $\Omega = B_R = \{x \in \mathbb{R}^N \mid |x| < R\}$.

LEMMA 5.3. *Let u_λ be the unique large positive* (*radial*) *solution of* (1.1) *for λ sufficiently large obtained in Corollary* 4.3. *Then u_λ has flat core $G_{\lambda,B}$. Moreover,*

$$\limsup_{\lambda\to\infty} \lambda^{1/2} d(\lambda, B) \le \frac{C(F)^{1/2}}{2}, \tag{5.15}$$

where $d(\lambda, B) = \operatorname{dist}(G_{\lambda,B}, \partial B_R)$.

PROOF. We know that u_λ satisfies the problem

$$-(r^{N-1}u_\lambda')' = \lambda r^{N-1} f(u_\lambda), \quad r \in (0, R), \quad u_\lambda'(0) = 0, \quad u_\lambda(R) = 0. \tag{5.16}$$

Now we introduce a change

$$\rho = g(r) = \begin{cases} \dfrac{1}{2-N}(R^{2-N} - r^{2-N}) & N \ge 3, \\ \log(R/r) & N = 2. \end{cases}$$

Observe that $0 < \rho < \infty$ if $0 < r < R$. Setting $v_\lambda(\rho) = u_\lambda(g^{-1}(\rho))$ in (5.16) leads to the problem

$$-v_\lambda'' = \lambda(g^{-1}(\rho))^{2(N-1)} f(v_\lambda), \quad 0 < \rho < \infty, \quad v(0) = v'(\infty) = 0, \tag{5.17}$$

where $' = d/d\rho$. Moreover, v_λ is the unique large positive solution of (5.17) (see [18]).

If we fix $0 < \theta < \infty$ independent of λ and $v = v_\lambda(\rho)$ stands for the unique large positive solution to (5.17), then we have that $v(\theta) \to b$ as $\lambda \to \infty$ and that there exists a unique $0 < \eta_\lambda < \theta$ such that $v(\eta_\lambda) = a$ and

$$-v'' \ge \lambda(g^{-1}(\theta))^{2(N-1)} f(v)$$

provided that $\eta_\lambda < \rho < \theta$ (since $f(v(\rho)) \ge 0$ for $\eta_\lambda < \rho < \theta$). The uniqueness of η_λ can be known from the structure of u_λ. In fact, we can easily show that $u_\lambda' \equiv 0$ and $u_\lambda \equiv b$ in $[0, \widetilde{r}_\lambda]$ for some $\widetilde{r}_\lambda \ge 0$ and $u_\lambda' < 0$ in $(\widetilde{r}_\lambda, R]$ (see [15]).

Thus, v_λ has the similar property. We know that $\eta_\lambda = g(r_\lambda)$, where $u_\lambda(r_\lambda) = a$. It follows from Proposition 3.1 that $\lambda^{1/2}(R - r_\lambda) \to t^0$ as $\lambda \to \infty$, where $t^0 > 0$ satisfies $y(t^0) = a$ and $y(t)$ is the unique solution of (3.2). (This can also be obtained from the arguments similar to that in the proof of Theorem 4.2 or that in the proof of Theorem A in [13]. In fact, if r is near R, $X^\lambda = \lambda^{1/2}(R - r)$ and $\widetilde{u}_\lambda(X^\lambda) = u_\lambda(r)$, we know

$$\widetilde{u}_\lambda(X^\lambda) \to y(t) \quad \text{in } C^1_{\mathrm{loc}}(0,\infty) \text{ as } \lambda \to \infty,$$

where y is the unique solution of (3.2).) By the first integral arguments similar to that in the proof of Lemma 5.2, we easily know from the property of y that

$$t^0 = 2^{-1/2} \int_0^a \frac{ds}{(F(b) - F(s))^{1/2}}.$$

Thus,

$$r_\lambda = R - (2\lambda)^{-1/2} \int_0^a \frac{ds}{(F(b) - F(s))^{1/2}} + o(\lambda^{-1/2})$$

for λ sufficiently large. Therefore, for $N \geq 3$,

$$\eta_\lambda = g(r_\lambda) = (2\lambda)^{-1/2} R^{1-N} \int_0^a \frac{ds}{(F(b) - F(s))^{1/2}} + o(\lambda^{-1/2}).$$

For $N = 2$, we also obtain

$$\eta_\lambda = g(r_\lambda) = (2\lambda)^{-1/2} R^{-1} \int_0^a \frac{ds}{(F(b) - F(s))^{1/2}} + o(\lambda^{-1/2}).$$

(Note that we use Taylor expansions here.)

