Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 13, 1999, 105–146

EQUIVARIANT DEGREE FOR ABELIAN ACTIONS. PART III: ORTHOGONAL MAPS

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Abstract. The main goal of this paper is to define an equivariant degree theory for orthogonal maps. We apply our degree to study of bifurcations and existence of solutions of equivariant nonlinear problems.

Introduction

This paper represents the third part of the study of the equivariant degree for abelian actions and constitutes another development of the theory given in [11]–[14]. Here we study orthogonal equivariant maps, in particular gradients and Hamiltonians, using the results of [13] and [14].

The basic setting is the following: let Γ be a compact abelian group acting linearly and via isometries on the finite dimensional space V. Let Ω be an open, bounded, invariant subset of V and $\Phi : \overline{\Omega} \to \mathbb{R}$ a C^1-invariant map, such that its gradient is non-zero on $\partial\Omega$.

Now, if $\Gamma = T^n \times \mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_s}$, with the torus T^n generated by $(\varphi_1, \ldots, \varphi_n)$, $\varphi_j \in [0, 2\pi]$, it is clear, from the fact that $\Phi(\gamma x) = \Phi(x)$, for all $\gamma \in \Gamma$, that

1991 *Mathematics Subject Classification.* Primary 58B05; Secondary 34C25, 47H15, 54F45, 55Q91, 58E09.

Key words and phrases. Equivariant degree, orthogonal maps, hamiltonian systems.

The research of the first author was partially supported by a CONACYT-KBN Grant: "Topological Methods in Nonlinear Analysis"

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$F(x) \equiv \nabla\Phi(x)$ is equivariant, i.e., $F(\gamma x) = \gamma F(x)$ and that if

$$A_j x = \frac{\partial}{\partial \varphi_j}(\gamma x)\bigg|_{\gamma=\mathrm{Id}},$$

one has

$$F(x) \cdot A_j x = 0, \quad j = 1, \dots, n.$$

Hence, $\nabla\Phi(x)$ is an orthogonal map, i.e. a Γ-equivariant map which satisfies these n orthogonality conditions.

The main goal of this paper is to define an equivariant degree theory for such maps, i.e. defined on invariant sets and with equivariant orthogonal homotopies.

In [3], Dancer has defined a Fuller-like degree, i.e. a rational, for gradient S^1-maps, using the restriction on the range of $\nabla\Phi(x)$ and a genericity argument. In [11] and [12] the case of an S^1-orthogonal map was studied with the S^1-degree of $F(x) + \lambda Ax$ on the set $[-1, 1] \times \Omega$, a rational in the first paper and a sequence of integers in the second. In these papers one had to assume that $F^\Gamma(x) \neq 0$ on Ω^Γ.

This last assumption was removed by Rybicki in [18] with the degree developed in [4] and [8] applied to $\widetilde{F}(x) + \lambda Ax$, where $\widetilde{F}(x)$ is a "normal map". Finally, Gęba, in [7], has defined a degree of Γ-gradients, for a general (non-necessarily abelian) Γ: the idea is to approximate the gradient by a gradient "normal" map and define, in this generic case, indices on the different isotropy subspaces via Poincaré sections, in a spirit similar to [3]. For an abelian Γ, our degree will coincide with Gęba's and will "classify" all possible degrees for orthogonal maps.

In the present paper we shall follow the suggestion, given in [10], to study for a general Γ, the problem

$$F(x) + \sum_1^n \lambda_j A_j x = 0.$$

In fact, by taking the scalar product of this equation with $F(x)$, one has $F(x) = 0$ and $\sum_1^n \lambda_j A_j x = 0$. Thus, if the $A_j x$ are linearly independent, one gets $\lambda_j = 0$ and one can use the Γ-degree of the above map on $I^n \times \Omega$. Of course this simple idea will not work if the $A_j x$ are not linearly independent. Thus, one needs to work up on the isotropy subspaces, with the right number of linearly independent vector fields and modifying the map $F(x)$ along the way.

Section 1 is devoted to the construction of the degree, first for gradients and then for orthogonal maps. As in [11], one "suspends" the map in order to get a fixed reference framework, $\prod_\nabla^\Gamma$ and $\prod_\perp^\Gamma$ respectively, for maps which are Γ-gradients on $I \times B$ or Γ-orthogonal from $I \times B$ into $\mathbb{R} \times V$. Here B is a large ball, centered at the origin and containing Ω. The associated map will be non-zero on

$\partial(I \times B)$ and its Γ-homotopy class will be the Γ-degree. The set $\prod_{\perp}^{\Gamma}$ is a group and the degree will have all the properties of a degree.

Section 2 constitutes our main result, i.e. that $\prod_{\perp}^{\Gamma}$ is a product of as many $\mathbb{Z}$ as there are isotropy subgroups of Γ. In Section 3 we extend the degree to infinite dimensions and compare it to the "normal map" approach. We also study the reduction of the symmetry and products. In Section 4 we compute the index of an isolated orbit and, in Section 5, we study bifurcation. Finally, Section 6 treats Hamiltonians.

In this paper we shall use freely the results of [13] and [14] but we shall recall the appropriate version of them as we proceed.

1. Construction of the degree

In this section we shall construct the equivariant degree, first for gradient maps and then for orthogonal maps.

(A) Gradient maps. Let $\Phi : \overline{\Omega} \to \mathbb{R}$ be a C^1-function such that $\Phi(\gamma x) = \Phi(x)$ for all γ in $\Gamma \cong T^n \times \mathbb{Z}_{m_1} \times \ldots \times \mathbb{Z}_{m_s}$ and with $F(x) \equiv \nabla\Phi(x)$, non-zero on $\partial\Omega$. As noted in the introduction one has that $F(\gamma x) = \gamma F(x)$ and $F(x) \cdot A_j x = 0$.

Let $B = B(0, R)$ be a large ball containing Ω. From the Dugundji–Gleason lemma, [11, p. 439], Φ has an invariant extension $\widetilde{\Phi}(x) : B \to \mathbb{R}$. By using mollifiers, one may assume that $\widetilde{\Phi}$ is C^1 and that $\nabla\widetilde{\Phi}(x) = \widetilde{F}(x)$ is arbitrarily close to $F(x)$. In fact, if $\varphi(\rho) : \mathbb{R}^+ \to \mathbb{R}^+$ is decreasing, C^∞, with values A for $\rho < \varepsilon_0$ and 0 for $\rho \geq 1$, where A is such that $\int_V \varphi(|x|)\,dx = 1$, then $\widetilde{\Phi}_\varepsilon(x) = \varepsilon^{-N} \int_V \varphi(|y - x|/\varepsilon)\widetilde{\Phi}(y)\,dy$, where $\dim V = N$, is C^∞ and invariant (since $|y - \gamma x| = |\gamma^T y - x|$ and γ is an isometry). Furthermore, since $\widetilde{\Phi}_\varepsilon(x) = \int_V \varphi(|z|)\widetilde{\Phi}(x + \varepsilon z)\,dz$, $\widetilde{\Phi}_\varepsilon$ approximates uniformly $\widetilde{\Phi}$ in B and its gradient $\widetilde{F}_\varepsilon$ does approximate F on $\overline{\Omega}_{\varepsilon_0} \equiv \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq \varepsilon_0\}$, for $\varepsilon < \varepsilon_0$. Since F is non-zero on $\partial\Omega$, one may choose ε_0 such that $F(x) \neq 0$ on $\overline{\Omega}\backslash\Omega_{\varepsilon_0}$ and replace Ω by Ω_{ε_0}.

As in [11], the next step is to construct an invariant neighbourhood N of $\partial\Omega$, on which $\widetilde{F}(x)$ is non-zero, and an invariant C^1-function φ, from B into $[0, 1]$, such that φ is 0 on $\overline{\Omega}$ and 1 outside $\Omega \cup N$: if N is an ε_1-neighbourhood, let N_1 and N_2 be $\varepsilon_1/3$ and $2\varepsilon_1/3$ neighbourhoods of $\partial\Omega$. One may choose φ_1 to have values 0 in $\Omega \cup N_1$ and 1 outside $\Omega \cup N_2$. By taking mollifiers φ_ε, then one will have the required properties for $\varepsilon < \varepsilon_1$.

Let next $0 < \varepsilon$ be such that $4\varepsilon|\nabla\varphi(x)| \leq |\widetilde{F}(x)|$, for all $x \in N$, and for $t \in [0, 1]$ define

$$\widehat{\Phi}(t, x) = \varepsilon(t^2 + t(2\varphi(x) - 1)) + \widetilde{\Phi}(x).$$

Then, $\nabla\widehat{\Phi} = (\varepsilon(2t + 2\varphi(x) - 1), \widetilde{F}(x) + 2\varepsilon t\nabla\varphi(x))^T$ and its zeros are such that $F(x) = 0$ for $x \in \Omega$ and $t = 1/2$.

It is clear that if one has a gradient Γ-homotopy on $\partial\Omega$, the corresponding gradients $\widetilde{\Phi}$ will be Γ-homotopic as maps from $\partial(I \times B)$ into $\mathbb{R} \times V \backslash \{0\}$. Hence, if $\prod_{\nabla}^{\Gamma}$ is the set of Γ-homotopy gradients from $S^V \cong \partial(I \times B)$ into $\mathbb{R} \times V \backslash \{0\}$, one may define

$$\deg{}_{\nabla}^{\Gamma}(\Phi; \Omega) \equiv [\nabla\widetilde{\Phi}]_{\nabla} \in \prod\nolimits_{\nabla}^{\Gamma}.$$

(B) Orthogonal maps. The construction for orthogonal maps follows similar lines: let $F : \Omega \to V$ be a Γ-equivariant map, with F non-zero on $\partial\Omega$ and $F(x) \cdot A_j x = 0$, $j = 1, \ldots, n$.

Choose B as above and let $\widetilde{F}_0(x)$ be any equivariant extension of F to B. Since $\widetilde{F}_0$ is not necessarily orthogonal to $A_j x$, we shall use the Gram–Schmidt orthogonalization in the following form: let

$$\widetilde{A}_1(x) = \begin{cases} A_1 x / \|A_1 x\| & \text{if } A_1 x \neq 0, \\ 0 & \text{if } A_1 x = 0, \end{cases}$$

$$\widehat{A}_j(x) = A_j x - \sum_{1}^{j-1} (A_j x, \widetilde{A}_i(x)) \widetilde{A}_i(x),$$

and

$$\widetilde{A}_j(x) = \begin{cases} \widehat{A}_j(x) / \|\widehat{A}_j(x)\| & \text{if } \widehat{A}_j(x) \neq 0, \\ 0 & \text{if } \widehat{A}_j(x) = 0. \end{cases}$$

Clearly the $\widetilde{A}_j(x)$ are orthogonal and $\widehat{A}_j(x) = 0$ if and only if $A_j x$ is a linear combination of $A_1 x, \ldots, A_{j-1} x$. Furthermore, since Γ is abelian, A_j is Γ-equivariant as well as $\widetilde{A}_j(x)$ and $\widehat{A}_j(\lambda x) = \lambda \widehat{A}_j(x)$, for λ in $\mathbb{R}$. All these facts can be easily proved by induction. Recall also that, if T^n acts on a complex coordinate z as *exp* $(i \sum N_j \varphi_j)$, then $A_j z = i N_j z$. Let

$$\widetilde{F}(x) = \widetilde{F}_0(x) - \sum_{1}^{n} (\widetilde{F}_0(x), \widetilde{A}_j(x)) \widetilde{A}_j(x).$$

Lemma 1.1.

(a) $\widetilde{F}(x)$ *is an orthogonal* Γ*-extension of* $F(x)$,

(b) $\widetilde{F}(x)$ *is continuous.*

Proof. By construction $\widetilde{F}(x)$ is orthogonal to $\widetilde{A}_j(x)$ for all j and hence to all $A_j x$, which are linear combinations of them. Furthermore, if x is in $\overline{\Omega}$, then $\widetilde{F}_0(x) = F(x)$ is orthogonal to all $A_j x$, hence to all $\widetilde{A}_j(x)$, and $\widetilde{F}(x) = F(x)$.

Thus, the more delicate part is the continuity of $(\widetilde{F}_0(x), \widetilde{A}_j(x)) \widetilde{A}_j(x)$. Let x_n be a sequence converging to x_0 such that $\widehat{A}_j(x_n)$ is non-zero and converges to 0 (the other cases are trivial). Then, since $\widetilde{A}_j(x_n)$ has norm 1, there is a subsequence such that $\widetilde{A}_j(x_n)$ converges to some v, with norm 1, and the above expression converges to $(\widetilde{F}_0(x_0), v)v$.

Now, since $\widehat{A}_j(x_0) = 0$, then $A_j x_0 = \sum_1^{j-1} \lambda_i A_i x_0$, i.e. x_0 belongs to $\ker(A_j - \sum_1^{j-1} \lambda_i A_i) \equiv V_1$. But V_1 is invariant under Γ and in fact $V_1 = V^T$, where T is the torus $(-\lambda_1\varphi, \dots, -\lambda_{j-1}\varphi, \varphi, 0, \dots, 0)$. Hence, from the equivariance, $\widetilde{F}_0(x_0)$ belongs to V_1 and one will have proved the continuity if one shows that v is in $V_2 = V_1^\perp$.

Assume first that j is the first index for which $\widetilde{A}_j(x_0) = 0$ and write any x in V as $x_1 + x_2$, with x_i in V_i. Since $\widetilde{A}_i$ is equivariant, one has that $\widehat{A}_i(x_1)$ is in V_1 and, since $A_j x_1$ is a linear combination of $A_1x_1, \dots, A_{j-1}x_1$, one has $\widehat{A}_j(x_1) = 0$. Now

$$(\widehat{A}_k x)_1 - \widehat{A}_k(x_1) = -\sum_1^{k-1}(A_k x_1, \widetilde{A}_i(x)_1)(\widetilde{A}_i(x)_1 - \widetilde{A}_i(x_1))$$
$$-\sum_1^{k-1}(A_k x_2, \widetilde{A}_i(x)_2)\widetilde{A}_i(x)_1 - \sum_1^{k-1}(A_k x_1, \widetilde{A}_i(x)_1 - \widetilde{A}_i(x_1))\widetilde{A}_i(x_1)$$

and

$$\|(\widetilde{A}_k x)_1 - \widetilde{A}_k(x_1)\| \le \|\widehat{A}_k(x)_2\|^2/\|\widehat{A}_k(x)\|^2$$
$$+ 2\|(\widehat{A}_k x)_1 - \widehat{A}_k(x_1)\|/\|\widehat{A}_k(x_1)\|,$$

where one uses $\left||a|^{-1} - |b|^{-1}\right| = |(b+a, b-a)|/|a||b|(|a|+|b|) \le |b-a|/|a||b|$.

From the fact that $||\widetilde{A}_k(x)_2|| \le C\|x_2\|$, it is easy to prove by induction that $\|(\widehat{A}_k x)_1 - \widehat{A}_k(x_1)\|$, $\|(\widetilde{A}_k x)_1 - \widetilde{A}_k(x_1)\| \le C_k\|x_2\|^2$, where C_k depends on $\|\widehat{A}_l(x)\|^{-1}$ and $\|\widehat{A}_l(x_1)\|^{-1}$ for $l < k$. Again by induction, these norms are close to those for x_0, hence non-zero. Hence, $\|(\widehat{A}_j x)_1\| \le C_j\|x_2\|^2$ and $(\widehat{A}_j x)_2 = A_j x_2 - \sum_{i=1}^{j-1} \lambda_i A_i x_2 + 0(\|x_2\|^2)$, when x tends to x_0. In this case, if for some subsequence, one has that $x_2/\|x_2\|$ converges to X_2, then $\widetilde{A}_j(x)$ converges to $v = w_2/\|w_2\|$, with $w_2 = A_j X_2 - \sum_1^{j-1} \lambda_i A_i X_2$, which is non-zero by definition of V_1.

If $\widehat{A}_j(x_0)$ is not the first zero vector, let

$$\widehat{A}_j(x) = A_j x - \sum_1 (A_j x, \widetilde{A}_i(x))\widetilde{A}_i(x) - \sum_2 (A_j x, \widetilde{A}_i(x))\widetilde{A}_i(x),$$

where the first sum corresponds to i with $\widetilde{A}_i(x_0) \neq 0$ and the second sum to $-\widehat{B}_2(x)$, with $\widetilde{A}_i(x_0) = 0$. Then, $\widehat{A}_j(x) = \widehat{B}_1(x) + \widehat{B}_2(x)$, with $\widehat{B}_1(x)$ orthogonal to $\widehat{B}_2(x)$.

