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MORSE HOMOLOGY AND DEGENERATE MORSE INEQUALITITIES

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Abstract. Based on Morse homology of Morse functions, we give a new proof of the Morse–Bott inequalities for functions with non-degenerate critical manifolds. A proof of the Morse inequalities for functions with isolated critical points as developed by Gromoll–Meyer is also presented with the same method.

Introduction

Let M be an n-dimensional smooth closed manifold, i.e. M is compact and without boundary and let f be a Morse function on M, i.e. f is smooth and each critical point of f is nondegenerate. Point $x \in M$ is called a critical point of f if $df(x) = 0$. It is called nondegenerate if at the local coordinate $(x_1, \ldots , x_n)$ at x, the Hessian $(\partial^2 f/\partial x_i \partial x_j)$ is nondegenerate. The number $i(x)$ of negative eigenvalues of $(\partial^2 f/\partial x_i \partial x_j)$ counted with multiplicity is called Morse index of x. The classical Morse theory relates numbers of critical points of a Morse function f to the topology of M. More precisely, let $m_k = \#\{x \mid df(x) = 0,\ i(x) = k\}$, b_k be the kth Betti number of M. The Morse polynomial of f and the Poincaré polynomial of M are defined by $M(t, f) = \sum m_k t^k$ and $P(t, M) = \sum b_k t^k$, respectively. With these notions, the Morse inequalities can be written in the following form.

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THEOREM 0.1. *Let M be a closed manifold and f be a smooth Morse function on M. Then there is a polynomial $Q(t)$ with nonnegative integer as its coefficients such that*

$$M(t,f) = P(t,M) + (1+t)Q(t).$$

In particular, this implies

$$m_k \geq \beta_k, \quad k = 1, \dots, n,$$

$$m_0 - m_1 + \dots + (-1)^n m_n = b_0 - b_1 + \dots + (-1)^n b_n = \chi(M).$$

There are many ways to prove this theorem, a standard reference is [17]. There is a proof based on the connecting trajectories of gradient flow of f between critical points of f. It can be described as follows. We fix a Riemannian metric on M and consider the gradient flow of f

$$u'(s) + f'(u) = 0. \tag{0.1}$$

It is well known that each solution u of (0.1) satisfies $\int_{-\infty}^{\infty} |u'(s)|^2\, ds < \infty$ and both of the following limits $\lim_{s\to-\infty} u(s) = x_1$ and $\lim_{s\to\infty} u(s) = x_2$ exist for some critical points x_1, x_2 of f. For fixed critical points x_1, x_2 of f, let

$$\mathcal{M}(x_1,x_2) = \{u \text{ satisfies (0.1) } \lim_{s\to-\infty} u(s) = x_1 \text{ and } \lim_{s\to\infty} u(s) = x_2\}.$$

It is well known that for generic f (or generic metric g on M), $\mathcal{M}(x_1,x_2)$ is a smooth manifold with a coherent orientation, and $\dim \mathcal{M}(x_1,x_2) = i(x_2) - i(x_1)$. $\mathcal{M}(x_1,x_2)$ is translation invariant, i.e. if $u \in \mathcal{M}(x_1,x_2)$, then for any $s \in \mathbb{R}$, $u(s+\cdot\,) \in \mathcal{M}(x_1,x_2)$. Hence $\dim \mathcal{M}(x_1,x_2) = 1$ if $i(x_2) - i(x_1) = 1$, thus $\mathcal{M}(x_1,x_2)/\mathbb{R}$ is of dimension 0. It can be shown that in this case, $\mathcal{M}(x_1,x_2)/\mathbb{R}$ is compact, so it is a finite set. We denote by $n(x_1,x_2)$ the algebraic number of this set according to the coherent orientation. Having these data, one can construct the Morse complex as follows. Let

$$K(f) = \{x \in M,\ df(x) = 0\}, \quad C_k(f) = \bigoplus_{\substack{x\in K(f),\\ i(x)=k}} \mathbb{Z}\langle x\rangle,$$

$$\partial_k : C_k(f) \to C_{k-1}(f), \quad \partial_k\langle x\rangle = \sum_{\substack{y\in K(f),\\ i(y)=k-1}} n(y,x)\langle y\rangle.$$

Set $(\mathcal{C}(f),\partial) = \bigoplus_{k=1}^n (C_k(f),\partial_k)$, then $\partial_{k-1}\circ\partial_k = 0$, hence $(\mathcal{C}(f),\partial)$ is a complex. Following [21], we call this complex Morse complex and its homology the Morse homology. A detail proof of the following theorem can be found in [21].

Theorem 0.2. *Let $f \in C^2(M)$ be a Morse function such that the Morse homology of f is well defined. Then, for each integer k, $H_k(\mathcal{C}(f), \partial) = H_k(M, \mathbb{Z})$.*

The Morse inequalities are simple consequences of this theorem. Morse theory from this point of view plays a crucial role in the Floer theory. Floer used this method to construct his infinite dimensional version of Morse theory, now known as Floer homology theory in symplectic geometry and gauge theory. They are very important tools in symplectic geometry and low dimensional topology, see [12]–[14]. Another approach to prove the Morse inequalities which is closely related to the above method was proposed by Witten in [22] based on a deformation of the De Rham complex. In fact, Witten's paper was one of main motivations for the Floer's theory.