Let us introduce now the auxiliary problem

$$-v'' = \lambda(g^{-1}(2\theta))^{2(N-1)} f(v), \quad \eta_\lambda < \rho < \theta, \quad v(\eta_\lambda) = a, \quad v'(\theta) = 0. \tag{5.18}$$

We observe now that (5.18) admits a unique positive solution $v = \underline{v}_\lambda(\rho,\theta)$ provided $\lambda > \lambda_0$ and $\underline{v}_\lambda(\theta) = b$, where

$$\begin{aligned}\lambda_0 = \bigg[\theta^{-1}\bigg(&2^{-1/2} R^{1-N} \int_0^a \frac{ds}{(F(b) - F(s))^{1/2}} + 2 \\ &+ (2(g^{-1}(2\theta))^{2(N-1)})^{-1/2} \int_a^b \frac{ds}{(F(b) - F(s))^{1/2}}\bigg)\bigg]^2.\end{aligned}$$

In fact, restricting to $\eta_\lambda < \rho < \theta$ the unique positive solution $\widehat{v}_\lambda$ of

$$\begin{aligned} -v'' &= \lambda(g^{-1}(2\theta))^{2(N-1)} f(v), \\ \eta_\lambda &< \rho < 2\theta - \eta_\lambda, \\ v(\eta_\lambda) &= v(2\theta - \eta_\lambda) = a \end{aligned} \tag{5.19}$$

with $\max \widehat{v}_\lambda = b$, we obatin $\underline{v}_\lambda$. Now we show that (5.19) has a unique positive solution $\widehat{v}_\lambda(x)$ with $\max \widehat{v}_\lambda = b$. In fact, the arguments similar to that in the proof of Lemma 5.2 imply that, if

$$\lambda > \left[\theta^{-1}\left(\lambda^{1/2}\eta_\lambda + (2(g^{-1}(2\theta))^{2(N-1)})^{-1/2}\int_a^b \frac{ds}{(F(b)-F(s))^{1/2}}\right)\right]^2,$$

(5.19) has a unique solution $\widehat{v}_\lambda(x)$ with $\widehat{v}_\lambda(\theta) = b$ satisfying

$$\int_a^{\widehat{v}_\lambda(x)} \frac{ds}{(F(b)-F(s))^{1/2}} = (2\lambda(g^{-1}(2\theta))^{2(N-1)})^{1/2}(x-\eta_\lambda) \quad \text{for } x \in (\eta_\lambda, \theta)$$

and

$$\int_a^{\widehat{v}_\lambda(x)} \frac{ds}{(F(b)-F(s))^{1/2}} = (2\lambda(g^{-1}(2\theta))^{2(N-1)})^{1/2}(2\theta-\eta_\lambda - x)$$

for $x \in (\theta, 2\theta - \eta_\lambda)$. Define

$$d(\lambda) = \eta_\lambda + (2\lambda(g^{-1}(2\theta))^{2(N-1)})^{-1/2}\int_a^b \frac{ds}{(F(b)-F(s))^{1/2}},$$

$$A = \int_0^a \frac{ds}{(F(b)-F(s))^{1/2}}, \quad B = \int_a^b \frac{ds}{(F(b)-F(s))^{1/2}}.$$

We easily know $d(\lambda) < \theta$ for $\lambda > \lambda_0$ and sufficiently large and thus

$$\widehat{v}_\lambda \equiv b \quad \text{in } [d(\lambda), 2\theta - d(\lambda)].$$