By induction one has that $\widehat{B}_2(x)$ goes to 0 and, from the previous argument, $\widehat{B}_1(x)$ goes also to 0, when x goes to x_0. Note that x_0 belongs to $\ker(A_j - \sum_{i=1} \lambda_i^j A_i)$ and, for each i in the second sum, to $\ker(A_i - \sum \lambda_i^k A_k)$, that is x_0

is in the fixed point subspace of a $(n+1)$-torus, T, where n is the cardinality of the second sum.

Now if, for subsequences, $\widehat{B}_1(x)/\|\widehat{B}_1(x)\|$ goes to $u_1, \widetilde{A}_i(x)$ to w_i, for i in the second sum, $\|\widehat{B}_1(x)\|/\|\widehat{A}_j(x)\|$ to $\alpha_1, \alpha_i(x)/\|\widehat{A}_j(x)\|$ to β_i, with $\alpha_i(x) = (A_j x, \widetilde{A}_i(x))$ which, by induction, goes to 0, then $\widetilde{A}_j(x)$ goes to $\alpha_1 u_1 + \sum_2 \beta_i w_i$, with $\alpha_1^2 + \sum \beta_i^2 = 1$. From the above argument, one has that u_1 is orthogonal to $\ker(A_j - \sum_1 \lambda_1^j A_i)$ and w_i to $\ker(A_i - \sum \lambda_i^k A_k)$. Hence v is orthogonal to V^T, proving the lemma. $\square$

Note that the above lemma can be used in order to prove that if a Γ-map F is such that $F(x)$ is not a linear combination of the $A_j x$, for any x in $\partial\Omega$, then F is Γ-homotopic on $\partial\Omega$ to an orthogonal map $F(x) - \sum(F(x), \widetilde{A}_j(x))\widetilde{A}_j(x)$ via a linear deformation: if it is zero, then $F(x)$ is a linear combination of the $\widetilde{A}_j(x)$ and hence of the $A_j x$. The lemma shows that the resulting map is continuous.

The rest of the construction of the degree is then easy: let N be an invariant neighbourhood of $\partial\Omega$, on which $\widetilde{F}(x)$ is non-zero, and let $\varphi(x)$ be an invariant partition of unity, with value 0 in $\overline{\Omega}$ and 1 in the complement of $\Omega \cup N$, then

$$\widehat{F}(t,x) = (2t + 2\varphi(x) - 1, \widetilde{F}(x))$$

is non-zero an $\partial(I \times B)$ and is an orthogonal Γ-map on $I \times B$.

Furthermore, it is clear that if F and G are homotopic on $\partial\Omega$, via an orthogonal Γ-map, then $\widehat{F}(t,x)$ and $\widehat{G}(t,x)$ are homotopic via an orthogonal Γ-map. Hence, one may define

$$\deg_{\perp}(F;\Omega) \equiv [\widehat{F}]_{\perp},$$

where $[\widehat{F}]_{\perp}$ is the homotopy class of $\widehat{F}$ in $\prod_{\perp}^{\Gamma}$, the set of all Γ-homotopy classes of orthogonal Γ-maps from $\partial(I \times B)$ into $\mathbb{R} \times V \setminus \{0\}$.

As in [11, Proposition 2.1], the class of $\widehat{F}$ is independent of the construction, i.e. of φ, N and $\widetilde{F}$. Furthermore, if Ω is a ball, one may take a radial extension (hence non-zero on $B \setminus \Omega$) and one has $[\widehat{F}]_{\perp} = \sum_0 [F]_{\perp}$, the suspension of $[F]_{\perp}$. We shall prove later on that $\sum_0$ is an isomorphism.

Another important fact is that the Equivariant Borsuk homotopy extension theorem, [11, p. 439], is valid for orthogonal Γ-maps.

Lemma 1.2. *$\prod_{\perp}^{\Gamma}$ is an abelian group.*

Proof. It is enough to check that the arguments given in [11, Propositions A.1 and A.4] are still valid. In particular, by using the equivariant Borsuk theorem, one may deform any F in $\prod_{\perp}^{\Gamma}$ to a map with values $(1,0)$ for $t = 0$ or 1. This enables one to define the sum in $\prod_{\perp}^{\Gamma}$. $\square$

Theorem 1. $\deg_{\perp}$ *has all the properties of a degree, i.e., non-triviality, additivity, excision, and the Hopf property (if Ω is a ball and F has a zero degree then there is a non-zero orthogonal Γ-extension).*

Proof. It is enough to go over the proofs of [11]. The fact that the suspension is an isomorphism gives the additivity without suspension. □

One has then the following situation:

$$\prod\nolimits_{\nabla}^{\Gamma} \xrightarrow{\perp_*} \prod\nolimits_{\perp}^{\Gamma} \xrightarrow{\Pi_*} \prod\nolimits_{S^V}^{\Gamma}(S^V),$$

where the first map consists in taking a gradient Γ-map as an orthogonal one and $\prod_*$ is the morphism given by forgetting the orthogonality. From [13] the last group is a product of $\mathbb{Z}$, one for each isotropy subgroup H with $T^n < H$. Furthermore, it is also $\prod_{S^{V^1}}^{\Gamma}(S^{V^1})$, where $V^1 = V^{T^n}$ and $[F] = [F^{T^n}, Z]$, where Z is in the orthogonal complement of V^1 (see [13, Corollary 5.1]). Now, since $A_j^T + A_j = 0$ (by differentiating the equality $(\gamma x, \gamma y) = (x, y)$), the map $(F^{T^n}(x), Z)$ is an orthogonal map. Hence $\prod_*$ is onto.

Note that Parusiński has proved that, if $\Gamma = \{e\}$, the map $\perp_*$ is one to one and onto, see [17]. This could lead to the conjecture that, in general, $\perp_*$ is also one to one and onto.

Remark 1. If one has $F(\lambda, x) : \mathbb{R}^k \times V \to V$, Γ-orthogonal to $A_j x$, or $\Phi(\lambda, x) : \mathbb{R}^k \times V \to \mathbb{R}$, Γ-invariant, such that on the boundary of a bounded, open and invariant subset Ω of $\mathbb{R}^k \times V$, one has $F(\lambda, x) \neq 0$, or $\nabla_x \Phi(\lambda, x) \neq 0$, then one may perform the same constructions and define two Γ-degrees, $\deg_{\nabla}^{\Gamma}(\Phi; \Omega)$ in $\prod_{\nabla}^{\Gamma}(S^{\mathbb{R}^k \times V}, S^V)$ and $\deg_{\perp}^{\Gamma}(\Phi; \Omega)$ in $\prod_{\perp}^{\Gamma}(S^{\mathbb{R}^k \times V}, S^V)$. This last set will be an abelian group and one will have a degree with the usual properties (additivity here will be up to one suspension). One may also define the maps $\perp_*$ and $\prod_*$ into $\prod_{S^{\mathbb{R}^k \times V}}^{\Gamma}(S^V)$, which has been studied in [13].

2. Main theorem

The following constitutes the main abstract result of the paper. Its proof will be by modifying the original map on subspaces with orbits of increasing dimension: in fact, if $\Gamma_x = H$ with $\dim \Gamma/H = k$, then the orbit Γx is a k-dimensional manifold with tangent space at x generated by k of the $A_j x$.

Theorem 2.

(1) $\prod_{\perp}^{\Gamma} \cong \mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z}$, *with one* $\mathbb{Z}$ *for each isotropy subgroup of* Γ.
(2) $[\widehat{F}]_{\perp} = \sum_H d_H [F_H]_{\perp}$, *with explicit generators* F_H. *If* $d_H \neq 0$, *then* $\widehat{F}$ *has a zero in* V^H.
(3) *Any sequence of* d_H *is the degree of some orthogonal* Γ*-map defined on* Ω, *provided* d_H *is taken to be* 0 *if* Ω^H *is empty.*

Proof. Let F be an orthogonal Γ-map, from B into V, which is non-zero on ∂B (in order to make lighter the writing, $I \times B$ is denoted by B).

Step 1. As indicated above, $[F^{T^n}]$, as an element of $\prod_{S^{V^1}}^{\Gamma}(S^{V^1})$, is $\sum_{T^n\le H} d_H[F_H]$. Note that $A_j x = 0$ on $V^1 = V^{T^n}$. Hence, $[F_1]_\perp \equiv [F]_\perp - [F^{T^n}, Z]_\perp$ has a non-zero orthogonal Γ-extension to B^{T^n}. Thus, $F_1(X,Z) = (F_1^{T^n}(X,Z), F_\perp(X,Z))$, with $F_1(X,0)\neq 0$ and $F_\perp(X,Z)$ is orthogonal to A_jZ.

Step 2. Recall that the action of T^n on the kth coordinate of Z is of the form $\exp i(\sum n_k^j\varphi_j)$. Assume, without loss of generality, that $n_1^1\neq 0$ and let $\lambda_j \equiv n_1^j/n_1^1$ for $j=2,\ldots,n$. Let

$$V_1 = V^{T^n}\times\{z_k : n_k^1\neq 0 \text{ and } n_k^j = \lambda_j n_k^1,\ j\ge 2\}.$$

Then, on V_1, one has $A_jx = \lambda_jA_1x$ and $V_1 = V^{T_1}$, where T_1 is the $(n-1)$-torus $(-\sum_2^n\lambda_j\varphi_j,\varphi_2,\ldots,\varphi_n)$. Let B_1 be the ball B^{V_1}, then, the map $F_1(x)+\lambda A_1x$ is non-zero on $\partial(I\times B_1)$, where λ is in $I=[-1,1]$, since $F_1(X,0)\neq 0$ and, from the fact that F_1 is orthogonal to A_1x, a zero of the above map is such that $F_1(x)=0$ and $\lambda A_1x = 0$. That is, if $Z\neq 0$, then $\lambda = 0$, since $A_1z_k = in_k^1z_k$. We are assuming here that $n_1^1>0$. If not one changes λA_1x to $-\lambda A_1x$. Thus, $F_1(x)+\lambda A_1x$ defines an element of $\prod_{S^{V_1\times\mathbb{R}}}^{\Gamma}(S^{V_1})\cong A\times\mathbb{Z}\times\ldots\times\mathbb{Z}$, see [13, Corollary 5.1], where $A=\prod_{S^{\mathbb{R}\times V^1}}^{\Gamma}(S^{V^1})$ and there is one $\mathbb{Z}$ for each isotropy subgroup H of Γ acting on V_1, with $\dim\Gamma/H = 1$. Since $F_1^{T^n}\neq 0$, one has that $[F_1+\lambda A_1z] = 0+\sum d_H[\widetilde{F}_H]$. Here $T_1\le H<T^n$ and $\widetilde{F}_H$ is the following map [13, p. 394]:

$$\begin{aligned}\widetilde{F}_H(\lambda,x) &= \widetilde{F}_H(\lambda,t,X_0,y_j,u_j,z_1,\ldots,z_j,\ldots)\\ &= \Big(2t+1-2\prod|x_j||z_1|^{\alpha}, |z_1|X_0, (Q_j-1)y_j|z_1|,\\ &\qquad |z_1|(P_j-1)u_j, |z_1|((2t-1)\eta+i\lambda)z_1,\ldots,|z_1|(R_j-1)z_j,\ldots\Big),\end{aligned}$$

where $\Gamma/H = (\Gamma/H_1)\ldots(\widetilde{H}_{j-1}/\widetilde{H}_j)\ldots$, with $\widetilde{H}_j = H_1\cap\ldots\cap H_j$, H_j the isotropy subgroup of x_j, the jth coordinate. Here X_0 is in V^{T^n}, $\Gamma/H_j\cong\mathbb{Z}_2$ for y_j, $\Gamma/H_j\cong\mathbb{Z}_m$ for u_j and $\Gamma/H_j\cong S^1$ for z_j. If $k_j = |\widetilde{H}_{j-1}/\widetilde{H}_j|$, then k_j is finite, except for z_1. The product in the first component is only for those k_j which are strictly bigger than 1. $Q_j = y_j^2$ if $k_j=2$ and $Q_j=2$ if $k_j=1$. P_j is an invariant monomial of $x_1,\ldots,x_j=u_j$, with exponent k_j in u_j if $k_j>1$ and $P_j=2$ if $k_j=1$. The same definition holds for R_j (for instance if $\Gamma=S^1$ and n_1 is the largest common divisor of all n_j, then $k_j=1$). For other cases see [12]. The exponent α is chosen in such a way that when $Q_j=P_j=R_j=1$ and hence $|z_j| = |z_1|^{q_j}$, for some q_j, then $\alpha+\sum q_j\neq 0$. This implies that the zeros of F_H are for $\lambda=0$, $t=1/2$, $|x_j|=|z_1|=1$ if $k_j>1$ and that one has, for $z_1=1$, exactly $\prod k_j$ zeros and exactly one in the fundamental cell for V^H: $\mathcal{C}_H\equiv\{x_j, 0\le|x_j|<R,\ 0\le \operatorname{Arg} x_j<2\pi/k_j\}$. Each zero, for z_1 in $\mathbb{R}^+$, has index 1: $\eta = \pm 1$ is chosen in such a way that, given the basic orientation, this index is 1.

Let

$$F_H(x) = \widetilde{F}_H(0,x) - (\widetilde{F}_H(0,x), \widetilde{A}_1(x))\widetilde{A}_1(x).$$

By construction F_H is an orthogonal Γ-map, with z_1-component $|z_1|(2t-1-i\alpha(x)n_1^1)z_1$ and the same first component as $\widetilde{F}_H$. Thus, the zeros of F_H are those of $\widetilde{F}_H$ and F_H defines an element of $\prod_\perp^\Gamma$. Furthermore, $\widetilde{F}_H(\lambda, x)$ is Γ-homotopic to $F_H(x) + \lambda A_1 x$: deform $\lambda - \alpha$ in the z_j-component to 0 and then α in the z_1-component to 0 and n_1^1 to 1. Note that $F_H(x) + \lambda A_1 x$ is zero only if $\lambda = 0$ and $F_H(x) = 0$, since F_H is orthogonal to $A_j x$. Hence, $F_H(x) + \lambda A_1 x$ can be taken as generators of $\prod_{S^{\mathbb{R}\times V_1}}^\Gamma(S^{V_1})$.

Complementing F_H by the identity of $V_1^\perp$, one has that

$$[F_2]_\perp \equiv [F_1]_\perp - \sum_{T_1 \le H < T^n} d_H [F_H]_\perp$$

is orthogonal to $A_j x$ and $F_2(x) + \lambda A_1 x$, on $\partial(I \times B_1) \cup B^{T^n}$, is Γ-extendable to a non-zero Γ-map $F(\lambda, x)$ on $I \times B_1$.

We claim that this fact implies that $F_2(x)$ itself has a non-zero orthogonal Γ-extension to B_1, i.e., that $[F_2]_\perp = 0$.

The proof of the claim follows the lines of [13, Theorem 3.1], by working on V_1^H, for H in decreasing order. Thus, if H is maximal (hence any $K > H$ must contain T^n), one may extend $[F_2']_\perp = [F_1]_\perp - d_H[F_H]_\perp$ in such a way that the resulting orthogonal map is non-zero on $\partial\mathcal{C}_H$: this is true on V^K, for $K > H$, since there F_1^K is non-zero, and by a dimension argument, since $\dim \partial\mathcal{C}_H = \dim V^H - 2$, as in [14, Lemma 4.1]. Thus, one may assume that $F_2'(x) + \lambda A_1 x$ is non-zero on $\partial(I \times \mathcal{C}_H)$ and has a zero degree with respect to $I \times \mathcal{C}_H$ (this is the obstruction degree which characterizes $[F_2' + \lambda A_1 x]_\Gamma$).

Now, in $\mathcal{C}_H$ one has the component z_1 in $\mathbb{R}^+$ and, since $F_2'(x) \neq 0$ for $z_1 = 0$, one may compute this obstruction degree on the ball $A \equiv I \times \mathcal{C}_H \cap \{z_1 > \varepsilon\}$, for some small ε. If $F_2' = (f_1, f_2, F_\perp)$, where $f_1 + if_2$ corresponds to the z_1-component, one may perform on ∂A the homotopy $F_2'(x) + \lambda(\tau A_1 x + (1-\tau)A_1 z_1)$: in fact, taking the scalar product with $F_2'(x)$, one has $|F_2'|^2 + \lambda(1-\tau)(F_2', A_1 z_1) = 0$ at a zero of the homotopy, that is, from the orthogonality: $|F_2'|^2 - \lambda(1-\tau)(F_\perp, A_1 y) = |F_2'|^2 + \lambda^2\tau(1-\tau)|A_1 y|^2$ on a zero. Hence, $F_2'(x) = 0$, $\lambda A_1 z_1 = 0$ and, since $z_1 > \varepsilon$, $\lambda = 0$; that is, the zeros are inside A. The resulting map $(f_1, f_2 + \lambda n_1^1 z_1, F_\perp)$ is linearly deformable on ∂A, to $(f_1, \lambda, F_\perp)$, since from the orthogonality one has $f_2 z_1 = -(F_\perp, A_1 y)$, assuming $n_1^1 > 0$. From the product theorem, one obtains that $\deg(f_1, F_\perp; \mathcal{C}_H \cap \{z_1 \ge \varepsilon\}) = 0$, i.e., $(f_1, F_\perp)$ has a non-zero extension, $(\widetilde{f}_1, \widetilde{F}_\perp)$, to $\mathcal{C}_H \cap \{z_1 \ge \varepsilon\}$. Defining, on this set, $\widetilde{f}_2 = -(\widetilde{F}_\perp, A_1 y)/z_1$, one obtains a non-zero orthogonal extension $\widetilde{F}'_2(x)$ of $F'_2(x)$, first on $\mathcal{C}_H$ and then, by the action of the group Γ, on V_1^H.