Followings are some generalizations of the Morse theory: Morse–Bott theory for non-degenerate critical manifolds [3], Gromoll–Meyer theory for isolated critical points [16], Morse theory for manifold with boundary, and Morse–Novikov theory for multi-valued functions [18], [19]. Proofs using the Witten's method for Morse–Bott inequalities, for Morse inequalities with boundary, for Morse–Novikov theory for multi-valued functions and its equivariant version have been given in [2], [5], [6], [9], respectively. The aim of this paper is to give a proof for Morse–Bott inequalities, Gromoll–Meyer theory and Morse theory for manifold with boundary using Theorem 0.2 and a perturbation technique. The key points that we will use are relative and local versions of the Morse homology, we use them to construct a filtration of the Morse complex. The proof is finished by spectral sequence from algebraic topology. In [1], one can find a proof of Morse–Bott inequalities and equivariant Morse inequalities by connecting trajectories method without perturbation. Our approach here is simpler.

1. Morse–Bott inequalities

Definition 1.1. Let f be a smooth function on M. A submanifold $N \subset M$ is called a non-degenerate critical submanifold of f if

(1) for all $x \in N$, $df(x) = 0$,
(2) $(\partial^2 f/\partial x_i \partial x_j)$ is non-degenerate along the normal bundle of N in M.

The number of negative eigenvalues of $(\partial^2 f/\partial x_i \partial x_j)(x)$, $x \in N$ along the normal bundle of N is called the Morse index at x, denote it by $i(x)$. We assume that N is connected, thus $i(x)$ is a constant for $x \in N$, it is called the Morse index of critical manifold N. We denote $i(N) = i(x)$, $x \in N$. Following are the well known Morse–Bott inequalities.

Theorem 1.2. *Let f be a smooth function on a closed manifold M, $K(f) = \{x \in M,\ df(x) = 0\}$. Assume that $K(f) = \bigcup N_j$ and each N_j is a non-degenerate critical manifold. Then there is a polynomial $Q(t)$ with a non-negative* *integer as its coefficients such that*

$$\sum_{N_j\subset K(f)} P(t,N_j)t^{i(N_j)} = P(t,M) + (1+t)Q(t).$$

The aim of this section is to present a proof of this theorem using the connecting trajectories of gradient flow. It is based on Theorem 0.2 and a perturbation of function f. Let $c_1 < c_2 < \dots < c_l$ be the critical values of f, that is for each c_i, there is an $x \in K(f)$, $f(x) = c_i$. We fix an ε_0 such that $0 < \varepsilon_0 < \min_j(c_{j+1} - c_j)$. Set $b_j = c_j + \varepsilon_0/2$, $1 \le j \le l$, $b_0 = c_1 - \varepsilon_0/2$. For any $\varepsilon > 0$, take a $g \in C^2(M)$ satisfying the conditions of Theorem 0.2 and $\|g - f\|_{C^2(M)} \le \varepsilon/2$. It is easy to see, for small ε,

$$dg(x) \neq 0 \text{ if } g(x) = b_j,\ 1 \le j \le l.$$

Let $(\mathcal{C}(g), \partial)$ be the Morse complex of g. For $j = 1, \dots, l$, we set

$$C_k(g,b_j) = \bigoplus_{\substack{x\in K(g)\\ g(x)<b_j\\ i(x)=k}} \mathbb{Z}\langle x\rangle, \qquad \mathcal{C}_j(g) = \bigoplus_k C_k(g,b_j),$$

then $(\mathcal{C}_j(g), \partial)$ is a subcomplex of $(\mathcal{C}(g), \partial)$, i.e. $\partial : \mathcal{C}_j(g) \to \mathcal{C}_j(g)$ and

$$(\mathcal{C}_1(g),\partial) \subset (\mathcal{C}_2(g),\partial) \subset \ldots \subset (\mathcal{C}_l(g),\partial) = (\mathcal{C}(g),\partial),$$

form a filtration of $(\mathcal{C}(g), \partial)$. Let $(\bigoplus_k C_k(g,b_j)/C_k(g,b_{j-1}), \partial)$ be the reduced relative complex. The following lemma is easy to prove.

LEMMA 1.3. *For $1 \le j \le l$, let*

$$C_k(g,b_{j-1},b_j) = \bigoplus_{\substack{x\in K(g)\\ i(x)=k\\ b_{j-1}<g(x)<b_j}} \mathbb{Z}\langle x\rangle,$$
$$\partial' : C_k(g,b_{j-1},b_j) \to C_{k-1}(g,b_{j-1},b_j),$$
$$\partial'\langle x\rangle = \sum_{\substack{y\in K(g)\\ i(y)=k-1\\ b_{j-1}<g(y)<b_j}} n(y,x)\langle y\rangle.$$

Then $\partial' \circ \partial' = 0$ and

$$\Big(\bigoplus_k \frac{C_k(g,b_{j+1})}{C_k(g,b_j)}, \partial\Big) \cong \Big(\bigoplus_k C_k(g,b_j,b_{j+1}), \partial'\Big).$$

In what follows we denote

$$(\mathcal{C}(g,b_{j-1},b_j),\partial) = \Big(\bigoplus_k C_k(g,b_{j-1},b_j),\partial'\Big) = \Big(\bigoplus_k \frac{C_k(g,b_j)}{C_k(g,b_{j-1})},\partial\Big).$$