It is clear that $\widehat{v}_\lambda(\rho)$ (for $\eta_\lambda < \rho < 2\theta - \eta_\lambda$) is a subsolution of the problem

$$-v'' = \lambda(g^{-1}(\rho))^{2(N-1)}f(v), \quad v(\eta_\lambda) = a, \quad v(2\theta-\eta_\lambda) = v_\lambda(2\theta-\eta_\lambda). \tag{5.20}$$

(Note that $v_\lambda(2\theta-\eta_\lambda) \to b$ as $\lambda \to \infty$.) Since b is a supersolution of (5.20), then we use the arguments similar to that in the proof of Theorem 2.3 to obtain a positive solution $\overline{v}_\lambda$ of (5.20) in $[\widehat{v}_\lambda, b]$. Since $v_\lambda(\rho)$ is the unique large positive solution of (5.17) and $v_\lambda(\rho)$ satisfies (5.20), we can conclude

$$\overline{v}_\lambda \equiv v_\lambda \quad \text{in } (\eta_\lambda, \theta), \tag{5.21}$$

$$a < \underline{v}_\lambda(\rho, \theta) \le v_\lambda(\rho) \quad \text{for } \eta_\lambda < \rho < \theta. \tag{5.22}$$

(To show (5.21), we first notice that $v_\lambda(2\theta-\eta_\lambda) = u_\lambda(g^{-1}(2\theta-\eta_\lambda)) \to b$ as $\lambda \to \infty$. $\overline{v}_\lambda$ and v_λ are corresponding to the solutions $\overline{u}_\lambda$ and u_λ of the problem

$$-(r^{N-1}u')' = \lambda r^{N-1}f(u) \quad \text{in } (g^{-1}(2\theta-\eta_\lambda), g^{-1}(\eta_\lambda)),$$
$$u(g^{-1}(\eta_\lambda)) = u_\lambda(g^{-1}(\eta_\lambda)) = a, \quad u(g^{-1}(2\theta-\eta_\lambda)) = u_\lambda(g^{-1}(2\theta-\eta_\lambda)).$$

Since u_λ is the unique large positive solution of (5.16), extending $\overline{u}_\lambda$ to be u_λ in $[0, g^{-1}(2\theta-\eta_\lambda)]$ and $(g^{-1}(\eta_\lambda), R]$, we have that $\overline{u}_\lambda$ is also a large positive solution of (5.16). Thus, $\overline{u}_\lambda \equiv u_\lambda$ for λ sufficiently large. This shows (5.21).)

Notice that $\underline{v}_\lambda(\rho,\theta)$ develops a flat core for each $\lambda > \lambda_0$ and

$$\underline{v}_\lambda(d(\lambda)) = b. \tag{5.23}$$

Since u_λ is decreasing, (5.23) implies $u_\lambda(r) \equiv b$ for $0 \le r \le g^{-1}(d(\lambda))$. Thus,

$$0 < \operatorname{dist}(G_{\lambda,B}, \partial B_R) \le R - g^{-1}(d(\lambda)). \tag{5.24}$$

If we put $\widehat{d}(\lambda) := R - g^{-1}(d(\lambda))$, it follows that

$$\widehat{d}(\lambda) = R^{N-1}[(2\lambda)^{-1/2}R^{1-N}A + (2\lambda)^{-1/2}(g^{-1}(2\theta))^{1-N}B + o(\lambda^{-1/2})] + o(\lambda^{-1/2})$$

for λ sufficiently large. Since

$$\lim_{\lambda\to\infty} \lambda^{1/2}\widehat{d}(\lambda) = 2^{-1/2}A + 2^{-1/2}\left(\frac{R}{g^{-1}(2\theta)}\right)^{N-1} B, \tag{5.25}$$

it is obtained from (5.24) and (5.25), after passing to the limit as $\theta \to 0^+$, that

$$\lim_{\lambda\to\infty} \sup \lambda^{1/2}\operatorname{dist}(G_{\lambda,B}, \partial B_R) \le \frac{C(F)^{1/2}}{2}, \tag{5.26}$$