For a general H, one assumes by induction that $[F'_2]_\perp = [F_1]_\perp - \sum_{K\leq H} d_K[F_K]_\perp$ has been extended, as a non-zero orthogonal map to all V_1^K, for $K < H$, that is, together with a dimension argument, one has a non-zero map on $\partial\mathcal{C}_H$, in particular for the corresponding $z_1 = 0$. Then, one repeats the above argument in order to obtain a non-zero orthogonal extension of F'_2 on V_1^H.

Step 3. On $V_1^\perp$ consider the first coordinate z_k with $n_k^1 \neq 0$ and repeat the above construction in order to get $\widetilde{V}_1 = V^{\widetilde{T}_1}$. Clearly $\widetilde{V}_1 \cap V_1 = V^{T^n}$ and one obtains a non-zero orthogonal extension on $\widetilde{V}_1$ of F^{T^n}. Since the generators for F_2 are trivial on $V_1^\perp$, one obtains a compatible extension. One repeats this construction until all coordinates with $n_k^1 \neq 0$ are exhausted and then with $V_2 = V^{T^n} \times \{z_k : n_k^1 = 0,\ n_k^2 \neq 0 \text{ and } n_k^j = \lambda_j n_k^2,\ j > 2;\ \lambda_j = n_{k_0}^j/n_{k_0}^2\}$, and so on.

Hence, if H is such that $\dim\Gamma/H = 1$ one has one z_1 with $\dim\Gamma/H_1 = 1$ and $|H_1/H| < \infty$, one has an extension $[F_2]_\perp$ of $[F]_\perp - \sum_{\dim\Gamma/H=1} d_H[F_H]_\perp$, which is orthogonal and non-zero on $\bigcup_{\dim\Gamma/H=1} V^H$.

Step 4. The next stage is for two-dimensional Weyl groups. Assume

$$\det\begin{pmatrix} n_1^1 & n_1^2 \\ n_2^1 & n_2^2 \end{pmatrix} = \det A \neq 0$$

and define, for $j \geq 3, \lambda_1^j$ and λ_2^j by

$$\begin{pmatrix} n_1^j \\ n_2^j \end{pmatrix} = A\begin{pmatrix} \lambda_1^j \\ \lambda_2^j \end{pmatrix}.$$

Let $V_2 = \{z_k : n_k^j = \lambda_1^j n_k^1 + \lambda_2^j n_k^2,\ j \geq 2\}$.

Then, on V_2, one has $A_j x = \lambda_1^j A_1 x + \lambda_2^j A_2 x$ for $j \geq 3$ and $V_2 = V^{T_2}$, where T_2 is the $(n-2)$-torus $(-\sum\lambda_1^j\varphi_j, -\sum\lambda_2^j\varphi_j, \varphi_3, \ldots, \varphi_n)$. In particular any isotropy subgroup H for V_2 has $\dim\Gamma/H \leq 2$. The action of T^n on z_k is $\exp\, i(n_k^1\psi_1 + n_k^2\psi_2)$, where $\psi_1 = \varphi_1 + \sum\lambda_1^j\varphi_j, \psi_2 = \varphi_2 + \sum\lambda_2^j\varphi_j$.

Consider the map $F_2(x) + \lambda_1 A_1 x + \lambda_2 A_2 x,\ \lambda_1, \lambda_2 \in I = [-1, 1]$, where $F_2(x) \neq 0$ if $\dim\Gamma/\Gamma_x \leq 1$ and F_2 is an orthogonal Γ-extension of $F(x)$. Hence, a zero of this map will give a zero of F_2 and hence $\lambda_1 = \lambda_2 = 0$: it is clear that $A_j x$ is tangent to the orbit Γx, here at most two dimensional, and that $F_2(x) \neq 0$ if Γx is one-dimensional. Hence, on zeros of F_2, $A_1 x$ and $A_2 x$ are linearly independent. We are assuming here that $\det A > 0$. If this is not the case, one changes $\lambda_1 A_1 x$ to $-\lambda_1 A_1 x$.

Thus, $[F_2(x) + \lambda_1 A_1 x + \lambda_2 A_2 x]_\Gamma$ is an element of $\prod^\Gamma_{S^{\mathbb{R}^2\times V_2}}(S^{V_2})$, the group of all Γ-homotopy classes of maps from $\partial(I^2 \times B_2)$ into $V_2 \setminus \{0\}$, where B_2 is the ball B^{V_2}. Now this group is $A \times \mathbb{Z} \times \ldots \times \mathbb{Z}$, with A corresponding to isotropy subgroups H on V_2 with $\dim\Gamma/H \leq 1$ and there is one $\mathbb{Z}$ for each H

with $\dim \Gamma/H = 2$, see [13, Theorem 5.1]. Then, $[F_2(x) + \lambda_1 A_1 x + \lambda_2 A_2 x]_\Gamma = 0 + \sum d_H[\widetilde{F}_H]_\Gamma$, where $T_2 \leq H$ and $\dim \Gamma/H = 2$. $\widetilde{F}_H$ is the following map:

$$\begin{aligned}\widetilde{F}_H(\lambda, x) = \Big(&2t + 1 - 2\prod |x_j||z_1|^\alpha |z_2|, |z_1 z_2|X_0, |z_1 z_2|(Q_j - 1)y_j,\\ &|z_1 z_2|(P_j - 1)u_j, |z_1 z_2|(i(n_1^1\lambda_1 + n_1^2\lambda_2) + (|z_2|^2 - 1))z_1,\\ &|z_1 z_2|(i(n_2^1\lambda_1 + n_2^2\lambda_2) + \eta(2t - 1))z_2, |z_1 z_2|(R_j - 1)z_j, \dots\Big)\end{aligned}$$

where x_j, X_0, y_j, u_j, Q_j, P_j, R_j are as in the first step. The exponent α has the role of fixing the zeros at $|x_j| = 1 = |z_1| = |z_2|$, $t = 1/2$, $\lambda_1 = \lambda_2 = 0$. The factor $|z_1 z_2|$ is such that $\widetilde{F}_H = (2t+1, 0)$ if z_1 or z_2 is 0. For z_1 and z_2 real and positive the index of each zero is equal to $\eta\mathrm{Sign}\det A$, that is $\widetilde{F}_H$ can be taken as generator, by the appropriate choice of η. Let

$$F_H(x) = \widetilde{F}_H(0, x) - (\widetilde{F}_H(0, x), \widetilde{A}_1(x))\widetilde{A}_1(x) - (\widetilde{F}_H(0, x), \widetilde{A}_2(x))\widetilde{A}_2(x).$$

By construction F_H is an orthogonal Γ-map. Writing $F_H(x) = \widetilde{F}_H(0, x) - \alpha|z_1 z_2|A_1 x - \beta|z_1 z_2|A_2 x$, one sees easily that the zeros of F_H are those of $\widetilde{F}_H(0, x)$ and that one has for them $\alpha = \beta = 0$. Furthermore, as a Γ-map, $F_H(x) + \lambda_1 A_1 x + \lambda_2 A_2 x$ is linearly deformable to $\widetilde{F}_H(0, x) + \lambda_1 A_1 x + \lambda_2 A_2 x$ (the zeros are for $\lambda_1 = \tau\alpha|z_1 z_2|, \lambda_2 = \tau\beta|z_1 z_2|$ and $\widetilde{F}_H(0, x) = 0$ for which $\alpha = \beta = 0$). Then, this last map is deformable to $\widetilde{F}_H(\lambda, x) = \widetilde{F}_H(0, x) + |z_1 z_2|(\lambda_1 A_1 Z + \lambda_2 A_2 Z)$, with $Z^T = (z_1, z_2)$. This means that one may take $F_H(x) + \lambda_1 A_1 x + \lambda_2 A_2 x$ as the generator in $\prod^\Gamma_{S^{\mathbb{R}^2 \times V_2}}(S^{V_2})$. Let then

$$[F_3]_\perp \equiv [F_2]_\perp - \sum_{T_2 \leq H,\, \dim \Gamma/H = 2} d_H[F_H]_\perp,$$

then F_3 is an orthogonal Γ-map and $F_3(x) + \lambda_1 A_1 x + \lambda_2 A_2 x$, on $\partial(I^2 \times B_2) \bigcup_{\dim \Gamma/H \leq 1} V^H$ is Γ-extendable to a non-zero map $F(\lambda, x)$ on $I^2 \times B_2$.

As before, we claim that this implies that $[F_3]_\perp = 0$: one proceeds on isotropy subspaces of increasing dimension by considering on the fundamental cell $\mathcal{C}_H$ an orthogonal map F_3' which, by induction and dimension arguments, is non-zero on $\partial\mathcal{C}_H$. In particular $F_3'(x) \neq 0$ for $0 \leq z_1 \leq \varepsilon$ or $0 \leq z_2 \leq \varepsilon$, and the obstruction degree d_H is the degree of $F_3'(x) + \lambda_1 A_1 x + \lambda_2 A_2 x$ on the ball $\mathcal{A} = I^2 \times \mathcal{C}_H \cap \{z_1, z_2 \geq \varepsilon\}$. If $F_3'(x) = (f_1 + if_2, g_1 + ig_2, F_\perp) = (F, F_\perp)$, then one may deform linearly $F_3'(x) + \lambda_1 A_1 x + \lambda_2 A_2 x$ to $F_3'(x) + \lambda_1 A_1 Z + \lambda_2 A_2 Z$, with $Z^T = (z_1, z_2)$: by taking the scalar product one obtains, on a zero of the homotopy $|F_3'|^2 + (1 - \tau)(\lambda_1(F, A_1 Z) + \lambda_2(F, A_2 Z)) = 0$. But, by the orthogonality, $(F, A_i Z) = -(F_\perp, A_i Y)$ and, on a zero, $F_\perp = -\tau(\lambda_1 A_1 Y + \lambda_2 A_2 Y)$, hence $|F_3'|^2 + \tau(1 - \tau)(\lambda_1^2|A_1 Y|^2 + 2\lambda_1\lambda_2(A_1 Y, A_2 Y) + \lambda_2^2|A_2 Y|^2) = 0$, which implies, since the quadratic form is non-negative, $F_3'(x) = 0, \lambda_1 A_1 Z + \lambda_2 A_2 Z = 0$ which implies $\lambda_1 = \lambda_2 = 0$, since on $\mathcal{A}$ the vectors $A_1 Z$ and $A_2 Z$

are linearly independent and the zeros of the deformation are inside $\mathcal{A}$. The resulting map

$$\left(f_1, g_1, A\begin{pmatrix}\lambda_1 z_1\\ \lambda_2 z_2\end{pmatrix} + \begin{pmatrix} f_2\\ g_2\end{pmatrix}, F_\perp\right)$$

is linearly deformable to

$$\left(f_1, g_1, A\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}, F_\perp\right),$$

since from the orthogonality

$$A\begin{pmatrix} z_1 f_2\\ z_2 g_2\end{pmatrix} = -\begin{pmatrix}(F_\perp, A_1 Y)\\ (F_\perp, A_2 Y)\end{pmatrix}$$

and a zero of $F_\perp$ on $\mathcal{A}$, will give $f_2 = g_2 = 0$.

This last map is a product and since the extension degree is 0 one has that $(f_1, g_1, F_\perp)$ has degree equal to 0 on $\mathcal{C}_H \cap \{z_1, z_2 > \varepsilon\}$ and therefore a non-zero extension $(\widetilde{f}_1, \widetilde{g}_1, \widetilde{F}_\perp)$ to this set. Defining $\widetilde{f}_2$ and $\widetilde{g}_2$ on this set via

$$A\begin{pmatrix} z_1 \widetilde{f}_2\\ z_2 \widetilde{g}_2\end{pmatrix} = -\begin{pmatrix}(\widetilde{F}_\perp, A_1 Y)\\ (\widetilde{F}_\perp, A_2 Y)\end{pmatrix},$$

one obtains a non-zero orthogonal extension $\widetilde{F}'_3(x)$ of $F'_3(x)$ first on $\mathcal{C}_H$ and then, by the action of the group Γ, on V_2^H.

The rest of the proof in then clear: exhaust all isotropy subgroups H with $\dim \Gamma/H = 2$ and then go on to higher dimensional Weyl groups.

Now, if $[F]_\perp = \sum d_H [F_H]_\perp$, then $[F^K]_\perp = \sum d_H [F_H^K]_\perp$ and, in fact, the sum reduces to those $H \geq K$, since $F_H^K \neq 0$ if K is not a subgroup of H, in which case $V^H \cap V^K$ is a strict subspace of V^H: there is at least one $x_j = 0$ and the first component of F_H^K is non-zero. For $K \leq H$, it is easy to see that F_H^K is the generator for the group $\prod_\perp^\Gamma (S^K, V^K \setminus \{0\})$. Hence, if $F^K \neq 0$ one has $d_H = 0$, for all $K \leq H$.

In order to complete the proof of the theorem, it remains to prove (3). Let H be an isotropy subgroup such that $\Omega^H \neq \phi$ and $\dim \Gamma/H = l$. Assume that one has the components $z_1, \dots, z_l$ such that the matrix A, with $A_{ij} = n_i^j$, $1 \leq i$, $j \leq l$, is non singular. For $j = l+1, \dots, n$ one defines $\lambda_1^j, \dots, \lambda_l^j$ via

$$\begin{pmatrix} n_1^j\\ \vdots\\ n_l^j\end{pmatrix} = A\begin{pmatrix}\lambda_1^j\\ \vdots\\ \lambda_l^j\end{pmatrix}$$

and $V_l = \{z_k : n_k^j = \sum_1^l \lambda_s^j n_k^s,\ j > l\}$. If $\dim V_l = N$ and one writes the action of T^n on V_l, in matricial form as $\sum_1^n n_k^j \varphi_j$, for $k = 1, \dots, N$, let C be the $N \times l$

matrix, with $C_{ij} = n_i^j$, (A corresponds to the first l rows of C), then one has

$$C\begin{pmatrix}\varphi_1\\ \vdots\\ \varphi_l\end{pmatrix} + C\begin{pmatrix}\lambda_1^{l+1}\\ \vdots\\ \lambda_l^{l+1}\end{pmatrix}\varphi_{l+1} + \ldots + C\begin{pmatrix}\lambda_1^{n}\\ \vdots\\ \lambda_l^{n}\end{pmatrix}\varphi_n = C\begin{pmatrix}\psi_1\\ \vdots\\ \psi_l\end{pmatrix},$$

where $\psi_j = \varphi_j + \sum_{l+1}^n \lambda_j^i\varphi_i$ gives a new parametrization of T^n (the action of $\psi_{l+1},\ldots,\psi_n$ on V_l is trivial) and $V_l = V^{T_l}$, where T_l is the $(n-l)$-torus given by ψ_j, $j = l+1,\ldots,n$.

Now, if $\dim V^\Gamma \geq 1$, let the point $(x_0^0, \widetilde{X}_0^0, y_j^0, u_j^0, z_j^0)$ be in Ω^H, where $(x_0^0, \widetilde{X}_0^0)$ is in V^Γ (by translation we shall assume it to be (0,0)) and (y_j, u_j) is in V^{T^n}. By perturbing a little one may assume that y_j^0, u_j^0, z_j^0 are non-zero provided they are components of V^H. Let $x_j' = x_j/|x_j^0|$ for these components and $x_0' = x_0/R$, where $\Omega \subset B_R$. Let

$$\begin{aligned} f_0(x) = \bigg(x_0' - 2\Big(a\prod|x_j'||z_1'|^\alpha - 1\Big), a\widetilde{X}_0, a(Q_j-1)y_j, a(P_j-1)u_j,\\ a(|z_2'|-1)z_1,\ldots,a(|z_l'|-1)z_{l-1}, ax_0'z_l, a(R_j-1)z_j,\ldots\bigg),\end{aligned}$$

with $a = |z_1'|\ldots|z_l'|$, where Q_j, P_j, R_j are as above but with the variables y_j', u_j', z_j', so that the only zeros of $f_0(x)$ in Ω (hence with $|x_0'| < 1$) are for $(x_0, \widetilde{X}_0) = (0,0)$, $|y_j'| = |u_j'| = |z_j'| = 1$, hence on the orbit of the chosen point. If one adds to the z_j-component for $j = 1,\ldots,l$, the term $i\lambda_j z_j$, one obtains a Γ-map $f_0(\lambda, x)$ which has a single component, corresponding to H in $\deg_\Gamma(f_0(\lambda,x); I^l\times\Omega)$: 1 or d if for some j one replaces P_j or R_j by P_j^d or R_j^d (conjugates for negative d): see [13, p. 411].