We will use the following filtration

$$0 \subset (\mathcal{C}_1(g),\partial) \subset \ldots \subset (\mathcal{C}_l(g),\partial) = (\mathcal{C}(g),\partial),$$

of $(\mathcal{C}(g),\partial)$ to compute the Morse homology of $H_*(\mathcal{C}(g),\partial)$ by spectral sequence theory. For our purpose, we only need the homology of first and second term of this filtration. For the spectral sequence theory of a filtration of a complex, we refer to [4]. From the short exact sequence of complex

$$0 \to \bigoplus_j (\mathcal{C}_j(g),\partial) \xrightarrow{\ i\ } \bigoplus_j (\mathcal{C}_{j+1}(g)\,\partial) \xrightarrow{\ j\ } \bigoplus_j (\mathcal{C}(g,b_j,b_{j+1}),\partial) \longrightarrow 0,$$

we have the following long exact sequence

$$\cdots \to H_{k+1}(\mathcal{A}) \xrightarrow{i_1} H_{k+1}(\mathcal{A}) \xrightarrow{j_1} H_{k+1}(\mathcal{B}) \xrightarrow{k_1} H_k(\mathcal{A}) \longrightarrow \cdots$$

of homology groups with $\mathcal{A} = \bigoplus_j (\mathcal{C}_j(g),\partial)$, $\mathcal{B} = \bigoplus_j (\mathcal{C}(g,b_j,b_{j+1}),\partial)$. We write the above exact sequence simply as

$$H(\mathcal{A}) \xrightarrow{i_1} H(\mathcal{A}) \xrightarrow{j_1} H(\mathcal{B}) \xrightarrow{k_1} H(\mathcal{A}).$$

Let $E_1 = H(\mathcal{B})$, $\delta_1 = j_1 \circ k_1 : H(\mathcal{B}) \to H(\mathcal{B})$, this is a boundary operator, hence we have the homology $E_2 = H(E_1,\delta_1)$ and a boundary operator $\delta_2 : H(E_2) \to H(E_2)$ and so on. Continuing this process, by spectral sequence theory, we conclude, after finite steps,

$$H_*(M) = H_*(\mathcal{C}(g),\partial) = H_*(E_m,\delta_m) = H_*(E_{m+1},\delta_{m+1}).$$

The following proposition is a relative version of Morse homology which is fundamental to our proof.

Proposition 1.4. *For small $\varepsilon > 0$, $0 \le j \le l-1$,*

$$H_*(\mathcal{C}(g,b_j,b_{j+1}),\partial) = \bigoplus_{\substack{N\subset K(f) \mid f(x)=c_j \\ x\in N}} H_{*-i(N)}(N),$$

hence

$$E_1 = H_*(\mathcal{B}) = \bigoplus_{N\subset K(f)} H_{*-i(N)}(N),$$

$$\sum \dim H_k(\mathcal{C}(g,b_j,b_{j+1}),\partial)t^k = \sum_{\substack{N\subset K(f) \mid f(x)=c_j \\ x\in N}} P(t,N)t^{i(N)}.$$

Remark. In fact, following the proof in next sectiom, we can show

$$H_*(\mathcal{C}(g,b_j,b_{j+1}),\partial) = H_*(f_{b_{j+1}}, f_{b_j}),$$

with $f_b = \{x \in M \mid f(x) \le b\}$, we will not use this fact later.

Lemma 1.5. *Let* $X_0 \xrightarrow{\partial_0} X_1 \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{n-1}} X_n$ *be a complex, i.e.* $\partial_k \circ \partial_{k-1} = 0$, $H_*(X)$ *be the homology. Then there is a polynomial* $Q(t)$ *with non-negative integer as its coefficients such that*

$$\sum \dim X_i t^i = \sum \dim H_i(X) t^i + (1+t) Q(t).$$

Proof. It is an easy exercise in linear algebra. □

With Proposition 1.4 in hand, now we can prove Theorem 2.2.

Proof of Theorem 1.2. Repeatedly use Lemma 1.5, there are polynomials $Q_j(t)$, its coefficients are nonnegative integers, $j = 1, \ldots, m-1$ such that

$$\begin{aligned}
\sum_{N \subset K(f)} P(t, N) t^{i(N)} &= \sum_{N \subset K(f)} \dim H_i(N) t^{i+i(N)} \\
&= \sum_i \dim H_i(E_1, \delta_1) t^i \\
&= \sum_i \dim H_i(E_2, \delta_2) t^i + (1+t) Q_1(t) \\
&= \sum_i \dim H_i(E_3, \delta_3) t^i + (1+t)(Q_1(t) + Q_2(t)) \\
&= \cdots \\
&= \sum_i \dim H_i(E_m, \delta_m) t^i + (1+t)(Q_1(t) \ldots + Q_{m-1}(t)) \\
&= \sum_i \dim H_i(\mathcal{C}(g), \partial) t^i + (1+t)(Q_1(t) \ldots + Q_{m-1}(t)).
\end{aligned}$$

By Theorem 0.2, $H_*(\mathcal{C}(g), \partial) = H_*(M)$, therefore

$$\sum_{N \subset K(f)} P(t, N) t^{i(N)} = P(t, M) + (1+t) Q(t),$$

with $Q(t) = Q_1(t) + \ldots + Q_{m-1}(t)$. This ends the proof. □

The remaining of this section is devoted to the proof of Proposition 1.4.