since $2^{-1/2}(A+B) = C(F)^{1/2}/2$. This completes the proof of Lemma 5.3. □

Lemma 5.4. *Let $\Omega = A(R_1,R_2) = \{x \in \mathbb{R}^N \mid 0 < R_1 < |x| < R_2\}$ and $u_\lambda(r)$ be the unique large positive solution of* (1.1) *in Ω for λ sufficiently large. If $G_{\lambda,A} = \{x \in A(R_1,R_2) \mid u_\lambda(|x|) = b\}$, then*

$$\lim_{\lambda\to\infty} \inf \lambda^{1/2}\operatorname{dist}(G_{\lambda,A}, \partial A) \ge \frac{C(F)^{1/2}}{2}. \tag{5.27}$$

Proof. Setting

$$\rho = g(r) = \begin{cases} \dfrac{1}{2-N}[r^{2-N} - R_1^{2-N}] & \text{for } N \ge 3, \\ \log\left(\dfrac{r}{R_1}\right) & \text{for } N = 2, \end{cases}$$

and

$$v_\lambda(\rho) = u_\lambda(g^{-1}(\rho)),$$

we can rewrite (1.1) as

$$-v'' = \lambda(g^{-1}(\rho))^{2(N-1)}f(v), \quad 0 < \rho < T, \quad v(0) = v(T) = 0,$$

where $' = d/d\rho$ and $T = g(R_2)$. Since u_λ is the unique large positive solution of (1.1) with $r_1 = R_1$, $r_2 = R_2$, then $v_\lambda(\rho)$ is the unique large positive solution of this problem. Moreover, there exist $0 < \eta_\lambda^1 < \eta_\lambda^2 < T$ such that

$$v_\lambda(\eta_\lambda^1) = v_\lambda(\eta_\lambda^2) = a$$

and $\eta_\lambda^1 = g(r_\lambda^1)$, $\eta_\lambda^2 = g(r_\lambda^2)$, where $R_1 < r_\lambda^1 < r_\lambda^2 < R_2$ such that $u_\lambda(r_\lambda^1) = u_\lambda(r_\lambda^2) = a$. By the arguments similar to that in the proof of Lemma 5.3, we have

$$\lambda^{1/2}(r_\lambda^1 - R_1) \to t^0 = 2^{-1/2}A, \quad \lambda^{1/2}(R_2 - r_\lambda^2) \to t^0 = 2^{-1/2}A \quad \text{as } \lambda \to \infty,$$

where t^0 and A are defined in the proof of Lemma 5.3. Thus, for λ sufficiently large,

$$r_\lambda^1 = R_1 + (2\lambda)^{-1/2}A + o(\lambda^{-1/2}), \quad r_\lambda^2 = R_2 - (2\lambda)^{-1/2}A + o(\lambda^{-1/2}).$$

Then, for $N \geq 3$,

$$\begin{aligned} &\eta_\lambda^1 = g(r_\lambda^1) = (2\lambda)^{-1/2}R_1^{1-N}A + o(\lambda^{-1/2}), \\ &\eta_\lambda^2 = g(r_\lambda^2) = (N-2)^{-1}(R_1^{2-N} - R_2^{2-N}) - (2\lambda)^{-1/2}AR_2^{1-N} + o(\lambda^{-1/2}). \end{aligned}$$

For $N = 2$, we have

$$\begin{aligned} &\eta_\lambda^1 = g(r_\lambda^1) = \log(1 + (2\lambda)^{-1/2}AR_1^{-1} + o(\lambda^{-1/2})) = (2\lambda)^{-1/2}AR_1^{-1} + o(\lambda^{-1/2}), \\ &\eta_\lambda^2 = g(r_\lambda^2) = \log(R_2R_1^{-1}) - (2\lambda)^{-1/2}AR_2^{-1} + o(\lambda^{-1/2}). \end{aligned}$$

(Note that we use Taylor expansions in the calculations.) Thus

$$\begin{aligned} &\eta_\lambda^1 + \eta_\lambda^2 \geq [2(N-2)]^{-1}(R_1^{2-N} - R_2^{2-N}) \quad &&\text{for } N \geq 3 \text{ and } \lambda \text{ large}, \\ &\eta_\lambda^1 + \eta_\lambda^2 \geq 2^{-1}(\log R_2 - \log R_1) &&\text{for } N = 2 \text{ and } \lambda \text{ large}. \end{aligned}$$