Since A is invertible, it is clear that $f_0(\lambda, x)$ is Γ-homotopic to $f_0(x) + \sum_1^{j=l}\lambda_j A_j x$, with zeros at the above orbit and $\lambda = 0$: in fact, A, if $\det A > 0$, is deformable to I and, if $\det A < 0$, changing λ_1 to $-\lambda_1$, one still has a generator. Replace a by a function of $|z_1'|,\ldots,|z_l'|$ with value 0 if some $|z_j'| < \varepsilon$ and value 1 if all $|z_j'| > 2\varepsilon$. Then $f_0(x) = (2t+2, 0)$ if some $|z_j'| < \varepsilon$.

Choose $\lambda_j(x)$ such that $(f_0(x) + \sum_1^l \lambda_j A_j x, A_k x) = 0$, for $k = 1,\ldots,l$: if all $|z_j'| > \varepsilon$, the $A_j x$ are linearly independent, hence the matrix $(A_j x, A_k x)$ is invertible. If some $|z_j'| < \varepsilon$, then $(f_0(x), A_k x) = 0$ and the only solution is $\lambda = 0$. The map $f_0(x) + \sum_1^l \lambda_j(x) A_j x$ is an orthogonal Γ-map, with zeros in Ω at the orbit of the original point (there $\lambda_j(x) = 0$) and with orthogonal degree non-trivial only at $d_H = 1$ (or d).

If $l = 0$, there are no z_i' and one follows the same construction with y_i and u_i.

If $\dim V^\Gamma = 0$, then one has to take Ω with $\Omega^\Gamma = \phi$ and, for V^H, the map

$$f_0(\lambda, x) = \Big((|z_2'|Q_j - 1)_j), (|z_2'|P_j - 1)u_j, (i\lambda_1 + |z_2'| - 1)z_1, \dots ,$$
$$(i\lambda_{l-1} + |z_l'| - 1)z_{l-1}, \Big(i\lambda_l/R + 2i\sum(|x_j'| - 1)^2$$
$$+ |z_1'| - 1)^d\Big) z_l, (|z_l'|R_j - 1)z_j, \dots \Big)$$

see [13, p. 411]: if $f_0(\lambda, x) = 0$ and $z_l = 0$ then $y_j = u_j = z_j = 0$, that is $x = 0$, which is not in Ω. While if $z_l \neq 0$, then $|z_j'| = 1$, $|y_j'| = |u_j'| = 1$ and $\lambda = 0$. As above $f_0(\lambda, x)$ is Γ-homotopic on $I^l \times \Omega$ to $f_0(0, x) + \sum_1^l \lambda_j A_j x$ (if $\det A > 0$, if not change λ_1 to $-\lambda_1$), and one may choose $\lambda_j(x)$ such that $f_0(\lambda(x), x)$ is an orthogonal Γ-map: replace $|z_l'|$ by a and $\sum(|x_j'| - 1)^2$ by $a\sum(|x_j'| - 1)^2$, where a is as above and ε such that the ball $B(0, 2\varepsilon)$ is not in Ω. Then $f_0(0, x)$ is orthogonal to $A_j x$ whenever $|z_j'| < \varepsilon$ for some j and $\lambda_i(x) = 1$ there.

The resulting map has orthogonal degree equal to d for H and 0 otherwise.

For any sequence d_H one either follows the construction of [13, p. 386], to get maps F^H as above with degree d_H and a map F such that

$$\deg_\perp(F; \Omega) = \sum \deg_\perp\Big(F^H; \Omega^H \setminus \bigcup_{K>H} \Omega^K\Big),$$

or one uses the argument of [12, p. 73]: if $\dim V^\Gamma \geq 1$, take as many $0 = x_0^0 < x_1 < \dots < x_N$ with $N = \sum |d_H|$ and $x_j - x_{j-1} = 4\varepsilon$. Take f_j the above map, where x_0' is changed to $x_0' - x_j$, then if φ_j has value 1 if $|x_0' - x_j| < \varepsilon$ and 0 if $|x_0' - x_j| > 2\varepsilon$, define $f(x)$ as $(\varphi_j f_j(x) + (1 - \varphi_j)(1, 0))$ for $|x_0' - x_j| \leq 2\varepsilon$ and (1, 0) outside. If $\dim V^\Gamma = 0$, one follows the construction of [12, p. 74]. $\square$

Remark 2. For the case of parameters, one should follow the same lines in order to compute $\prod_\perp^\Gamma(S^{\mathbb{R}^k \times V}, S^V)$: If $F(\mu, x)$ is an element of this group, then F^{T^n} belongs to $\prod_{S^{\mathbb{R}^k \times V}}^\Gamma(S^V)$ and $[F_1]_\perp = [F]_\perp - [F^{T^n}, Z]_\perp$ has a non-zero orthogonal Γ-extension to B^{T^n} and, on $\mathbb{R}^k \times V_1$, the map $F_1(\mu, x) + \lambda A_1 x$ defines an element of $\prod_{S^{\mathbb{R}^{k+1} \times V_1}}^\Gamma(S^{V_1})$. However, the generators of this last group are not explicit, except for the case $k = 1$. Hence, it is not clear that these generators can be written as $F_H(\mu, x) + \lambda A_1 x$. Then, the extension on $\partial\mathcal{C}_H$ meets obstructions on the walls of the fundamental cell (the dimension argument doesn't work anymore) and the suspension (which replaces the product theorem) is not an isomorphism if $\dim V^H$ is too low. Thus, we shall not pursue this study here, except in the special case of bifurcation.

3. Operations

In this section we study the relationship with the normal map approach,the extension to infinite dimension, the reduction of the group and products.

3.1. Normal maps. As in [11] and [13], this construction, together with the index computations, will enable to relate our degree to Rybicki's in [18] and Gęba's in [7]. It will turn out that they coincide, if $\Gamma = S^1$ in the first case and if Γ is abelian in the second case.

Let H be an isotropy subgroup and define $\psi : (V^H)^\perp \to [0,1]$, be such that $\psi(x_\perp)$ is 1 if $|x_\perp| < \varepsilon$ and 0 if $|x_\perp| > 2\varepsilon$. If $F(x) = (F^H(x_H, x_\perp), F_\perp(x_H, x_\perp))$ is an element of $\prod_\perp^\Gamma$ or it is non-zero on $\partial\Omega$, then $F(x)$ is orthogonally Γ-homotopic to the map $(F^H(x_H, (1-\psi)x_\perp), (1-\psi)F_\perp(X_H, (1-\psi)x_\perp) + \psi x_\perp)$, since $A_j x$ is orthogonal to $x_\perp$ and to $F(x)$. Since $F_\perp(x_H, 0) = 0$ and $F^H(x_H, 0)$ is non-zero on $\partial\Omega^H$, one chooses ε so small that $F^H(x_H, x_\perp) \neq 0$ for $|x_\perp| < 2\varepsilon$.

In the case of a gradient, if $F(x) = \nabla\Phi(x)$, let

$$\widetilde{\Phi}(x) = \psi(x_\perp)(\Phi(x_H) + |x_\perp|^2/2) + (1 - \psi(x_\perp))\Phi(x_H, x_\perp).$$

Then $\nabla\widetilde{\Phi}(x) = (F^H(x) + \psi(F^H(x_H) - F^H(x)), (1-\psi)F_\perp(x) + \psi x_\perp + (\Phi(x_H) - \Phi(x) + |x_\perp|^2/2)\nabla\psi)$.

If $|x_\perp| > 2\varepsilon$, then $\nabla\widetilde{\Phi}(x) = F(x)$ while if $|x_\perp| < \varepsilon$, one has $\nabla\widetilde{\Phi}(x) = (F^H(x_H), x_\perp)$. If on $\partial\Omega^H$ one has that $|F^H(x_H)| > \eta$, one chooses ε so small that on $\partial\Omega^H \times \{x_\perp : |x_\perp| \leq 2\varepsilon\}$, one has $|F^H(x) - F^H(x_H)| < \eta/2$. Thus, $\nabla\Phi$ is Γ-homotopic to $\nabla\widetilde{\Phi}$.

Working in stages, as in [13, Theorem 5.4], one gets that F is orthogonally Γ-homotopic to F_N, where $F_N(x_H, x_\perp) = (F_N^H(x_H), x_\perp)$ for any H provided $|x_\perp| < \varepsilon$, i.e. a normal map. Similarly, for the case of gradients, $\nabla\Phi$ is Γ-homotopic to $\nabla\Phi_N$.

In [18] and [7], the authors use this homotopy to reduce the definition of the degree to that of a normal map and a direct sum on all isotropy subgroups. For each such subgroup the index is then defined in a generic situation, via Poincaré sections. As pointed out in the Introduction, our approach classifies all possible degrees.

3.2. Extension to infinite dimension. If $F(x) = x - K(x)$, or $\Phi(x) = |x|^2/2 - \Psi(x)$, with K compact and Γ-orthogonal to $A_j x$, or $\nabla\Psi(x)$ compact, then the extension of the degree to this case requires, following the classical approximation by finite-dimensional maps K_N, that these maps can be taken to be Γ-orthogonal and that the suspension by any representation V_0 is one to one. Since K is compact and $F(x) \neq 0$ on $\partial\Omega$, one has a uniform approximation of K on $\partial\Omega$ by K_N, so that the degree of $x - K(x)$ will be that of $x - K_N(x)$: the averaging on the compact group Γ and the orthogonalization of Lemma 1.1 (restricted to the finite dimensional subspace) will give a small perturbation. On the other hand, the suspension by V_0 is one to one on the generators of $\prod_\perp^\Gamma$ and an orthogonal map, as well as a gradient. Hence, from [13, Theorem 9.1] one may take the direct limit of these groups.

Theorem 3.1. *In the above situation, both degrees are well defined for infinite dimensional spaces and compact maps, or gradients. The degree for orthogonal maps has the same properties listed in Theorem* 2, *except that any map has almost all d_H's equal to* 0.

3.3. Reduction of the group. Let $\Gamma_0 < \Gamma$, with $\Gamma_0 \cong T^{n_0} \times \ldots$. Let

$$P_\perp : \prod\nolimits_\perp^\Gamma \longrightarrow \prod\nolimits_\perp^{\Gamma_0},$$

be the restriction morphism.

According to [14, Lemma 6.1], any isotropy subgroup H_0 of Γ_0 is of the form $H_0 = H \cap \Gamma_0$, where H is an isotropy subgroup of Γ. Furthermore, there is a minimal $\underline{H} \leq H$, such that $H_0 = \underline{H} \cap \Gamma_0$ and $V^{H_0} = V^{\underline{H}}$. One also has that $\dim \Gamma_0/H_0 = k_0 \leq \dim \Gamma/H = k$ and, in case of equality, if $\widetilde{H}_0^0 > H_0$ and $\widetilde{H}_0 > H$ are the maximal isotropy subgroups with $\dim \Gamma_0/\widetilde{H}_0^0 = \dim \Gamma/\widetilde{H}_0 = k$, then $|\widetilde{H}_0^0/H_0|$ divides $|\widetilde{H}^0/H|$ and

$$P_*\Big[F_H + \sum_1^k \lambda_i A_i x\Big]_\Gamma = |\widetilde{H}_0/H|/|\widetilde{H}_0^0/H_0|\Big[F_{H_0} + \sum_1^k \lambda_i A_i x\Big]_{\Gamma_0}$$

(see [14, Proposition 6.1]), where P_* is the restriction morphism

$$\prod\nolimits_{S^{\mathbb{R}^k \times V}}^{\Gamma}(S^V) \longrightarrow \prod\nolimits_{S^{\mathbb{R}^k \times V}}^{\Gamma_0}(S^V).$$

Here we shall prove:

Theorem 3.2.

$$P_\perp\Big(\sum_{H<\Gamma} d_H[F_H]_\perp\Big) = \sum_{H_0<\Gamma_0}\bigg(\sum_1 d_H\frac{|\widetilde{H}_0/H|}{|\widetilde{H}_0^0/H_0|}\bigg)[F_{H_0}]_\perp,$$

where the sum $\sum_1$ is over all H with $H_0 = H \cap \Gamma_0$ and $\dim \Gamma/H = \dim \Gamma_0/H_0$. In particular $P_\perp([F_H]_\perp) = 0$ if $k_0 < k$.

Proof. From the proof of Theorem 2, it is clear that one may take the generators for the parametrized problem as $F_H + \sum \lambda_i A_i x$. If $k = k_0$, then one has $A_1 x, \ldots, A_k x$ linearly independent for x with $\Gamma_x = H$ and $\Gamma_{0x} = H_0$, hence one may take these generators, for which [14, Proposition 6.1], applies and one has part of the answer.

Note that if $k_0 < k$, for some H, then, since $\underline{H} < H$, one has $\dim \Gamma/\underline{H} \geq \dim \Gamma/H$ and, since $V^{\underline{H}} = V^{H_0}$, the only possibility is that $n_0 < n$ and the action of T^{n_0} on $V^{\underline{H}}$ reduces the number of linearly independent $A_j x$ from k to k_0. Assume then that $A_1 x, \ldots, A_{k_0} x$ correspond to Γ_0 and are linearly independent if $\Gamma_{0x} = H_0$, while $A_1 x, \ldots, A_k x$ correspond to Γ and are linearly independent if $\Gamma_x = H$ (and a fortiori if $\Gamma_x = \underline{H}$). Consider the map $F_H(x) + \widetilde{A}_{k_0+1}(x)$ on V^{H_0}. By construction, it is orthogonal to $A_j x$, $j = 1, \ldots, k_0$ and its zeros are such that $F_H(x) = (F_H^H(x_H), Z) = 0$ and $A_{k_0+1}x$ is a linear combination of $A_1x, \dots, A_{k_0}x$. But then, $Z = 0$, x_H has isotropy H, by construction of F_H, and $A_1x_H, \dots, A_kx_H$ are linearly independent. Hence, the map has no zeros, but $P_\perp[F_H]_\perp = [F_H + \widetilde{A}_{k_0+1}(x)]_\perp = 0$. $\square$

3.4. Products. Let V_1, V_2 be two Γ-representations, $\Omega = \Omega_1 \times \Omega_2$ be an open, bounded invariant product, $f_i(x_i)$ be orthogonal Γ-maps, which are non-zero on $\partial\Omega_i$, $i = 1, 2$. As in [14, Lemma 6.2], it is easy to see that, if F_i are the maps constructed in $\prod_\perp^\Gamma(V_i)$, then

$$[F_1, F_2]_\perp = \sum\nolimits_0 \deg_\perp((f_1, f_2); \Omega_1 \times \Omega_2),$$

where $\sum_0$ is a trivial suspension. Furthermore, [14, Lemma 6.3], any isotropy subgroup H for the product is of the form $H = H_1 \cap H_2$, where, as before, there are minimal $\underline{H}_i$ with $V_i^H = V_i^{\underline{H}_i}$. If $k_i = \dim \Gamma/H_i$, $k = \dim \Gamma/H$, then $k_i \le k \le k_1 + k_2$.

THEOREM 3.3. *If $\widetilde{H}_j^0$ is the maximal isotropy subgroup containing H_j, $\Gamma/\widetilde{H}_j^0 \cong T^{k_j}$, then, if $[F_i]_\perp = \sum d_H^i [F_H^i]_\perp$, one has*

$$[F_1, F_2]_\perp = \sum d_{H_1} d_{H_2} \frac{|\widetilde{H}_1^0/H_1||\widetilde{H}_2^0/H_2|}{|\widetilde{H}_1^0 \cap \widetilde{H}_2^0/H_1 \cap H_2|}[F_{H_1 \cap H_2}]_\perp$$

where the sum is over all H_1, H_2, with $\dim \Gamma/H_1 + \dim \Gamma/H_2 = \dim \Gamma/(H_1 \cap H_2)$.