Lemma 1.6. *Let* g *be a Morse function on* M *satisfying the following conditions:*

(1) $|f - g| < \varepsilon$ *for small* ε,

(2) $\mathcal{M}(x, y)$ *is a manifold for any* $x, y \in K(g)$ *with* $b_j < g(x)$, $g(y) < b_{j+1}$.

Then $H_*(\mathcal{C}(g, b_{j-1}, b_j), \partial)$ *is well defined, and for any such functions* g_1 *and* g_2,

$$H_*(\mathcal{C}(g_1, b_{j-1}, b_j), \partial) \cong H_*(\mathcal{C}(g_2, b_{j-1}, b_j), \partial).$$

The proof is similar to that of the Morse homology and is independent of choices of the Morse functions, see [21], we omit it here. The key point in the lemma is that we only make assumptions on critical points located in set

$\{x \mid b_j < f(x) < b_{j=1}\}$, thus we can deform the function g in order to compute the homology. For simplicity of notation, we fix a critical value c_j of f, and set

$$c = c_j, \quad a = b_j, \quad b = b_{j+1}, \quad K_c(f) = \{x \in K(f) \mid f(x) = c\} = \bigcup_k N_k.$$

We construct a function g as follows. Let $\varepsilon > 0$ be a constant to be fixed later, $\rho_k(x)$ be the distance function from x to N_k which is defined in a neighbourhood of N_k, and let η_0 be a small positive number such that $\eta_0 < \min_{i \neq j} \operatorname{dist}(N_i, N_j)/4$. For each k, we take a cut off function

$$\eta_k(x) = \begin{cases} 1 & \text{if } \rho_k(x) \leq \eta_0, \\ 0 & \text{if } \rho_k(x) \geq 2\eta_0, \end{cases}$$

and a Morse function $f_k(x)$ on N_k satisfying the conditions of Theorem 0.2 for manifold N_k. Finally, set

$$g_\varepsilon(x) = f(x) + \varepsilon \cdot \sum_k \eta_k(x) f_k(P_k x), \tag{1.1}$$

with P_k being the projection from a neighbourhood of N_k to N_k. The following lemmas characterize the critical points of g_ε in set $\{x \mid a < g_\varepsilon(x) < b\}$ and connecting orbits of those critical points of the gradient flow of g_ε.

Lemma 1.7. *There is an $\varepsilon_1 > 0$ such that for $0 < \varepsilon < \varepsilon_1$, g_ε is a Morse function and*

$$K(g_\varepsilon, a, b) = \{x \in K(g_\varepsilon) \mid a < g_\varepsilon(x) < b\} = \bigcup_k \{x \in N_k \mid df_k(x) = 0\},$$

and $i(g_\varepsilon, x) = i(N_k) + i(f_k, x)$ if $x \in N_k$, $df_k(x) = 0$, where $i(f_k, x)$ is the Morse index of f_k on N_k.

Proof. First, there is a $d_0 > 0$ such that

$$|df(x)| \geq d_0 \quad \text{if for each } k,\ \rho_k(x) \geq \eta_0.$$

So for small ε, we have

$$|dg_\varepsilon(x)| \geq d_0/2 \quad \text{if, for each } k,\ \rho_k(x) \geq \eta_0. \tag{1.2}$$

If $\rho_{k_1}(x) < \eta_0$, then $\eta_{k_1}(x) \equiv 1$, $\eta_k(x) \equiv 0$, $k \neq k_1$. Hence

$$g_\varepsilon(x) = f(x) + \varepsilon f_{k_1}(P_{k_1} x).$$

By Morse lemma, we can assume $g_\varepsilon(x) = (q(x_1)x_2, x_2)/2 + \varepsilon f_{k_1}(P_{k_1} x)$, where $x = (x_1, x_2)$, $x_1 \in N_{k_1}$, x_2 is the coordinate of the fiber of normal bundle of N_{k_1}, $q_{x_1}(x_2)$ is a nondegenerate quadratic form in x_2. $\partial g_\varepsilon / \partial x_2(x) = 0$ implies $x_2 = 0$, and $\partial g_\varepsilon / \partial x_1(x) = 0$ implies $df_{k_1}(x_1) = 0$. The index formula follows from definition immediately. $\square$

LEMMA 1.8. *There is an $\varepsilon_2 > 0$ such that, for $0 < \varepsilon < \varepsilon_2$, $x, y \in K(g_\varepsilon, a, b)$, we have*

(1) *If $x, y \in N_k$, and $\gamma(t)$ is a solution of*

$$\dot\gamma(t) = -g'_\varepsilon(\gamma(t)), \quad \lim_{t\to-\infty} \gamma(t) = x, \quad \lim_{t\to\infty} \gamma(t) = y, \tag{1.3}$$

then $\rho_k(\gamma(t)) \le \eta_0$.