Now we consider the problem

$$-v'' = \lambda R_1^{2(N-1)} f(v), \quad \eta_\lambda^1 < \rho < \eta_\lambda^2, \quad v(\eta_\lambda^1) = v(\eta_\lambda^2) = a. \tag{5.28}$$

The arguments similar to that in the proof of Lemma 5.3 imply that, for $\lambda > \lambda_0$ with

$$\lambda_0^{1/2} = \begin{cases} \dfrac{4(N-2)[(2R_1^{2(N-1)})^{-1/2}B + 2^{-1/2}R_1^{1-N}A + 2]}{R_1^{2-N} - R_2^{2-N}} & \text{for } N \geq 2, \\ \dfrac{4[(2R_1^2)^{-1/2}B + 2^{-1/2}R_1^{-1}A + 2]}{\log R_2 - \log R_1} & \text{for } N = 2, \end{cases}$$

(5.28) has a unique solution v_λ^- such that

$$v_\lambda^- \equiv b \quad \text{in } [d(\lambda), (\eta_\lambda^1 + \eta_\lambda^2) - d(\lambda)],$$

where $d(\lambda) := (2\lambda)^{-1/2}R_1^{1-N}B + \eta_\lambda^1 < (\eta_\lambda^1 + \eta_\lambda^2)/2$ and B is defined in the proof of Lemma 5.3.

On the other hand, v_λ^- is a subsolution of the problem

$$-v'' = \lambda(g^{-1}(\rho))^{2(N-1)} f(v), \quad \eta_\lambda^1 < \rho < \eta_\lambda^2, \quad v(\eta_\lambda^1) = v(\eta_\lambda^2) = a. \tag{5.29}$$

Since b is a supersolution of (5.29), we can obtain a positive solution $\overline{v}_\lambda$ of (5.29) in $[v_\lambda^-, b]$ by the arguments similar to that in the proof of Theorem 2.3. It is clear that $\overline{v}_\lambda$ is a large positive solution of (5.29) and hence $\overline{v}_\lambda \equiv v_\lambda$ in $(\eta^1_\lambda, \eta^2_\lambda)$. This implies that

$$a < v^-_\lambda(\rho) \le v_\lambda(\rho), \quad \eta^1_\lambda < \rho < \eta^2_\lambda. \tag{5.30}$$

We know that for $\lambda > \lambda_0$ and sufficiently large,

$$v^-_\lambda(\rho) = b \quad \text{for } \rho \in [d(\lambda), (\eta^1_\lambda + \eta^2_\lambda) - d(\lambda)].$$

Then v_λ has flat core and thus u_λ has flat core.

Let us consider the auxiliary problem

$$-w'' = \lambda(g^{-1}(\theta))^{2(N-1)} f(w), \quad \eta^1_\lambda < \rho < \theta, \quad w(\eta^1_\lambda) = a, \quad w(\theta) = b, \tag{5.31}$$

where θ is again an arbitrary fixed number so that $\eta^1_\lambda < \theta < (\eta^1_\lambda + \eta^2_\lambda)/2$. By the arguments similar to that in the proof of Lemma 5.3, we know that (5.31) exhibits a unique positive solution $w = \underline{w}_\lambda(\rho, \theta)$ for λ large enough. Moreover, if $\rho(\lambda) := (2\lambda)^{-1/2}(g^{-1}(\theta))^{1-N} B + \eta^1_\lambda$, then $\underline{w}_\lambda = b$ for $\rho(\lambda) \le \rho \le \theta$, while $a < \underline{w}_\lambda < b$ in $\eta^1_\lambda < \rho < \rho(\lambda)$. On the other hand, since v_λ solves (5.29) in $\eta^1_\lambda < \rho < \eta^2_\lambda$ then it defines a subsolution to (5.31) in $\eta^1_\lambda < \rho < \theta$. Since b is a supersolution of (5.31), we can use the arguments similar to that in the proof of Theorem 2.3 to obtain that there exists a solution w_λ of (5.31) between v_λ and b. The uniqueness of $\underline{w}_\lambda$ implies that $w_\lambda \equiv \underline{w}_\lambda$ in $[\eta^1_\lambda, \theta]$. Thus,

$$a < v_\lambda(\rho) \le \underline{w}_\lambda(\rho, \theta) \le b \quad \text{for } \eta^1_\lambda < \rho < \theta. \tag{5.32}$$