PROOF. It is clearly enough to compute the class $[F_{H_1}, F_{H_2}]_\perp$ for the generators. Writing V^H as $(V_1^{H_1} \times V_2^{H_2}) \times (V_1^{H_1})^\perp \times (V_2^{H_2})^\perp$ one has, for the action of

$$\Gamma/H = \Gamma/H_1 \times H_1/H_1 \cap H_2,$$

k_1 coordinates of $V_1^{H_1}$, $z_1, \dots, z_{k_1}$, giving $A_1x_1, \dots, A_{k_1}x_1$ linearly independent, and $k - k_1$ coordinates of $V_2^{H_2}, \widetilde{z}_1, \dots, \widetilde{z}_{k-k_1}$ for the action of H_1 on that space. Note that, given the order chosen in V^H, the coordinates of $(V_1^{H_1})^\perp$ and of $(V_2^{H_2})^\perp$ do not contribute, in a non-trivial way, to the fundamental cell. Now, as in the proof of Theorem 2, one may write the action of T^n on $V_1^{H_1}$ as $C(\psi_1, \dots, \psi_{k_1})^T$, hence $A_jx_1 = 0$ for $j > k_1$ by changing the parametrization of T^n from the φ to the ψ. Assume that $\psi_{k_1+1}, \dots, \psi_{k-k_1}$ give A_jx_2 linearly independent for the action of H_1 on $V_2^{H_2}$, then, one may suppose, changing the parametrization, that $A_jx_2 = 0$ for $j > k$ and that $A_jx_2, k_1 < j \le k$, are linearly independent, (there are also $k_1 + k_2 - k$ linearly independent vectors A_jx_2 for $j \le k_1$).

Now, if $k = k_1 + k_2$, then $[F_1 + \sum_1^{k_1} \lambda_j A_j x_1, F_2 + \sum_{k_1+1}^k \lambda_j A_j x_2]$ has been computed in [14, Proposition 6.3], giving $\alpha[F_H + \sum_1^k \lambda_j A_j x]$ where α is the coefficient of the theorem. On the other hand, if $k < k_1 + k_2$, one has to add to $F_2 + \sum_{k_1+1}^k \lambda_j A_j x_2$ the sum $\sum \lambda_j A_j x_2$, for j in a subset J of $k_1 + k_2 - k$

elements of $\{1,\dots,k_1\}$. But, for this second sum, one may deform λ_j to 0 and then to a fixed $\varepsilon_j \neq 0$, without affecting the class but giving a zero extension degree. In that case, $[F_1,F_2]_\perp = 0$. $\square$

Remark 3 (Composition). In [14] we derived a formula for part of the equivariant degree of a composition. In the case of orthogonal maps, it is easy to see that the composition of such maps is not necessarily orthogonal. However, following the case of gradients, one may study the situation of a map $h(x) = Df(x)^T g(f(x))$, where g is orthogonal, f is C^1 and only equivariant (for instance $g(y) = \nabla\Phi(y)$, then $h(x) = \nabla_x(\Phi(f(x)))$. Then, from the relations $Df(\gamma x)\gamma = \gamma Df(x)$ and $Df(x)A_j x = A_j f(x)$, (obtained by differentiating $f(\gamma x) = \gamma f(x)$), one has that $h(x)$ is an orthogonal Γ-map.

If $f(\partial\Omega) \subset \partial\Omega_1$, $0 \notin \partial\Omega_1$ and $0 \notin g(\partial\Omega_1)$, one may look at $deg_\Gamma(f;\Omega)$, $\deg_\perp(g;\Omega_1)$ and $\deg_\perp(h;\Omega)$ provided h is non-zero on $\partial\Omega$ (for instance if $Df(x)$ is invertible on $\partial\Omega$). In this case, by choosing the neighbourhood N of $\partial\Omega$ such that $N \subset f^{-1}(N_1)$, N_1 neighbourhood of $\partial\Omega_1$ where $\widetilde{g}$ is non-zero, one has, for $F(t,x) = (2t+2\varphi(x)-1, \widetilde{f}(x))$ and $G(t_1,y) = (t_1+2\psi(y), \widetilde{g}(y))$, with $t_1 \in [-1,1]$, that

$$[2t+2\varphi(x)-1, D\widetilde{f}(x)^T\widetilde{g}(\widetilde{f}(x))]_\perp = [DF(t,x)^T G(F(t,x))]_\perp,$$

(one may assume that $\widetilde{f}$ is C^1).

Note that the presence of $DF(t,x)^T$ does not allow to distribute the class of $[DF^T GoF]_\perp$ with respect to $[F]_\Gamma$ or with respect to $[G]_\perp$ except, as done in [14], if $[G]_\perp = \sum d_H[G_H]_\perp$ provided DF is invertible on the boundary of the cylinder. Since for $\deg_\perp(h;\Omega)$, $DF = \left(\begin{smallmatrix} 2 & (\nabla\varphi)^T \\ 0 & D_x\widetilde{f}(x)\end{smallmatrix}\right)$, invertibility of DF means that $D\widetilde{f}(x)$ is invertible in B, let us consider the following particular case:

Proposition 3. *Assume $\Omega = B$, $f(0) = 0$, $Df(x)$ invertible in $B, g(y) \neq 0$ if $|y| \geq R_1$ and $|f(x)| \geq R_1$ if $x \in \partial B$. Then*

$$\deg_\perp((Df)^T g(f(x)); B) = \deg_\perp(g(y); B(0,R_1)).$$

Proof. Since $\Omega = B$ and $\Omega_1 = B(0,R_1)$, the construction of F and G are not necessary: one may compute directly the class of $h(x)$ and of $g(y)$. Note also that in the non-equivariant case, if the zeros of h are isolated then $Dh(x) = (Df)^T(x)Dg(y)Df(x)$ whenever $g(f(x)) = 0$. Hence in this case the Brower degree of h is that of g. Note that the invertibility of Df implies that the zeros of f have to be in V^Γ and thus 0 is the only zero of $f(x)$.

Now one may deform orthogonally $h(x)$ on ∂B to the following map $Df(x)^T |f(x)|^2 g(R_1 f(x)/|f(x)|)$, via $(\tau + (1-\tau)|f|^2)h(x)$ first and then via $|f(x)|^2 g\cdot (f(x)(\tau + (1-\tau)R_1/|f(x)|))$. The new map has its only zero at $x = 0$. Then, one may deform x on ∂B to εx, for ε small and use the homotopy where $f(\varepsilon x)$ is replaced by $\tau f(\varepsilon x) + (1-\tau)Df(0)\varepsilon x$ and $Df(\varepsilon x)$ by $\tau Df(\varepsilon x) + (1-\tau)Df(0)$:

since $Df(0)$ commutes with any $\gamma \in \Gamma$ (and hence with A_j) the deformation is clearly Γ-orthogonal and, for ε small enough, the path from $Df(0)$ to $Df(\varepsilon x)$ consists of invertible matrices, that is the only zero is at $x = 0$. Furthermore, in $GL_\Gamma(V)$ the matrix $Df(0)$ is deformable to $A \equiv \operatorname{diag}(\varepsilon_\Gamma, \varepsilon_{Z_2}, \dots, I)$, where $\varepsilon_\Gamma = \operatorname{diag}(\operatorname{Sign}\det Df(0)^\Gamma, I)$ on V^Γ, ε_{Z_2} is a similar matrix on $V^H \cap (V^\Gamma)^\perp$, for each H with $\Gamma/H \cong \mathbb{Z}_2$, and the last I is on the other irreducible representations, see [10, Theorem 1.2, p. 407]. Finally, by undoing the above homotopies, one has that $[h]_\perp = [Ag(AX)]_\perp = \sum d_H [AG_H(Ax)]_\perp$. But, from the form of $G_H(x) = (\,\cdot\,, X_0, (y^2-1)y, \dots)$ one has that $AG_H(Ax) = G_H(x)$. □

4. Poincaré sections and index of an isolated orbit

As in [14], we shall study first the following situation: let H be an isotropy subgroup such that $\dim \Gamma/H = k$, then there are complex coordinates $z_1, \dots, z_k$ with isotropy $H_0 > H$ and $|H_0/H| < \infty$. Assume that the orthogonal map F, from B into V, is non-zero on ∂B and on each set given by $z_j = 0$, for each $j = 1, \dots, k$. If one takes all $\widetilde{H} < H_0$ such that $|H_0/\widetilde{H}| < \infty$, then there is a minimal one $\underline{H}$, an $(n-k)$-torus, [14, p. 377]. Furthermore, if $\widetilde{C}$ is the $N \times n$ matrix with $\widetilde{C}_{ij} = n_i^j$, $i = 1, \dots, N = \dim V^{\underline{H}}$, $j = 1, \dots, n$, then $\widetilde{C}$ has rank k and has an invertible submatrix A, for instance n_i^j for $i = 1, \dots, k$, $j = 1, \dots, k$ corresponding to $z_1, \dots, z_k$ and $\varphi_1, \dots, \varphi_k$. Then if $(\lambda_1^j, \dots, \lambda_k^j)^T = A^{-1}(n_1^j, \dots, n_k^j)^T$ for $j > k$, as in the proof of Theorem 2, the subspace $V^{\underline{H}}$ is given by those coordinates z_l which satisfy $n_l^j = \sum_1^k \lambda_s^j n_l^s$ for $j > k$ (if, for some j and l, one doesn't have equality then $\widetilde{C}$ would have rank bigger than k). Note that $A_j x = \sum_1^k \lambda_l A_l x$, for $j > k$ and x in $V^{\underline{H}}$, and $A_1 x, \dots, A_k x$ are linearly independent if x has its coordinates $z_1, \dots, z_k$ non-zero.

Proposition 4.1. *Let F be as above, then $[F]_\perp = \sum_{H_j < H_0} d_j [F_j]_\perp$. If $B_k^j \equiv B^{H_j} \cap \{z_1, \dots, z_k \in \mathbb{R}^+\}$, then for $H_i > \underline{H}$, the corresponding d_i are given by the formula*

$$\deg\left(\left(F + \sum_1^k \lambda_l A_l x\right)^{H_i}; B_k^i\right) = \sum_{H_i < H_j < H_0} d_j |H_0/H_j|.$$

Proof. If K is not a subgroup of H_0, then for some j, $j = 1, \dots, k$, one has that $z_j = 0$ in V^K. Hence, from Theorem 2, the corresponding d_K is 0. Also, one has that $[F^{\underline{H}}]_\perp = \sum d_j [F_j^{\underline{H}}]_\perp$, where the sum is on those j with $\underline{H} < H_j < H_0$ (for the others $[F^{\underline{H}_j}]_\perp = 0$). From the construction of Theorem 2 and [14, Theorem 2.1], one has the above formula. □

Note that the above formula can be arranged as a lower triangular invertible matrix which will yield d_j for $\underline{H} < H_j < H_0$. The other components d_j, with $\dim \Gamma/H_j > k$, have to be computed in special cases as for an isolated orbit. Note also that if F comes from the construction of the orthogonal degree for a map f, then, by the product theorem for the ordinary degree of $(2t+2\varphi(x)-1, \widetilde{f}^{H_i})$ on $I\times(\Omega^{H_i}\cap\{z_1,\dots,z_k\in\mathbb{R}^+\})$, one has that:

$$\deg\bigg(\bigg(F+\sum_1^k\lambda_l A_l x\bigg)^{H_i};B_k^i\bigg)=\deg\bigg(\bigg(f+\sum_1^l\lambda_l A_l x\bigg)^{H_i};\Omega_k^i\bigg)$$

where $\Omega_k^i=\Omega^{H_i}\cap\{z_1,\dots,z_k\in\mathbb{R}^+\}$.

Let us then consider the case of an isolated orbit: assume that Γx_0 is an isolated zero-orbit of the orthogonal map f on Ω, with $\Gamma_{x_0}=H$ and $\dim\Gamma/H=k$. Then, as above, there are $z_1,\dots,z_k$ with isotropy H_0 and non-zero at x_0, with $|H_0/H|<\infty$. As in [14, p. 379], one may choose a neighbourhood Ω of the orbit such that the corresponding $\varphi(x)$ is 1 whenever x has a coordinate $x_l=0$ and x_0 has the same coordinate $x_l^0\neq0$. Hence, $(2t+2\varphi(x)-1,\widetilde{f}(x))^K\neq0$ for any K which is not a subgroup of H. Thus,

$$[F]_\perp=\operatorname{Index}_\perp(f;\Gamma x_0)=\sum_{H_j<H}d_j[F_j]_\perp.$$

One may assume that A_jx_0 are linearly independent for $j=1,\dots,k$ and that $z_j^0\in\mathbb{R}^+$, for $j=1,\dots,k$. Then, from Proposition 4.1, for $H_j>\underline{H}$, one may compute d_j from

$$\deg\bigg(\bigg(f+\sum_1^k\lambda_l A_l x\bigg)^{H_i};\Omega_k^i\bigg)=|H_0/H|\operatorname{Index}\bigg(\bigg(f+\sum_1^k\lambda_l A_l x\bigg)^{H_i};x_0\bigg)$$

(see [14]).

Lemma 4.1. *Assume that f is C^1 at x_0 and let $A\equiv Df(x_0)$, then:*

(1) *A is H-equivariant and for any $K<H$, $A^K=\operatorname{diag}(A^H,A_{\perp K})$, with $A^K=Df^K(x_0)$. For $K<\underline{H}$, then $A^K=\operatorname{diag}(A^H,A_{\perp\underline{H}},A'_{\perp K})$ and $A'_{\perp K}$ is self-adjoint as a complex matrix and H-orthogonal.*
(2) *A_jx_0, for $j=1,\dots,k$, are in* $\ker A$ *and are orthogonal to* $\operatorname{Range}A$. *In particular, if* $\dim\ker A=k$, *then $A_{\perp K}$ is invertible for any $K<H$, $A|_{B_k}$ is invertible and Γx_0 is hyperbolic in the sense of* [14, p. 383].

Proof. Since $Df(\gamma x)\gamma=\gamma Df(x)$, then A is H-equivariant and has the block-diagonal structure. In particular, if $K<\underline{H}$, then, since $\underline{H}<T^n$, $A'_{\perp K}$ is a complex matrix and $\dim\underline{H}/K\geq1$. Hence, if $\widetilde{A}_jx$ are the generators for the action of H, for $j=k+1,\dots,n$, then on any irreducible representation of $(V^{\underline{H}})^\perp$ one has at least one $\widetilde{A}_j$ which is invertible.

Note that if one reparametrizes T^n by letting $\psi_j=\varphi_j+\sum_{l+1}^n\lambda_j^i\varphi_i$, as in the proof of Theorem 2, then $\underline{H}$ corresponds to $\psi_1,\dots,\psi_k\equiv0$, $[2\pi]$ and one may choose $\psi_{k+1},\dots,\psi_n$ acting trivially on $V^{\underline{H}}$ and $\widetilde{A}_j$ corresponds to the derivative with respect to ψ_j, $j=k+1,\dots,n$.

Now, since f is Γ-orthogonal it is also H-orthogonal. If $f^K = (f^{\underline{H}}, f_\perp)$, then $f^K(x) \cdot \widetilde{A}_j x = f_\perp(x) \cdot \widetilde{A}_j x_\perp = 0$, for $x = x^{\underline{H}} + x_\perp$. Since $f_\perp(x^{\underline{H}}) = 0$, one has $(Df_\perp(x^{\underline{H}})x_\perp + R(x_\perp)) \cdot \widetilde{A}_j x_\perp = 0$, with $R(x_\perp) = o(|x_\perp|)$. Dividing by $|x_\perp|^2$ and taking the limit when $x_\perp$ tends to 0, one has that $Df_\perp(x^{\underline{H}})x_\perp \cdot \widetilde{A}_j x_\perp = 0$ and, in particular $A'_{\perp K}$ is H-orthogonal.

Take K corresponding to an irreducible representation such that $\widetilde{A}_j \equiv \widetilde{A}$ is invertible on it. Let $B \equiv A'_{\perp K}$, then one has $B\widetilde{A} = \widetilde{A}B$ and $BX \cdot \widetilde{A}X = 0$ for any X in that representation. Then, since $B(X + X_0) \cdot \widetilde{A}(X + X_0) = 0$, for all X, X_0, one has $\widetilde{A}^T B + B^T \widetilde{A} = 0$. But, as we have seen before, $\widetilde{A}^T = -\widetilde{A}$, hence $B^T = \widetilde{A}B\widetilde{A}^{-1} = B\widetilde{A}\widetilde{A}^{-1} = B$.

Now, since the action of H on X is as S^1, B is in fact of the form $\begin{pmatrix} \mathcal{A} & -\mathcal{B} \\ \mathcal{B} & \mathcal{A} \end{pmatrix}$ as a real matrix. Then, $B = B^T$ implies $\mathcal{A} = \mathcal{A}^T$ and $\mathcal{B} = -\mathcal{B}^T$, that is $(\mathcal{A} + i\mathcal{B})^* = \mathcal{A} + i\mathcal{B}$.