(2) *If $x \in N_{k_1}$, $y \in N_{k_2}$, and $k_1 \ne k_2$, then there is no solution $\gamma(t)$ of (1.3).*

PROOF. (1) Assume that the conclusion is not true, then there is a sequence $\varepsilon_l \to 0$ as $l \to \infty$, and for each l, we have a solution $\gamma_l(t)$ of (1.3), and $t_l \in \mathbb{R}$ such that $\eta_0 \le \rho_k(\gamma(t_l)) \le 2\eta_0$. Since (1.3) is translation invariant and M is compact, we can assume that $t_l = 0$ for each l and $\gamma_l(t_l) = x_l \to x_0$ as $l \to \infty$. Then $\gamma_l \to \gamma$ $C^2_{\rm loc}(\mathbb{R})$, where $\gamma(t)$ is the solution of

$$\dot\gamma(t) = -f'(\gamma(t)), \quad \gamma(0) = x_0.$$

Since $df(x_0) \ne 0$, $\gamma(t)$ is not a constant solution, hence there are $x_1, y_1 \in K(f)$ such that

$$\lim_{t\to-\infty} \gamma(t) = x_1, \quad \lim_{t\to\infty} \gamma(t) = y_1,$$

with

$$f(x_1) \ge f(x) \quad \text{and} \quad f(y_1) \le f(y).$$

Therefore

$$0 < f(y_1) - f(x_1) \le f(y) - f(x).$$

This is impossible since $f(x) = f(y)$ and $x \ne y$.

(2) Assume that there is a solution $\gamma(t)$ of (1.3) such that $x \in N_{k_1}$, $y \in N_{k_2}$. Let $T_1 = \max\{t \mid \rho_{k_1}(\gamma(t)) \le \eta_0\}$, $T_2 = \min\{t \mid \rho_{k_2}(\gamma(t)) \le \eta_0\}$, then

$$\begin{aligned} g_\varepsilon(x) - g_\varepsilon(y) &= -\int_{-\infty}^{\infty} \frac{dg_\varepsilon(\gamma(t))}{dt}\, dt = -\int_{-\infty}^{\infty} (g'_\varepsilon(\gamma(t)), \dot\gamma(t))\, dt \\ &\ge \int_{T_1}^{T_2} |g'_\varepsilon(\gamma(t))|^2\, dt \ge \frac{{d_0}^2}{4}(T_2 - T_1). \end{aligned}$$

On the other hand,

$$C(T_2 - T_1) \ge \int_{T_1}^{T_2} |\gamma'(t)|\, dt \ge \eta_0,$$

where $C = \max|f'(x)| + 1$. Hence

$$g_\varepsilon(x) - g_\varepsilon(y) \ge \frac{d_0^2}{4C}\eta_0, \tag{1.4}$$

which is independent of ε. Since $f(x) = f(y)$, we have

$$g_\varepsilon(x) - g_\varepsilon(y) = \varepsilon \cdot (f_{k_1}(x) - f_{k_2}(y)),$$

contradicting (1.4). $\square$

REMARK. Above arguments in fact prove the following stronger result: assume same condition as in (1), then

$$\lim_{\varepsilon\to 0}\max_{t\in\mathbb{R}}\rho_k(\gamma(t)) = 0,$$

uniformly for solution $\gamma(t)$ of (1.3).

LEMMA 1.9. *Let $x, y \in N_k$ and $\gamma(t)$ be a solution of* (1.3)*, then for small ε, we have $\rho_k(\gamma(t)) \equiv 0$, hence $\gamma(t)$ is a solution of*

$$\dot\gamma(t) = -\varepsilon f_k'(\gamma(t)), \quad \lim_{t\to-\infty}\gamma(t) = x, \quad \lim_{t\to\infty}\gamma(t) = y.$$

This lemma can be considered as a finite dimensional version of Lemma 5.2 in [15], the proof is similar.

PROOF. By Lemma 1.8, we know that if $0 < \varepsilon < \varepsilon_2$, then for any solution $\gamma(t)$ of (1.3), we have $\rho_k(\gamma(t)) \le \eta_0$, $t \in \mathbb{R}$. By the Morse lemma, we can assume that

$$g_\varepsilon(x) = \frac{1}{2}(q(x_1)x_2, x_2) + \varepsilon f_k(P_k x),$$

$q(x_1)$ is a symmetric matrix. Set $\gamma(t) = (x_1(t), x_2(t))$ and $h(t) = |x_2(\gamma(t))|^2$, we claim that for ε small, there is a constant $c > 0$ such that

$$\ddot h(t) \ge c\cdot h(t), \quad t\in\mathbb{R}. \tag{1.5}$$

Indeed, we have

$$\ddot h(t) = \frac{d}{dt}\langle x_2(t), \dot x_2(t)\rangle = |\dot x_2(t)|^2 + \langle x_2(t), \ddot x_2(t)\rangle. \tag{1.6}$$

Since $-\dot\gamma(t) = g_\varepsilon'(\gamma(t))$, we have

$$\begin{aligned}
-\dot x_1(t) &= \frac{1}{2}(q'(x_1)x_2, x_2) + \varepsilon f'(x_1) + o(|x_2|),\\
-\dot x_2(t) &= q(x_1)x_2 + o(|x_2|),\\
-\ddot x_2(t) &= q(x_1)\dot x_2(t) + q'(x_1)\dot x_1 x_2 + o(1)\dot x_2.
\end{aligned}$$

Substituting this into (1.6), noting that $\max_t |x_2(t)| \to 0$ as $\varepsilon \to 0$, we have

$$\ddot h(t) \ge 2|q(x_1)x_2|^2 - \varepsilon\max|f_j'(x_1)||x_2|^2 + o(|x_2|^2) \ge c\cdot h(t),$$

if ε is small. As $\lim_{t\to-\infty} h(t) = \lim_{t\to\infty} h(t) = 0$, there is a $t_0 \in \mathbb{R}$ such that $h(t_0) = \max_{t\in\mathbb{R}} h(t)$, hence $\ddot h(t_0) \le 0$. By (1.5), we conclude that $0 \le h(t_0) = \max_t h(t) \le 0$, that is $h(t) \equiv 0$. $\square$