(5.32) implies that

$$u_\lambda(r) \le \underline{w}_\lambda(g(r), \theta) < b$$

provided that $r \in A(R_1, R_2)$ and $R_1 < r < g^{-1}(\rho(\lambda))$. This means that

$$\operatorname{dist}(G_{\lambda,A}, \Gamma_1) \ge g^{-1}(\rho(\lambda)) - R_1, \tag{5.33}$$

where $\Gamma_1 = \{x \in \partial A \mid |x| = R_1\}$. Observing that

$$\begin{aligned} g^{-1}(\rho(\lambda)) - R_1 &= [(2-N)\rho(\lambda) + R_1^{2-N}]^{1/(2-N)} - R_1 \\ &= R_1^{N-1}\rho(\lambda) + o(\rho(\lambda)) \\ &= R_1^{N-1}[(2\lambda)^{-1/2}(g^{-1}(\theta))^{1-N} B \\ &\quad + (2\lambda)^{-1/2} R_1^{1-N} A + o(\lambda^{-1/2})] + o(\rho(\lambda)) \\ &= (2\lambda)^{-1/2}\left(\frac{R_1}{g^{-1}(\theta)}\right)^{N-1} B + (2\lambda)^{-1/2} A + o(\lambda^{-1/2}), \end{aligned}$$

for λ sufficiently large, we conclude from (5.33) that

$$\liminf_{\lambda\to\infty} \lambda^{1/2} \operatorname{dist}(G_{\lambda,A}, \Gamma_1) \ge 2^{-1/2}\left(\frac{R_1}{g^{-1}(\theta)}\right)^{N-1} B + 2^{-1/2} A.$$

Such estimate rapidly leads, by letting $\theta \to 0^+$, to the desired result. Namely,

$$\liminf_{\lambda\to\infty} \lambda^{1/2}\mathrm{dist}(G_{\lambda,A},\Gamma_1) \geq \frac{C(F)^{1/2}}{2}.$$

We can use the same idea to claim that

$$\liminf_{\lambda\to\infty} \lambda^{1/2}\mathrm{dist}(G_{\lambda,A},\Gamma_2) \geq \frac{C(F)^{1/2}}{2},$$

where $\Gamma_2 = \{x \in \partial A \mid |x| = R_2\}$. In fact, considering the auxiliary problem

$$-w'' = \lambda(g^{-1}(T))^{2(N-1)}f(w), \quad \theta < \rho < \eta^2_\lambda, \quad w(\theta) = b, \quad w(\eta^2_\lambda) = a,$$

where θ is an arbitrary fixed number so that $(\eta^1_\lambda + \eta^2_\lambda)/2 < \theta < \eta^2_\lambda$, we have that this problem has a unique positive solution $w = \underline{w}_\lambda(\rho,\theta)$ for λ large enough. Moreover, if $\rho(\lambda) := \eta^2_\lambda - (2\lambda)^{-1/2}(g^{-1}(T))^{1-N}B$, then $\underline{w}_\lambda = b$ for $\theta \leq \rho \leq \rho(\lambda)$, while $a < \underline{w}_\lambda(\rho) < b$ for $\rho(\lambda) < \rho < \eta^2_\lambda$. The same arguments as the above imply that

$$\mathrm{dist}(G_{\lambda,A},\Gamma_2) \geq R_2 - g^{-1}(\rho(\lambda)).$$

Our claim can be obtained by simple calculations. (Note that we need to use the formulae of η^2_λ for $N \geq 3$ and $N = 2$ given above respectively in the calculations. Moreover, we know that $\eta^2_\lambda \to T$ as $\lambda \to \infty$ and $g^{-1}(T) \to R_2$.) □