For the second part of the lemma, differentiating with respect to φ_j the relation $f(\gamma x_0) = 0$ one has $AA_j x_0 = 0$. Furthermore, from $f(x) \cdot A_j x = 0$. one obtains, for all x and x_0

$$Df(x_0)x \cdot A_j x_0 + f(x_0) \cdot A_j x = 0.$$

In particular, if $f(x_0) = 0$, then $A_j x_0$ is orthogonal to Range A. Also, if $\dim \ker A = k$, since $A_j x_0$ are independent, then $V = \ker A \oplus \operatorname{Range} A$, the algebraic multiplicity of A is k and $\ker D_{(\lambda,x)}(f(x) + \sum \lambda_j A_j x)|_{(0,x_0)}$ is generated by $(0, A_j x_0)$. Since the other two properties of hyperbolicity are clearly satisfied, one has, from [14, Proposition 3.2], the rest of the lemma. □

Theorem 4. *Let Γx_0 be an isolated orbit of dimension k and assume that $\dim \ker Df(x_0) = k$. Then, the orthogonal index of the orbit is well defined and is equal to the product:*

$$[F(x); x_0]_\perp = i_\perp(f^{\underline{H}}(x_{\underline{H}}); x_0) i_\perp(Df_\perp(x_0)\overline{X}; 0),$$

where $\underline{H}$ is the minimal isotropy subgroup contained in H such that $|H/\underline{H}| < \infty$ and $Df_\perp(x_0)\overline{X}$ is the linearization on $(V^{\underline{H}})^\perp$, which is complex self-adjoint and H-orthogonal. Furthermore,

$$i_\perp(f^{\underline{H}}) = d_H[F_H]_\perp + \sum_{H/H_i \cong \mathbb{Z}_2} d_{H_i}[F_{H_i}]_\perp + \sum_{H/\widetilde{H}_i \cong \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2} d_{\widetilde{H}_i}[F_{\widetilde{H}_i}]_\perp,$$

with $d_H = (-1)^{n_H}$, where n_H is the number of negative eigenvalues of $Df^H(x_0)$, $d_{H_i} = d_H((-1)^{n_{H_i}} - 1)/2$, where n_{H_i} is the number of negative eigenvalues of $Df_\perp^{H_i}(x_0)$ and $d_{\widetilde{H}_i}$ is given by d_H and d_{H_j} by the formula in [14, p. 381]. *Also*

$$i_\perp(Df_\perp(x_0)\overline{X}) = [F_\Gamma]_\perp - \sum n_i(K_i)[F_{K_i}]_\perp - \sum_{s=2}^{n-k} \prod n_j(K_j)[F_{\cap K_j}]_\perp,$$

where K_i are the irreducible representations of H in $(V^{\underline{H}})^{\perp}$, i.e. $H/K_i \cong S^1$ and $Df_{\perp}(x_0)$, which is block-diagonal on these representations, has a complex Morse number $n(K_i)$. In the second sum one has the product $n(K_{i_1})\dots n(K_{i_s})$ with $\dim H/K_{i_1}\cap\dots\cap K_{i_s} = s$. *Finally,* $[F_{H_i}]_{\perp}[F_{K_j}]_{\perp} = [F_{H_i\cap K_j}]_{\perp}$. *The generators used here are those of Theorem 2 with $\eta = 1$.*

Note that $d_{H_i} = 0$ unless V^{H_i} contains a real coordinate y with a $\mathbb{Z}_2$-action of Γ (and y not in V^H). In fact, if z is a complex coordinate in $V^{H_i}\cap(V^H)^{\perp}$, then real eigenvalues come in pairs. Note also that if two complex coordinates z_1 and z_2 have the same isotropy subgroup K of H, with $\dim H/K = 1$, then either z_1 and z_2 or z_1 and $\overline{z}_2$ belong to the same irreducible representation of H. By taking conjugates if necessary, we shall assume that one has only the first case and that the formula includes the sum of the Morse numbers for the z and the $\overline{z}$: as real representations they are the same, via the linear map: $z \to \overline{z}$.

Proof. The first step in the computation of the index is to find the Poincaré indices for K with $\underline{H} < K < H$. For $k = 0$, one has to use [14, Theorem 3.2], while for $k \geq 1$, one uses [14, Theorem 3.3]. Hence, $i_H = (-1)^{n_H}$ where n_H is the number of real negative eigenvalues of $Df(x_0)^H$. In fact $i_H(f) = \varepsilon(-1)^{n_H}$, where ε is a factor which depends on the orientation chosen and on the sign of the determinant of the matrix A with $A_{ij} = n_i^j$, $i,j = 1,\dots,k$. But, from Proposition 4.1, one has $i_H(f) = d_H i_H(F_H)$. By construction $i_H(F_H)$ is 1, since sign $\det(DF_H)_k = 1$, hence $\varepsilon = 1$. From the product formula, $i_K = i_H(-1)^{n_K}$, for $K < H$, where n_K is the number of real negative eigenvalues, counted with multiplicities, of $A_{\perp K}$, hence $i_K = i'_K$ if $K < K'$ and K'/K doesn't contain a $\mathbb{Z}_2$-factor. From Proposition 4.1, one gets $d_H = i_H, d_{K_j} = (i_{K_j} - i_H)/2$ if $H/K_j \cong \mathbb{Z}_2$ and d_K is completely determined by the above integers if $H/K \cong \mathbb{Z}_2\times\dots\times\mathbb{Z}_2$ and $d_K = 0$ otherwise.

Before computing d_K for K with $\dim H/K > 0$, let us look at some examples.

1. Let $\mathbb{Z}_2$ act on y as an antipodal map and S^1 act on z by $e^{i\varphi}$. Then, the map $f(y,z) = (-y, (|z^2|-1)z)$ is equivariant with respect to $\Gamma = \mathbb{Z}_2\times S^1$ and has the isolated zero-orbit $y = 0$, $|z| = 1$, with $H = \mathbb{Z}_2$ and $K = \{e\}$. $Df^H(0, z = 1) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and since $k = 1$ one has $i_H = -1$, $i_K = 1$. Note that $f = \nabla\Phi$, with $\Phi(y,z) = -y^2/2 + (|z|^2/2 - 1)|z|^2/2$.

2. Let $\Gamma = S^1$ act on (z_1, z_2) by $(e^{i\varphi}z_1, e^{2i\varphi}z_2)$ and let $f(z) = f_0(z) - \lambda(z)Az$, with $f_0(z) = (z_2\overline{z}_1, (|z_2|^2-1)z_2)$, $Az = (iz_1, 2iz_2)$ and $\lambda(z) = f_0(z)\cdot Az/|Az|^2 = (z_2\overline{z}_1^2 - z_1^2\overline{z}_2)/2i(|z_1|^2 + 4|z_2|^2)$, which is real. Note that if $F = (f_1,\dots,f_N)\in\mathbb{C}^N$ and $Ax = (in_1z_1,\dots,in_Nz_N)$, then $F\cdot Ax = \operatorname{Im}\,(n_1f_1\overline{z}_1 + \dots + n_Nf_N\overline{z}_N)$. Thus, here if $f(z) = 0$, then from the orthogonality, $z_2\overline{z}_1 = 0$ and $(z_1 = 0,\ |z_2| = 1)$ is an isolated orbit, for which $Df^H(0,1) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$. It is then easy to compute the

index of $f(z) + \lambda Az$ at $\lambda = 0$, $z_1 = 0$, $z_2 = 1$, by deforming $\lambda(z)$ to 0, getting $i_H = -1$, $i_K = 1$, with $H = \mathbb{Z}_2, K = \{e\}$.

3. If f is in normal form, then $f^{\perp}(x^H, x_\perp) = x_\perp$, for $|x_\perp| < \varepsilon$ and $Df^{\perp}(x^H, 0) = \mathrm{Id}$, hence $i_K = i_H$ for any $K < H$. In this case $d_H = i_H$, $d_K = 0$ for $K < H$, $K \geq \underline{H}$. Then, $[F^{\underline{H}}]_\perp - d_H[F_H^{\underline{H}}]_\perp \equiv [F_1^{\underline{H}}]$ is non-zero and, since F is normal, one may complement $F_1^{\underline{H}}$ by $x_{\underline{H}}^{\perp}$, obtaining that $[F]_\perp - d_H[F_H]_\perp = 0$, i.e., $d_K = 0$ for any other K. Thus, Gęba's degree and ours coincide.

We may now go to the second step of the proof. Let $\lambda_j(x) = D_{x_\perp} f_\perp(x_H)x_\perp \cdot \widetilde{A}_j(x)$, for $j = 1, \ldots, k$, where $\widetilde{A}_j(x)$ are orthonormal as in Lemma 1.1 but starting the orthogonalization process from $j = n$, i.e. in reverse order. Here we are assuming that one has reparametrized the torus T^n in such a way, as in Lemma 4.1, that $A_j x_{\underline{H}} = 0$ for $j = k+1, \ldots, n$. Hence, for $j > k$, $\widetilde{A}_j(x)$ are in $(V^{\underline{H}})^\perp$ and orthogonal to $D_{x_\perp} f_\perp(x_H)x_\perp$ since $A_j x_H = 0$ and $D_{x_\perp} f_\perp(x_H)x_\perp$ is H-orthogonal. Furthermore, for $j \leq k$, since $\widetilde{A}_j(x) = \widetilde{A}_j(x_H) + 0(x_\perp)$, then, in the neighbourhood of x_0 where $A_j x_0$ are linearly independent, one has $\lambda_j(x) = 0(|x_\perp|^2)$ and $\widetilde{A}_j(x)_\perp = 0(x_\perp)$. Consider the homotopy

$$(f^H(x_H, \tau x_\perp), \tau f_\perp(x_H, \tau x_\perp)) + (1 - \tau^2)\Big(D_{x_\perp} f_\perp(x_H)x_\perp - \sum_1^k \lambda_j(x)\widetilde{A}_j(x)\Big),$$

on the tubular neighbourhood of the orbit Γx_0. It is clear that the first term in the homotopy is Γ-orthogonal, while the second term is built so that it is orthogonal to $A_j x$. The equivariance is clear. If the neighbourhood Ω of Γx_0 is of the form $\{(x_H, x_\perp) : \mathrm{dist}(x_H, \Gamma x_0) < \eta, |x_\perp| < \varepsilon\}$, since the homotopy reduces, for $x_\perp = 0$, to $(f^H(x_H, 0), 0)$ which is non-zero on the boundary of Ω (since Γx_0 is isolated) and the second component is linearized to $D_{x_\perp} f_\perp(x_H)x_\perp + \tau^2 o(x_\perp) + (1-\tau^2)0(|x_\perp|^3)$, hence one may choose ε so small that this component is non-zero for $|x_\perp| = \varepsilon$ (recall that $D_{x_\perp} f_\perp(x_H)$ is invertible at x_0 and hence in Ω).

Now, $D_{x_\perp} f_\perp(x_H)$ has the form $(B(x_H), \overline{B}(x_H))$, where $\overline{B}$ is complex self-adjoint and has a block diagonal structure on the equivalent irreducible representations of H. On each block, $\overline{B}(x_H)$ is similar to a diagonal real matrix $\Lambda(x_H)$, with a well defined Morse index n_K (i.e. the number of negative eigenvalues. Note that, as a real matrix, the Morse number of $\overline{B}(x_H)$ is $2n_K$). If v is an eigenvector of $\overline{B}(x_H)$, then γv is an eigenvector of $\overline{B}(\gamma x_H)$ with the same eigenvalue, hence if $\overline{B}(x_H) = U(x_H)\Lambda(x_H)U^*(x_H)$, with U unitary, then $U(\gamma x_H) \equiv \gamma U(x_H)\gamma^*, \Lambda(\gamma x_H) = \gamma\Lambda(x_H)\gamma^* = \Lambda(x_H)$ will diagonalize $\overline{B}(\gamma x_H)$, since Λ and γ are diagonal, hence commute. Note that $U(x_H)$ is continuous in x_H if the eigenvalues of $B(x_0)$ are simple. In general, for x_H in $\mathcal{C}_H$, the fundamental cell for H, and close to x_0, define $\widetilde{U}(x_H) = U(x_0)$ and $\widetilde{U}(\gamma x_H) = \gamma U(x_0)\gamma^*$. Let $\widetilde{\Lambda}(x_H) = \widetilde{U}^*(x_H)\overline{B}(x_H)\widetilde{U}(x_H)$, then $\widetilde{\Lambda}(\gamma x_H) = \gamma\widetilde{\Lambda}(x_H)\gamma^*$ is close to $\Lambda(x_0)$ for

x_H close to x_0, but not necessarily diagonal. Now, the space of unitary complex matrices is path-connected, hence one may choose a path $U_\tau(x_0)$ from $U(x_0)$ to I and therefore, from $\widetilde{U}(\gamma x_H)$ to I and from $\overline{B}(x_H)$ to $\widetilde{\Lambda}(x_H)$ which is linearly deformable to $\Lambda(x_0)$. By modifying $\lambda_j(x)$ along the deformations, one obtains an equivariant and Γ-orthogonal homotopy to

$$(f^H(x_H), B(x_H)X, \Lambda\overline{X}) - \sum_1^k \widetilde{\lambda}_j(x)\widetilde{A}_j(x),$$

where $x_\perp$ is written as $X + \overline{X}$ and $\widetilde{\lambda}_j(x) = B(x_H)X \cdot \widetilde{A}_j(x)$ since Λ is real and diagonal hence orthogonal to A_jx for all j and to the corresponding components of $\widetilde{A}_j(x)$. This last fact implies that one may take $\overline{X}$ to 0 in $\widetilde{A}_j(x)$ and still get an orthogonal homotopy. Hence one has arrived to the map:

$$(f^H(x_H), B(x_H)X, \Lambda\overline{X}) - \sum_1^k \widetilde{\lambda}_j(x_{\underline{H}})\widetilde{A}_j(x_{\underline{H}})$$

or, equivalently to $(f^{\underline{H}}(x_{\underline{H}}), \Lambda\overline{X})$, which is a product map. Note that, if one had linearized f at $x_{\underline{H}}$, instead of x_H, then the matrix $Df_{\overline{X}}(x_{\underline{H}})$ would be $\underline{H}$-equivariant and would give larger blocks, however the end result would be the same.

Now, the orthogonal Γ-index of $f^{\underline{H}}(x_{\underline{H}})$ at x_0 has been computed in the first step. It remains to compute the orthogonal index of $\Lambda\overline{X}$ at 0 and to apply the product Theorem 3.3.

It is clear that Λ may be deformed to $\operatorname{diag}(-I, I)$, where one deforms linearly each eigenvalue of $\overline{B}(x_0)$ to -1 or 1 according to its sign. The I-part acts as a suspension and does not affect the degree, while any $-z$ can be changed to $(1-|z|^2)z$ and one gets the sum of degrees on sets of the form $\{|z_j| < 1/2$, $j = 1, \dots, l, |z_j| > 1/2$ for $j > l\}$. For $|z_j| < 1/2$ one may deform back to z_j and obtain a suspension. Hence one is reduced to compute the orthogonal degree on sets of the form $\widetilde{\Omega} \equiv \{|z_j| > 1/2, j = 1, \dots, l\}$ of the map $(\dots, (1-|z_j|^2)z_j, \dots)$. Let H_j be the isotropy subgroup of z_j (by construction $\Gamma/H_j \cong S^1$), let $K = \bigcap_1^l H_j$, with $\dim \Gamma/K = s$, and let K_0 be the intersection of s of the H_j such that $\dim \Gamma/K_0 = s$ (say the first s). Then, from Proposition 4.1, the orthogonal degree with respect to $\widetilde{\Omega}$ is given by $[F]_\perp = \sum_{K<K_j<K_0} d_j[F_j]$, where d_j is given by the relations:

$$\begin{aligned}\deg\,([(1-|z_1|^2)z_1, \dots, (1-|z_l|^2)z_l]^{K_i} + \sum_1^s \lambda_l(A_lx)^{K_i}; \widetilde{\Omega}_k^i)& \\ = \sum_{K_i<K_j<K_0} d_j|K_0/K_j|,&\end{aligned}$$

as a map $(\lambda_1, \ldots, \lambda_s, z_1 > 0, \ldots, z_s > 0, z \in \widetilde{\Omega}^{K_i})$. Since on $\widetilde{\Omega}$ all z_j are non-zero, all the degrees on the left are 0, except for $K_i = K$. For K, since $A_1 x^{K_0}, \ldots, A_s x^{K_0}$ are linearly independent, one may deform $A_l z_j$ to 0 for $j > s$ (if of course $s < l$) and to $(1 - |z_l|^2) z_l$ one may add $i\tau z_l$. Hence d_j are all 0 if $s < l$, while if $s = l$, one has to compare the indices of the maps $(2t + 1 - 2\prod |z_i|, (|z_2|^2 - 1)z_1, \ldots, (|z_s|^2 - 1)z_{s-1}, \eta(2t - 1)z_s) + \sum \lambda_j A_j x$ and $(2t + 2\varphi(x) - 1, (1 - |z_1|^2)z_1, \ldots, (1 - |z_s|^2)z_s) + \sum \lambda_j A_j x$, where the first map is the generator and η is chosen such that the index is 1 and where $\varphi(x)$ is 1 if one of the z_i has norm less than 1/4 and is 0 if all z_i have norm larger than 1/2. An easy deformation, for z_i real and positive, of the first map to $(1 - z_1, z_2 - 1, \ldots, z_s - 1, \eta(2t - 1))$ and of the second to $(2t - 1, 1 - z_1, \ldots, 1 - z_s)$ will give an index of the first map equal to the number $-(-1)^s \eta\varepsilon$ Sign det A and of the second equal to $(-1)^s \varepsilon$ Sign det A, where ε is an orientation factor. Hence $i(f) = d_{K_0} i(F_{K_0})$ and $d_{K_0} = -\eta$.