PROOF OF PROPOSITION 1.4. Let g_ε be defined by (1.1). By Lemmas 1.7–1.9, for small ε, $H_*(\mathcal{C}(g_\varepsilon, b_{j-1}, b_j), \partial)$ is well defined. Lemma 1.6 implies

$$H_*(\mathcal{C}(g, b_{j-1}, b_j), \partial) \cong H_*(\mathcal{C}(g_\varepsilon, b_{j-1}, b_j), \partial).$$

Now we compute $H_*(\mathcal{C}(g_\varepsilon, b_{j-1}, b_j), \partial)$. For $x, y \in K(g_\varepsilon, b_{j-1}, b_j)$, by Lemma 1.9, either

$$\mathcal{M}(x, y) = \{\gamma \mid \dot\gamma = -\varepsilon f_k'(\gamma)\} \quad \text{if } x, y \in K(f_k),$$

or

$$\mathcal{M}(x, y) = \emptyset \quad \text{if } x \in K(f_k),\ y \in K(f_{k_1}),\ k \neq k_1.$$

Let $\mathcal{C}(\varepsilon f_k, N_k, \partial_k)$ be the Morse complex of εf_k on N_k, then we have

$$\begin{aligned} H_*(\mathcal{C}(g_\varepsilon, b_{j-1}, b_j), \partial) &\cong \bigoplus_{N_k \subset K(f_{c_j})} H_{*-i(N_k)}(\mathcal{C}(\varepsilon f_k, N_k, \partial)) \\ &= \bigoplus_{N_k \subset K(f_{c_j})} H_{*-i(N_k)}(N_k), \end{aligned}$$

this is Proposition 1.4. □

2. The Gromoll–Meyer theory

In this section, we deal with Morse theory for functions with isolated critical points as developed by Gromoll and Meyer [16]. First we recall some definitions. Let $f \in C^2(M)$, $x \in M$ be a critical point of f which is isolated, that is there is a neighbourhood U of x in M such that x is the only critical point of f in U. Clearly a non-degenerate critical point is isolated. In order to describe the local behavior of an isolated critical point of f, we need the notion of Gromoll–Meyer pair. Let $\gamma(t, x_0)$ be the solution of

$$\dot\gamma(t) = -f'(\gamma(t)), \gamma(0) = x_0. \tag{2.1}$$

Definition 2.1. Let $f \in C^2(M)$ and $x \in M$ be an isolated critical point of f. A pair of subsets (W, W_-) is called a (G-M) pair of x if the following conditions hold:

(1) W is a closed neighbourhood of x with the mean value property w.r.t. $\gamma(t, x)$, that is, if there are $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$, $\gamma(t_1, x), \gamma(t_2, x) \in W$, then $\gamma(t) \in W$ for $t \in (t_1, t_2)$.
(2) W_- is an exit set of W, i.e. for all $x_0 \in W$ and all $t_1 > 0$ such that if $\gamma(t_1, x_0) \notin W$, then there exists $t_0 \in [0, t_1)$ satisfying $\gamma([0, t_0), x_0) \subset W$ and $\gamma(t_0, x_0) \in W_-$.
(3) W_- is closed and is a union of a finite number of submanifolds that are transversal to the flow γ.

Let (W, W_-) be a (G-M) pair of x, the homology groups $H_*(W, W_-)$ are called the critical groups of x. For an isolated critical point x of f, there always exists a (G-M) pair (W, W_-) and $H_*(W, W_-)$ is independent of the choices of (G-M) pairs (W, W_-), denote it by $C_*(f, x)$, see [7]. Let $m_i(f, x) = \operatorname{rank} C_*(f, x)$, it is called Morse-type number of x, and $P(t, f, x) = \sum_i m_i(t, x) t^i$.

Theorem 2.2. *Let M be a closed manifold and $f \in C^2(M)$ such that each critical point of f is isolated. Then there exists a polynomial $Q(t)$ with non-negative integer as its coefficients such that*

$$\sum_{\substack{x\in M\\ df(x)=0}} P(t,f,x) = P(t,M) + (1+t)Q(t).$$

Remark. There is a Gromoll–Meyer theory for C^1 function and for functionals defined in infinite dimensional manifolds, see [7].

The proof of the theorem is similar to that of Theorem 1.2. Let $c_1 < c_2 < \ldots < c_l$ be the critical values of f. Fix an ε_0 such that $0 < \varepsilon_0 < \min_j(c_{j+1} - c_j)$. Set $b_j = c_j + \varepsilon_0/2,\ j = 1, \ldots, l$ and $b_0 = c_1 - \varepsilon/2$. For any $\varepsilon > 0$, we take a Morse function g satisfying the conditions of Theorem 0.2 and $\|f - g\|_{C^2} \le \varepsilon$. We define sequences of complexes $(\mathcal{C}(g, b_j), \partial)$, $(\mathcal{C}(g, b_j, b_{j+1}), \partial)$, $j = 0, \ldots, l-1$ as before. Thus we have a filtration

$$0 \subset (\mathcal{C}(g, b_1), \partial) \subset \ldots \subset (\mathcal{C}(g, b_l), \partial) = (\mathcal{C}(g), \partial).$$

In order to prove this theorem, we need to replace Proposition 1.4 by the following proposition. The other parts of the proof are the same as that of Theorem 1.2. Set $K_c(f) = \{x \in M \mid df(x) = 0,\ f(x) = c\}$.