Proof of Theorem 5.1. For any $x_0 \in \partial\Omega$ and a ball B being chosen to be tangent to $\partial\Omega$ at x_0 and $B \subset \Omega$, we consider the problem

$$-\Delta z = \lambda f(z) \quad \text{in } B, \quad z = 0 \quad \text{on } \partial B. \tag{5.34}$$

The arguments similar to that in the proof of Corollary 4.3 imply that (5.34) has a unique large positive (radial) solution z_λ for λ sufficiently large. Lemma 5.3 implies that for λ sufficiently large, flat core $G_{\lambda,B}$ of z_λ exists. On the other hand, we know that z_λ is a subsolution of (1.1) by extending it to be 0 on $\Omega \setminus B$, b is a supersolution of (1.1). By the arguments similar to that in the proof of Theorem 2.3, we obtain a positive solution $u_\lambda \in [z_\lambda, b]$ of (1.1). It is clear that u_λ is the unique large positive solution of (1.1). Therefore, $u_\lambda \geq z_\lambda$ in Ω and

$$\mathrm{dist}(x_0, G_\lambda) \leq \mathrm{dist}(x_0, G_{\lambda,B}).$$

Since $\mathrm{dist}(x_0, G_{\lambda,B}) = \mathrm{dist}(G_{\lambda,B}, \partial B)$, thus Lemma 5.3 implies

$$\limsup_{\lambda\to\infty} \lambda^{1/2}\mathrm{dist}(x_0, G_\lambda) \leq \frac{C(F)^{1/2}}{2}. \tag{5.35}$$

This implies

$$\limsup_{\lambda\to\infty} \lambda^{1/2} \max_{x\in\partial\Omega} \mathrm{dist}(x, G_\lambda) \leq \frac{C(F)^{1/2}}{2}. \tag{5.36}$$

To get the estimate of $\lim_{\lambda\to\infty} \inf \lambda^{1/2} \min_{x\in\partial\Omega} \operatorname{dist}(x, G_\lambda)$, we construct an annulus $A_e = \{x \in \mathbb{R}^N \mid \widehat{a} < |x - y_e| < R_e\}$ with $y_e \in \mathbb{R}^N$ such that $\Omega \subset A_e$ and A_e tangent to $\partial\Omega$ at x_0. Now we consider the problem

$$-\Delta z = \lambda f(z) \quad \text{in } A_e, \quad z = 0 \quad \text{on } \partial A_e. \tag{5.37}$$

Corollary 4.3 implies that (5.37) has a unique large positive (radial) solution z_λ for λ sufficiently large. Lemma 5.4 implies that for λ sufficiently large, flat core G_{λ,A_e} of z_λ exists. By extending u_λ to be 0 in $A_e \setminus \Omega$, we easily know that u_λ is a subsolution of (5.37). Moreover, b is a supersolution of (5.37). Thus the arguments similar to that in the proof of Theorem 2.3 imply that there exists a positive solution of (5.37) in $[u_\lambda, b]$. It is clear that this solution is the unique positive large solution z_λ of (5.37). Thus $u_\lambda \le z_\lambda$ in Ω for λ sufficiently large. Thus,

$$\operatorname{dist}(x_0, G_{\lambda,A_e}) \le \operatorname{dist}(x_0, G_\lambda).$$

Moreover, Lemma 5.4 implies

$$\lim_{\lambda\to\infty} \inf \lambda^{1/2} \operatorname{dist}(x_0, G_\lambda) \ge \frac{C(F)^{1/2}}{2}. \tag{5.38}$$

This also implies that

$$\lim_{\lambda\to\infty} \inf \lambda^{1/2} \min_{x\in\partial\Omega} \operatorname{dist}(x, G_\lambda) \ge \frac{C(F)^{1/2}}{2}. \tag{5.39}$$

Now our conclusions of Theorem 5.1 can be easily obtained from (5.36) and (5.39). □

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