Thus, $[F]_\perp = -[F_{K_0}]_\perp$, where the generator F_{K_0} is chosen with $\eta = 1$. Collecting all terms, one obtains:

$$i_\perp(\Lambda \overline{X}) = [F_\Gamma]_\perp - \sum n_i [F_{K_i}]_\perp - \sum_{s>1} \Big(\prod n_i\Big) [F_{\bigcap K_i}]_\perp,$$

where K_i is the isotropy subgroup of the ith coordinate in $(V^{\underline{H}})^\perp$ (by construction $\dim \Gamma/K_i = 1$), the first sum takes into account the K_i which are different and n_i is the number of those coordinates, with the same K_i, for which Λ contributes a -1. The second sum is over those K_i such that $\dim \Gamma/K_{i_1} \cap \ldots \cap K_{i_s} = s$. The map $F_\Gamma = (2t - 1, \overline{X})$ corresponds to the degree on the set where all z are small.

It is then enough to apply Theorem 3.3 for the product, noting that $\widetilde{H}_2^0 = H_2 = K_j$ or $K_{i_1} \cap \ldots \cap K_{i_s}$ so that one has the usual product of integers. Finally, since $H_i < H$, if $K < H$ gives an irreducible representation of H in $(V^{\underline{H}})^\perp$ and a block for $\overline{B}(x_0)$ with complex Morse index $n(K)$, then for any K_j isotropy subgroup of Γ of a coordinate z_j in the block, one has $H_i \cap K_j = H_i \cap K$ and $\sum n_j = n(K)$ with $\dim \Gamma/H_i \cap K = k + 1$. □

REMARK 4. Instead of using the product theorem, one could have followed the arguments of [13], that is replace $\overline{B}(x_H)$ by terms, on each of its blocks, of the form $(1 - \psi_i)\overline{B}_i(x_H)\overline{X}_i + \psi_i \overline{X}_i$, with the corresponding modifications of $\widetilde{\lambda}_j(x_H)$. Since $[F^{\underline{H}}]_\perp = \sum d_j [F_j^{\underline{H}}]_\perp$ for $\underline{H} < H_j < H$, one may consider

$$\widetilde{F}_j = (F_j^{\underline{H}}, (1 - \psi_i)\overline{B}_i(x_H)\overline{X}_i + \psi_i(\overline{X}_i) + \sum \lambda_j(x) \widetilde{A}_j(x)_i),$$

then $[F]_\perp = \sum d_j[\widetilde{F}_j]_\perp$ and it is enough to compute $[\widetilde{F}_j]_\perp$, which turns out to be

$$[F_j]_\perp - \sum_i n(K_i)[F_{H_j\cap K_i}]_\perp - \sum_{s>1}\prod n(K_i)[F_{H_j\cap K_{i_1}\cap\ldots\cap K_{i_s}}]_\perp$$

via a direct, but very lengthy, computation of the generators.

5. Bifurcation

Consider a family $f(\lambda, x)$ of orthogonal C^1-maps, with $f(\lambda, 0) = 0$, $\lambda \in \mathbb{R}^k$, $x \in V$. As seen in Lemma 4.1, if one writes $f(\lambda, x) = Df(\lambda, 0)x + R(\lambda, x) = B(\lambda)x + R(\lambda, x)$, then $B(\lambda)x$ and $R(\lambda, x)$ are both equivariant and orthogonal. $B(\lambda)$ has a block-diagonal structure on the irreducible representations of Γ and is complex self-adjoint on $(V^{T^n})^\perp$.

Assume that $B(\lambda)$ is invertible for $\lambda \neq 0$ in a neighbourhood of 0, then $\deg_\perp((|x| - \varepsilon, f(\lambda, x)); B_{2\varrho} \times B_{2\varepsilon})$ is well defined, where $B_{2\varrho} = \{\lambda : |\lambda| < 2\varrho\}$ and $B_{2\varepsilon} = \{x : |x| < 2\varepsilon\}$. Furthermore, one may deform linearly R to 0 (this is an orthogonal deformation). Then,

$$\deg_\perp((|x| - \varepsilon, B(\lambda)x); B_{2\varrho} \times B_{2\varepsilon}) = \deg_\perp((\varrho^2 - |\lambda|^2, B(\lambda)x); B_{2\varrho} \times B_{2\varepsilon})$$

will give the standard results on local and global bifurcation (see [10]).

If $k = 1$, this degree is $\deg_\perp(B(-\varrho)x; B_{2\varepsilon}) - \deg_\perp(B(\varrho)x; B_{2\varepsilon})$, from the product theorem. Hence, one has to compare the orthogonal indices at 0 of $B(\pm\varrho)x$ given in Theorem 4. For an invertible orthogonal matrix B, let $\sigma_\Gamma \equiv \operatorname{Sign}\det B^\Gamma$, $\sigma_H \equiv \operatorname{Sign}\det B^H_\perp$, if $\Gamma/H \cong \mathbb{Z}_2$ and $B^H_\perp$ is B restricted to $(V^\Gamma)^\perp \cap V^H$ and n_K be the complex Morse number of $B^K_\perp$, where K is the isotropy for some coordinate z, with $\dim \Gamma/K = 1$, and $B^K_\perp$ is B restricted to $(V^{T^n})^\perp \cap V^K$. Then,

$$\begin{aligned} i_\perp(Bx) = (-1)^{\sigma_\Gamma}\Bigg\{&[F_\Gamma]_\perp + \sum_{\Gamma/H_i\cong\mathbb{Z}_2}((-1)^{\sigma_{H_i}} - 1)/2[F_{H_i}]_\perp \\ &+ \sum_{\Gamma/\widetilde{H}_i\cong\mathbb{Z}_2\times\ldots\times\mathbb{Z}_2} d_{\widetilde{H}_i}[F_{\widetilde{H}_i}]_\perp\Bigg\} \\ &\times\Big\{[F_\Gamma]_\perp - \sum n_{K_i}[F_{K_i}]_\perp - \sum\prod n_{K_j}[F_{\bigcap K_j}]_\perp\Big\}. \end{aligned}$$

Theorem 5.1. *If $k = 1$, one has global bifurcation, i.e. the continuum of non-trivial solutions emanating from $(0, 0)$ is unbounded or returns to $(\lambda, 0)$ with $\lambda \neq 0$:*

- *in V^Γ, if det B^Γ changes sign, or*
- *in V^{H_i}, if det $B^{H_i}_\perp$ changes sign, or*
- *in V^{K_i}, if n_{K_i} changes.*

Furthermore, if the continuum is bounded and the bifurcation points on it are isolated, then the sum of the jumps of the orthogonal indices is 0*. Finally, if* $\det B^{\Gamma}$*,* $\det B_{\perp}^{H_i}$ *and* n_{K_i} *do not change, then there is an orthogonal nonlinearity* $\widetilde{R}(\lambda, x)$ *such that* $B(\lambda)x + \widetilde{R}(\lambda, x)$ *is zero only at* $x = 0$*.*

Note that this result generalizes the results for an S^1-action given in [12] and [18]. Note also that the corresponding theorem in [14, Theorem 5.1], was proved by using the J-homomorphism and phrased differently with respect to the action of $-y$, but one may recover part of the present result by applying [14] to $B(\lambda)x + \mu Ax$.

Note that n_K changes if $B_{\perp}^{K} = \lambda B$ and B has a non-zero signature, for example, if $B = I$. Finally, for the correct application of this result, it is important to assimilate complex conjugate representations (they are the same real representations) as the following example shows. Let Γ act on z_1 as $e^{i\varphi}$ and on z_2 as $e^{-i\varphi}$. Consider the orthogonal Γ-map $(\lambda z_1 + t\overline{z}_2, -\lambda z_2 + t\overline{z}_1)$, with $t = |z_1|^2 + |z_2|^2$, (here $Az = i(z_1, -z_2)$). It is easy to see that this map has no zeros except $z_1 = z_2 = 0$, i.e. there is no bifurcation, although the Morse numbers for z_1 and z_2 change but their sum remains invariant.

The last part of the theorem will be proved below together with the general case.

Let us turn now to the case of several parameters. Consider the equation

$$f(\lambda, x) = Ax - T(\lambda)x - g(\lambda, x) = 0,$$

where x is in B, A is a Fredholm operator of index 0 from B into E, both Hilbert Γ-spaces, $B \subset E$, $\|T(\lambda)\|$ tends to 0 as λ goes to 0 and $g(\lambda, x) = o(\|x\|)$ uniformly in λ. The map $f(\lambda, x)$ is assumed to be Γ-orthogonal (with respect to the scalar product in E). Then, the Liapunov–Schmidt reduction, see [10, p. 346], implies that for λ small enough:

$$f(\lambda, x) = (A - QT(\lambda))H(\lambda, x_1, x_2) \oplus B(\lambda)x_1 + G(\lambda, x) - (I - Q)T(\lambda)H(\lambda, x_1, x_2)),$$

where $x = x_1 + x_2$, with x_1 in $\ker A$, x_2 in a complement, Q is a projection from E into $\operatorname{Range} A$ and

$$\begin{aligned} H(\lambda, x_1, x_2) &= x_2 - (I - KQT(\lambda))^{-1} KQ(T(\lambda)x_1 + g(\lambda, x)), \\ B(\lambda) &= -(I - Q)T(\lambda)(I - KQT(\lambda))^{-1} P, \\ G(\lambda, x) &= -(I - Q)(I - T(\lambda)KQ)^{-1} g(\lambda, x), \end{aligned}$$

where K is the pseudo-inverse of A and $Px = x_1$.

The equation $f(\lambda, x) = 0$, with $g(\lambda, x)$ Lipschitz continuous in x, is equivalent to $H(\lambda, x_1, x_2) = 0$, which is uniquely solved for $x_2(x_1, \lambda)$ with a contraction argument, and the bifurcation equation

$$B(\lambda)x_1 + G(\lambda, x_1 + x_2(x_1, \lambda)) = 0.$$

Lemma 5.1. *Under the above hypothesis, one may choose P and Q such that the bifurcation equation is Γ-orthogonal.*

Note that the gradient case was treated in [10, p. 358].

Proof. As above, the orthogonality of $f(\lambda, x)$ implies that of A, $T(\lambda)$ and $g(\lambda, x)$. In particular, $A - T(\lambda)$ has a diagonal structure on equivalent irreducible representations of Γ and, on $(E^{T^n})^\perp$, its restriction has a complex self-adjoint form $\widetilde{A} - \widetilde{T}(\lambda)$ and the above space has the decomposition $\ker \widetilde{A} \oplus \operatorname{Range} \widetilde{A}$. As in [10, p. 413], one may choose P and Q equivariant, hence K and $B(\lambda)$ will be equivariant and will commute with A_j. Furthermore, one may choose an orthogonal projection $\widetilde{P}$ on $\ker \widetilde{A}$, with $\widetilde{Q} = I - \widetilde{P}$, hence the part of $B(\lambda)$ on $\ker A \cap (E^{T^n})^\perp$ will be $\widetilde{B}(\lambda) = -\widetilde{P}\widetilde{T}(I - \widetilde{K}(I - \widetilde{P})\widetilde{T})^{-1}\widetilde{P}$ which commutes with A_j and is self-adjoint (expand the inverse in power series). Hence $B(\lambda)$ is orthogonal.

On the other hand,

$$-(G(\lambda, x), A_j x_1) = (g, A_j x_1) + (\widetilde{Q} g, \widetilde{K}\widetilde{T}(I - \widetilde{Q}\widetilde{K}\widetilde{T})^{-1} A_j x_1),$$

by using the fact that A_j is 0 on E^{T^n} and has also a diagonal structure. Since g is orthogonal, one may replace the first term by $-(g, A_j x_2)$. But $x_2(x_1, \lambda)$ is such that $Qg = (A - QT)(x_1 + x_2)$, hence, using the fact that A is orthogonal and Q commutes with A_j, one obtains $(QTx_1, A_j x_2)$. The same substitution in the second term yields

$$((I - \widetilde{T}\widetilde{K}\widetilde{Q})^{-1}\widetilde{T}\widetilde{K}(A - QT)x_2, A_j x_1) - (x_1, \widetilde{T}\widetilde{Q}\widetilde{K}\widetilde{T}(I - \widetilde{Q}\widetilde{K}\widetilde{T})^{-1} A_j x_1),$$

where the first term reduces to $(\widetilde{T}x_2, A_j x_1)$ and the second is 0 since it is of the form $(x_1, LA_j x_1)$, with L self-adjoint (as we have seen orthogonality is equivalent to self-adjointness for linear operators). Thus, one has $(Tx_1, A_j x_2) + (Tx_2, A_j x_1) = 0$, since T is Γ-orthogonal. $\square$

Assume that $B(\lambda)$ is invertible for $\lambda \neq 0$ small, then if $B = E$, $A = I - K$ with K, $T(\lambda)$ and g compact, so that the orthogonal degree is

$$J_\perp^\Gamma(f) \equiv \deg_\perp((\|x\| - \varepsilon, f(\lambda, x)); B_{2\varepsilon} \times B_\varrho)$$

is well defined provided $f(\lambda, x)$ is non-zero if $x \neq 0$ and $\|\lambda\| = \varrho$, or by remaining in the local context, one may deform linearly (hence orthogonally) $f(\lambda, x)$ to $Ax_2 \oplus B(\lambda)x_1 + G(\lambda, x_1 + x_2(x_1, \lambda))$ on the set $\{\|x\| = \varepsilon,\ \|\lambda\| = \varrho\}$, if one chooses ε small enough: solving the first part one gets $x_2 = 0(\|x_1\|\|\lambda\|)$ and $B(\lambda)x_1$ dominates the other terms. Then, on the same set, one may deform G to 0. In particular, $J_\perp^\Gamma(f) = J_\perp^\Gamma(Ax_2 \oplus B(\lambda)x_1)$.

It is clear that the term Ax_2 will act only as an orientation factor and as an indicator for the different isotropy subspaces. It is needed in the global results.

Recall, from [10], that one has *no linearized orthogonal local bifurcation* if there is an orthogonal Γ-nonlinearity $G(\lambda, x_1)$ such that the only zero of the bifurcation equation is $x_1 = 0$. Similarly, there is *no linearized orthogonal global bifurcation* if there is a nonlinearity $g(\lambda, x), \Gamma$-orthogonal, such that the continuum of non-trivial solutions emanating from $(0, 0)$ is bounded and does not return to $(\lambda, 0)$, with $\lambda \neq 0$ (it could reduce to $(0, 0)$).

From the fact that the Borsuk extension theorem is valid for orthogonal maps, one has, as in [10, Propositions 6.1 and 6.3].

Lemma 5.2.

(1) *One has no-linearized orthogonal local bifurcation if and only if the map* $B(\lambda)\eta : S^{k-1} \times S^{d-1} \to V \setminus \{0\}$ *has a non-zero orthogonal extension* $B(\lambda, \eta)$, *to* $B^k \times S^{d-1}$, *where* $S^{k-1} = \partial B^k = \{\lambda : \|\lambda\| = \varrho\}$, $S^{d-1} = \{\eta \in \ker A : \|\eta\| = 1\}$ *and* V *is a complement of* $\operatorname{Range} A$, *of dimension* d.
(2) *If* $k < d_0$, *the dimension of* $\ker A^\Gamma$, *and if* $J_\perp^\Gamma(C(\lambda)^\Gamma X_0, x_0, \widetilde{B}(\lambda)Z) = 0$ *implies that* $J_\perp^\Gamma(C(\lambda)^\Gamma X_0, \widetilde{B}(\lambda)Z) = 0$, *where* (X_0, x_0) *span* $\ker A^\Gamma$ *and* Z *is in the complement, then one has no linearized orthogonal local bifurcation if an only if* $J_\perp^\Gamma(B(\lambda)x) = 0$.
(3) *If* $k < 2 \dim E^\Gamma - 2$ (*with equality possible if* $d_0 < \dim E^\Gamma$), *then there is no linearized orthogonal global bifurcation if and only if* $J_\perp^\Gamma((A - T(\lambda))x) = 0$.