Proposition 2.3.

$$H_*(\mathcal{C}(g, b_i, b_{i+1}), \partial) = \bigoplus_{x\in K_{c_i}} H_*(W(x), W_-(x)).$$

We need several lemmas to finish the proof. In the following, for each $x \in K(f)$, we fix a (G-M) pair $(W(x), W_-(x))$, and we always assume that the function g satisfies the conditions of Theorem 0.2. First, we have

Lemma 2.4. *Let $g \in C^2(M)$ such that $|g - f|_{C^2} < \varepsilon$, then for small ε, $dg(y) \neq 0$, $b_{j-1} < g(y) < b_j$ and $y \notin \bigcup_{x\in K_{c_j}(f)} W(x)$.*

Let g be a function as above.

Lemma 2.5. *Let $x_1, x_2 \in K_{c_j}(f)$, $x, y \in K(g)$ such that $b_{j-1} < g(x)$, $g(y) < b_j$ and $x \in W(x_1)$, $y \in W(x_2)$. Let $\gamma(t)$ be a solution of*

$$\dot\gamma(t) = -g'(\gamma(t)), \quad \lim_{t\to-\infty}\gamma(t) = x, \quad \lim_{t\to\infty}\gamma(t) = y. \tag{2.2}$$

Then for small ε, we have

1. *$\gamma(t) \in W(x_1)$, $t \in \mathbb{R}$ if $x_1 = x_2$.*
2. *There is no such $\gamma(t)$ satisfying* (2.2) *if $x_1 \neq x_2$.*

The above two lemmas are similar to Lemma 1.7 and Lemma 1.8, we omit the proof here. Now for $x \in K_{c_j}(f)$, we set

$$C_k(g,x) = \bigoplus_{\substack{y\in W(x),\ y\in K(g),\\ i(y)=k}} \mathbb{Z}\langle y\rangle, \quad \mathcal{C}(g,x) = \bigoplus_k C_k(g,x),$$

$$\partial : C_k(g,x) \to C_{k-1}(g,x), \quad \partial\langle y\rangle = \sum_{\substack{z\in K(g),\ z\in W(x),\\ i(z)=k-1}} n(y,z)\langle z\rangle.$$

From Lemma 2.5, we get

LEMMA 2.6. *$(\mathcal{C}(g,x),\partial)$ is a complex and*

$$H_*(\mathcal{C}(g,b_{j-1},b_j),\partial) \cong \bigoplus_{x\in K_{c_j}} H_*(\mathcal{C}(g,x),\partial)).$$

Thus in order to prove Proposition 2.3, it suffices to show

PROPOSITION 2.7. *For any $x \in K_{c_j}(f)$,*

$$H_*(\mathcal{C}(g,x),\partial) = H_*(W(x),W_-(x)).$$

This can be considered as a local version of the Morse homology. Before the proof, we need the notions of isolated invariant set and isolated neighbourhood and its Conley index for the gradient flow $\gamma(t,x)$, which are generalizations of an isolated critical point and its (G-M) pair for a function, cf. [11], [20].

DEFINITION 2.8. A set $S \subset M$ is called invariant w.r.t. to $\gamma(t,x)$ if $\gamma(t,S) = S$ for every $t \in \mathbb{R}$. It is called isolated if there is a neighbourhood N of S such that

$$S = I(N) = \bigcap_{t\in\mathbb{R}} \gamma(t,N).$$

A Conley index for an isolated invariant set S is a pair of compact sets $N_- \subset N$ such that $S = I(\mathrm{cl}N \setminus N_-) \subset \mathrm{int}(N \setminus N_-)$ and

$$x \in N_-,\ \gamma([0,t],x) \in N \text{ implies } \gamma(t,x) \in N_-,$$

$$x \in N \setminus N_- \text{ implies } \quad \exists t > 0,\ \gamma([0,t],x) \in N.$$

For every isolated invariant set S, there is a Conley index (N,N_-). Moreover, the homotopy type of (N/N_-) is indepedent of the choices of the index pairs. Hence $H_*(N,N_-)$ is well defined for each S.

PROOF OF PROPOSITION 2.7. First we fix an $x \in K_{c_j}(f)$ and its (G-M) pair $(W(x),W_-(x))$. Then for any $\varepsilon > 0$, we take a function $g \in C^2(M)$ such that g satisfies conditions of Theorem 0.2 and $|f-g|_{C^2} < \varepsilon$. For small ε, Lemma 2.3, Lemma 2.4 can be applied. Let $\gamma(t)$ be the solution of

$$\dot\gamma(t) = -g'(\gamma(t)), \quad \gamma(0) = y.$$

Set

$$S = \{y \in W(x),\ \gamma(t,y) \in W(x) \text{ for } t \in \mathbb{R}\},$$

then $S \subset W(x)$ is an isolated invariant set for $\gamma(t,y)$. We claim that for small ε, $(W(x), W_-(x))$ is a Conley index for S. Indeed, we know that $\{x\}$ is an isolated invariant set for the gradient flow of f, and $(W(x), W_-(x))$ is a Conley index for $\{x\}$, cf. [8]. By stabilty of Conley index, we conclude that if $|f-g|_{C^2} < \varepsilon$ and ε is small, $(W(x), W_-(x))$ is a Conley index for S. Now we fix such a function g. Let

$$M_k = \{y \in W(x),\ y \in K(g),\ i(y) = k\}, \quad k = 1, \dots, n,$$

then $(M_0, \dots M_n)$ is a Morse decomposition of the invariant set S, that is

$$\forall z \in S\ \exists j_1 < j_2 \text{ such that } \lim_{t\to-\infty} \gamma(t,z) \in M_{j_1},\ \lim_{t\to\infty} \gamma(t,z) \in M_{j_2}.$$

Hence there is a sequence of sets $N_0 \subset \dots \subset N_n$ such that (N_j, N_{j-1}) is a Conley index for M_j, $1 \le j \le n$, and (N_n, N_0) is a Conley index for the invariant set S, cf. [8], [20]. It is easy to see

$$H_j(N_j, N_{j-1}) = C_j(g,x) = \bigoplus_{\substack{y\in W(x)\\ y\in K(g)\\ i(y)=j}} \mathbb{Z}\langle y\rangle,$$

$$H_k(N_j, N_{j-1}) = 0, \quad k \ne j.$$

As in [20], we can show that

$$\partial : C_k(g,x) \to C_{k-1}(g,x) \quad \text{and} \quad \partial' : H_k(N_k, N_{k-1}) \to H_{k-1}(N_{k-1}, N_{k-2})$$

are the same, where ∂' is the operator in the long exact sequence of the homology groups of triple (N_k, N_{k-1}, N_{k-2}). $H_*(W(x), W_-(x)) = H_*(\mathcal{C}(g,x), \partial)$ follows from [20] too. This completes the proof of Proposition 2.7. □

With Proposition 2.3, the proof of Theorem 2.1 can proceed as that of Theorem 1.2, we omit the details.

3. Manifolds with boundary

Now we consider the case that M is a compact n-dimensional manifold with smooth boundary ∂M. Let $f \in C^2(M)$, $\widehat{f} = f\,|_{\partial M} \in C^2(\partial M)$, we assume that both f and $\widehat{f}$ have isolated critical points.

Definition 3.1. A function $f \in C^2(M)$ is said to satisfy the general boundary condition if

(1) f has no critical points on ∂M,
(2) $\widehat{f}$ has only isolated critical points.

Let $\partial M_- = \{x \in \partial M \mid \langle f'(x), n(x)\rangle \leq 0\}$, where $n(x)$ is the out normal of ∂M at x. Let $x \in K(f)$ and $(W(x), W_-(x))$ be a (G-M) pair for x, $y \in \partial M_- \cap K(\widehat{f})$ and $(W(y), W_-(y))$ be a (G-M) pair for y, set $M(t, x, f) = \sum_i \operatorname{rank} H_i(W(x), W_-(x))t^i$, $M(t, y, \widehat{f}) = \sum_i \operatorname{rank} H_i(W(y), W_-(y))t^i$. The following are Morse inequalities of manifold with boundary.

THEOREM 3.2. *Let $f \in C^2(M)$ satisfy the general boundary condition, then there is a polynomial $Q(t)$, its coefficients are non-negative intergers such that*

$$\sum_{x\in K(f)} M(t, x, f) + \sum_{y\in K_{\widehat{f}}\cap\partial M_-} M(t, y, \widehat{f}) = P(t, M) + (1 + t)Q(t).$$

In the following, we will give a proof of this theorem using the same method. We assume $\partial M_- = \emptyset$, the general case can be reduced to this case, cf. [10]. The proof is essentially the same as that of Theorem 2.2, we give a sketch.

PROOF. By assumption we have $(f'(x), n(x)) > 0$ for $x \in \partial M$. This property is preserved under small perturbations of f. We take a C^2 small perturbation g of f such that g is a Morse function and

$$\mathcal{M}(x, y) = \{\gamma(t) \mid \dot\gamma = -g'(\gamma),\ \lim_{t\to-\infty} \gamma(t) = x,\ \lim_{t\to\infty} \gamma(t) = y\},$$

is a manifold of dimension $\operatorname{ind}(y) - \operatorname{ind}(x)$. Now we can define Morse complex $(\mathcal{C}(g), \partial)$ as before. Set

$$S = \{x \in M,\ \gamma(t, x) \in M,\ \text{for all } t \in \mathbb{R}\}.$$

This is an invariant set for the gradient flow γ of g and $(M, \emptyset)$ is a Conley index for S. $(M_0, \dots, M_n)$ is a Morse decomposition of the invariant set S where

$$M_k = \{y \in M,\ y \in K(f),\ i(y) = k\}, \quad k = 1, \dots, n.$$

The same arguments as in proof of Proposition 2.7 show

$$H_*(\mathcal{C}(g), \partial) = H_*(M).$$

On the other hand, we have the following filtration

$$0 \subset (\mathcal{C}(g, b_1), \partial) \subset \ldots \subset (\mathcal{C}(g, b_l), \partial) = (\mathcal{C}(g), \partial)$$

of $(\mathcal{C}(g), \partial)$, which gives a spectral sequence convergent to $H_*(\mathcal{C}(g), \partial)$ with the first term $H_*(\mathcal{B}) = \bigoplus_{x\in K(f)} H_*(W(x), W_-(x))$. Similar to the proof of Theorem 1.2, we conclude Theorem 3.2. □

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