Now, $B(\lambda)$ has the form $\operatorname{diag}(B^\Gamma, B_j^{\mathbb{R}}, B_l^{\mathbb{C}}, \widetilde{B}_s)$ where $B_j^{\mathbb{R}}$ corresponds to equivalent irreducible representations of Γ with Γ acting as $\mathbb{Z}_2, B_l^{\mathbb{C}}$ where Γ acts as $\mathbb{Z}_p$, and $\widetilde{B}_s$ where Γ acts as S^1 and $\widetilde{B}_s = \widetilde{B}_s^*$, because of the orthogonality. Since $B(\lambda)$ is invertible for $\lambda \neq 0$, each $\widetilde{B}_s(\lambda)$ has a constant complex Morse number n_s (if $k > 1$). As noted after Theorem 5.1, complex conjugate representations are assimilated.

Let $GLS(\mathbb{C}^{n+m})$ be the set of self-adjoint invertible matrices with Morse index n. Consider the mapping $\mathcal{B} : GLS(\mathbb{C}^{n+m}) \to \Omega(GL(\mathbb{C}^{n+m}), -I, I)$, the set of paths in $GL(\mathbb{C}^{n+m})$ from $-I$ to I, given by $\mathcal{B}(B) = (1 - \mu^2)iB + \mu I$.

Lemma 5.3. *Mapping* $\mathcal{B}$ *induces an isomorphism from* $\prod_{k-1}(GLS(\mathbb{C}^{n+m}))$ *onto* $\prod_k(U(n + m))$, *provided* $0 < k - 1 \leq 2m, 2n$ *and gives the Bott periodicity.*

Proof. Since the spectrum of B is real and non-zero, it is clear that $\mathcal{B}(B)$ is invertible for all μ. Let T be unitary such that $B = T^*\Lambda T$, with $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_{n+m})$. Let $\widetilde{\Lambda} = \operatorname{diag}(\varepsilon_1, \dots, \varepsilon_{n+m})$, with $\varepsilon_j = \operatorname{sign} \lambda_j$. Let $\widetilde{B} = (i\pi/2)T^*\widetilde{\Lambda}T$, then $e^{(1-\mu)\widetilde{B}}$ is a path in $U(\mathbb{C}^{n+m})$ from $-I$ to I and in fact it is a minimal geodesic, [16, p. 127]. Furthermore, $\mathcal{B}(B)$ is linearly deformable to $e^{(1-\mu)\widetilde{B}}$, since they are simultaneously diagonalizable, then the eigenvalues for both paths are in the upper half plane if $\lambda_j > 0$. Conversely, for any skew-hermitian matrix $\widetilde{B}$, giving a minimal geodesic, one may construct B, for instance with $|\lambda_j| = 1$, such that $\mathcal{B}(B)$ and $e^{(1-\mu)\widetilde{B}}$ are in the same homotopy class, where B and $-i\widetilde{B}$ have the same Morse number.

Furthermore, the assignment of the negative eigenspace of B (or of $-i\widetilde{B}$) is a strong deformation retract of $GLS(\mathbb{C}^{n+m})$ to $G_n(\mathbb{C}^{n+m})$ the complex Grassmanian of n-planes in $\mathbb{C}^{n+m}$, see [16, p. 127], [2, Lemma 4.3]. Bott's theorem, [16, Theorem 23.3], gives that, if $n = m$, the map $\widetilde{B} \to e^{(i-\mu)\widetilde{B}}$ induces an isomorphism from $\prod_{k-1}(G_n(\mathbb{C}^{2n}))$ onto $\prod_k(SU(2n))$, if $k-1 \leq 2n$, and onto $\prod_k(U(2n))$, if $k \neq 1$. Also, the suspensions by I and $-I$ induce isomorphisms from $\prod_{k-1}(GLS(\mathbb{C}^{n+m}))$ onto $\prod_{k-1}(GLS(\mathbb{C}^{n+m+1}))$, provided $k-1 \leq 2m$ and $k-1 \leq 2n$, respectively ([9, Theorem 8.2.6, p. 102]), where this result is phrased in terms of Grassmanians. Note that changing B into $-B$, interchanges n and m. Thus, by suspending by $-I^{m-n}$ if $n < m$, or by I^{n-m} if $n > m$, then

$$\prod_{k-1}(GLS(\mathbb{C}^{n+m})) \cong \prod_{k-1}(GLS(\mathbb{C}^{2\alpha})) \cong \prod_{k}(U(2\alpha)) \cong \prod_{k}(U(n+m))),$$

if k satisfies the conditions of the lemma and $\alpha = \max(n,m)$, in particular one is in the stable range for $U(n+m)$, [16, Lemma 23.4]. Note also that, since $U(n+m)$ is a strong deformation retract of $GL(\mathbb{C}^{n+m})$, then the path spaces based on them have the same property. This gives the first part of the lemma.

Finally, using long exact sequences, [20, Theorem 10.16], one has that if $n \leq m$, then $\prod_{k-1}(G_n(\mathbb{C}^{n+m})) \cong \prod_{k-1}(V_{n+m,n}) \times \prod_{k-2}(U(n))$, where $V_{n+m,n}$ is the Stiefel manifold. Hence, for $k-1 \leq 2n, 2m$, one has $\prod_{k-1}(GLS(\mathbb{C}^{n+m})) \cong \prod_{k-2}(U(n))$, [9, p. 83]. Hence, if $0 < k-1 \leq 2n \leq 2m$, one gets an isomorphism from $\prod_{k-2}(U(n))$ onto $\prod_k(U(n+m))$. These groups are 0 if k is even and $\mathbb{Z}$ if k is odd. If $k = 1$, then $GLS(\mathbb{C}^{n+m})$ is connected and $SU(n+m)$ is simply connected. The set of self-adjoint invertible matrices has its connected components characterized by their Morse index: B is deformable to $\operatorname{diag}(-I, I)$ by deforming Λ to that matrix and T to Id. $\square$

We are now ready for the main result of this section. Because of space considerations, we shall stick to the stable case. Recall that $\widetilde{B}_s$ are complex self-adjoint invertible matrices with Morse index n_s and dimension $n_s + m_s$.

Theorem 5.2. *Assume $k-1 \leq 2n_s, 2m_s$ for all s, then there is no linearized orthogonal local bifurcation if and only if:*

(1) *There is no linearized equivariant local bifurcation in* $(\ker A)^{T^n}$.
(2) *If k is odd, if $\widetilde{B}_s(\lambda)$ is deformable in $GLS(\mathbb{C}^{n_s+m_s})$ to* $\operatorname{diag}(-I_{n_s}, I_{m_s})$. *If k is even,* (1) *is the only condition.*

Note that if $k = 1$, then (2) says that $\widetilde{B}_s(\pm\varrho)$ have the same Morse index, i.e. the situation of Theorem 5.1. Furthermore, if B^Γ is a $d_0 \times d_l$ matrix, $B_j^{\mathbb{R}}$ is $d_j \times d_j$ and $B_l^{\mathbb{C}}$ is a complex $d_l \times d_l$ matrix, then, if $k < d_0, d_j$, $k \le 2d_l$ for all j, l, one may apply [10, Theorem 6.1, p. 436], to verify (1). In particular, if k is odd, one needs B_0, B_j to be deformable to I, since $B_l^{\mathbb{C}}$ is always deformable to I.

Note also that, if one is not in the stable case, one may add the required number of $\pm Z_s$, in order to get $k - 1 \le 2n_s, 2m_s$. Then, if k is odd and $\widetilde{B}_s$ is not stably deformable to $\operatorname{diag}(-I, I)$, one has local (in fact global) bifurcation. The same sort of results holds for $(\ker A)^{T^n}$.

Proof. If $B(\lambda)\eta$ has an orthogonal extension $B(\lambda, \eta)$, from $S^{k-1} \times S^d$ to $B^k \times S^d$, then $(B(\lambda)\eta)^{T^n} \equiv B_0(\lambda)\eta_0$ has the equivariant extension $B(\lambda,\eta)^{T^n} = B_0(\lambda, \eta_0)$, from $S^{k-1} \times S^l$ to $B^k \times S^l$, where $\eta_0 \in S^l$ in $(\ker A)^{T^n}$. Hence, from [10, Proposition 6.1, p. 431], (1) is verified. Furthermore, if T is any of the $(n-1)$-tori used in Step 2 of the proof of Theorem 2, one has a similar orthogonal extension for $(B(\lambda)\eta)^T = (B_0(\lambda)\eta_0, \widetilde{B}(\lambda)\widetilde{\eta})$, where η_0 belongs to $(\ker A)^{T^n}$ and $\widetilde{\eta}$ to its complement. The group T^n acts as S^1 on $\widetilde{\eta}$ and, for some j, one has $A_j\widetilde{\eta}_s = in_s\widetilde{\eta}_s$, with $n_s > 0$ by taking conjugates, and $\widetilde{\eta}_s$ is in the respective space of equivalent irreducible representations.

Now, $(\|\eta_0\|B_0(\lambda, \eta_0/\|\eta_0\|), \widetilde{B}(\lambda)\eta)$, has a non-zero orthogonal extension from $S^{k-1} \times \partial(B^l \times B^{\widetilde{d}})$ to $B^k \times \partial(B^l \times B^{\widetilde{d}})$, since, on the first set this map is linearly (and orthogonally) deformable to $(B_0(\lambda)\eta_0, \widetilde{B}(\lambda)\widetilde{\eta})$. Furthermore, it is easy to check that the arguments of [10, Proposition 6.2, p. 432 and Remark 6.3, p. 434], are valid in the orthogonal case, by looking at the explicit construction and using the fact that the orthogonality is used only on $\widetilde{\eta}$ and that the Borsuk extension theorem is valid for orthogonal maps. Hence, the above map has this extension property if and only if it extends orthogonally from $\partial(B^k \times B_0^l) \times S^{\widetilde{d}-1}$ to $B^k \times B_0^l \times S^{\widetilde{d}-1}$. Let then $(B_0(\lambda, \eta_0, \widetilde{\eta}), \widetilde{B}(\lambda, \eta_0, \widetilde{\eta}))$ be this extension. But then, $(B_0(\lambda, \eta_0, \widetilde{\eta}), \widetilde{B}(\lambda, \eta_0, \widetilde{\eta}) + \mu A_j\widetilde{\eta})$ is equivariant and non-zero on $B^{k+1} \times B_0^l \times S^{\widetilde{d}-1}$, where in B^{k+1} we have added μ, with $|\mu| \le 1$. This map is an extension of its restriction on $\partial(B^{k+1} \times B_0^l) \times S^{\widetilde{d}-1}$. Again, from [10, Remark 6.3], which is true in this context, this last map has a non-zero Γ-extension if and only if $(\|\eta_0\|B_0(\lambda, \eta_0/\|\eta_0\|),\ \widetilde{B}(\lambda)\widetilde{\eta} + \mu A_j\widetilde{\eta})\Gamma$-extends from $S^k \times \partial(B_0^l \times B^{\widetilde{d}})$ to $B^{k+1} \times \partial(B_0^l \times B^{\widetilde{d}})$. Furthermore, one may adapt [10, Theorem 6.1, p. 436], to conclude that, in the stable case of the theorem, the family of matrices from S^k into $GL(\mathbb{C}^{d_s})$ given by $\widetilde{B}_s(\lambda) + \mu A_j$ must be deformable to the identity, since $k + 1 \le 2(n_s + m_s)$, provided n_s and m_s are not 0. Then, from Lemma 5.3, one has that $\widetilde{B}_s(\lambda)$ is deformable to $\operatorname{diag}(-I_n, I_m)$ in $GLS(\mathbb{C}^{n_s+m_s})$. Note that this is always the case if k is even. If n_s or m_s is 0, then, from the conditions of the theorem, one has $k = 1$, one has to replace $U(n+m)$ by $SU(n)$ and $\widetilde{B}_s(\lambda) + \mu A_j$

is trivial in $\prod_1(U(n))$ if and only if its complex determinant has zero winding number, which is the net number of eigenvalues of $\widetilde{B}_s(\lambda)$ which change sign as λ goes from $-\varrho$ to ϱ, that is, up to a sign, the difference of the Morse numbers. This argument has been used extensively in our previous papers.

The converse follows from the fact that $(B_0(\lambda)\eta_0, \widetilde{B}(\lambda)\widetilde{\eta})$ is orthogonally deformable to $(B_0(\lambda)\eta_0, \varepsilon_s\widetilde{\eta}_s)$ on $S^{k-1} \times S^{d-1}$, where $\varepsilon_s = \operatorname{diag}(-I_{n_s}, I_{m_s})$ which has the orthogonal non-zero extension $(\|\eta_0\|B_0(\lambda, \eta_0/\|\eta_0\|)), \varepsilon_s\widetilde{\eta}_s)$ to $B^k \times S^{d-1}$. The conclusion follows from the orthogonal Borsuk extension result and Lemma 5.2. □

If $f(\lambda, x)$ is a gradient and $\Gamma = \{e\}$, Bartsch, in [2], has found the same increase of the number of parameters given in Lemma 5.3, for the Conley index and the real Grassmanians and real Bott periodicity. In his case, as usual with Conley index, one has no continua.

6. Periodic solutions of Hamiltonian systems

As an illustration of the preceding results, we shall give an idea of how to study the problem of finding 2π-periodic solutions to

$$f(X) \equiv JX' + \nabla H(X) = 0,$$

where $X \in \mathbb{R}^{2N}$, J is the standard symplectic matrix and H is C^2. (Note that by rescaling time, there is no loss of generality when one looks for 2π-periodic solutions instead of a fixed period T).

Assume that the abelian group Γ_0 acts symplectically on $\mathbb{R}^{2N}$, i.e. it commutes with J or, if $X = (Y, Z)$ with Y and Z in $\mathbb{R}^N$, then the action on Y and Z are the same. If one of the complex irreducible representations of Γ_0 associates one coordinate of Y to its similar in Z, then J, on this pair, takes the form of a multiplication by i. Assume that H is invariant under Γ_0, hence $\nabla H(X)$ is equivariant, as well as the term JX'. Hence, if $B = H^1(S^1)$ and $E = L^2(S^1)$, for 2π-periodic functions, the equation is Γ-equivariant, for $\Gamma = S^1 \times \Gamma_0$, where the action of S^1 is by time translation.

The infinitesimal generators for Γ will be $AX \equiv X'$ for the action of S^1 and A_jX, $j = 1, \dots, n$, if the rank of Γ_0 is n. It is easy to see that $(f(X), AX) = \int_0^{2\pi}(JX' \cdot X' + \nabla H(X) \cdot X')\,dt = 0$, while $\nabla H(X) \cdot A_jX = 0$ (since H is Γ_0-invariant) and $(JX', A_jX) = \int_0^{2\pi} -(X^T JA_jX)'\,dt/2 = 0$, where we have used the relations $J^T = -J$, $A_j^T = -A_j$, $JA_j = A_jJ$ (since Γ_0 commutes with J). Thus, $f(X)$ is Γ-orthogonal.

Note that for the equation $X'' + \nabla V(X) = 0$, one may take the same generators $AX = X'$ and A_jX, if V is Γ_0-invariant. Of course B is then $H^2(S^1)$.

As in [12, p. 119], assume there is an open, bounded $\omega \subset \mathbb{R}^{2N}$, invariant under Γ_0, such that any 2π-periodic solution in $\overline{\omega}$ is in fact in ω. Let then

$\Omega \equiv \{X \in H^1(S^1) : \|X\|_1 < R,\ X(t) \in \omega\}$, where R is chosen so large that any periodic solution in ω has $\|X\|_1 < R/2$ (R depends on bounds on ∇H on $\overline{\omega}$ and Sobolev constants). Then $f(X) \neq 0$ on $\partial\Omega$ and the orthogonal degree of f with respect to Ω is defined. A word of caution is necessary here: we are dealing with two infinite dimensional spaces (a setting different from the one given in the present paper). The standard ways of reducing to a single space, i.e. looking at the integral equation or working in $H^{1/2}(S^1)$, have the inconvenient of obscuring the orthogonality. A complete theory should follow either the steps of [6] and study difference of degrees (as it is easily seen if $\nabla H(X) = AX$, for a constant matrix A, then the complex Morse index of $inJ + A$ is N for large n, that is most of the components of the orthogonal degree are non-zero). However, it is simpler to restrict oneself to a large ball in $H^1(S^1)$, hence $X(t)$ will be bounded, as well as $D^2H(X)$. Write $X(t) = \sum X_n e^{int}$, with $X_n = \overline{X}_{-n}$ in $\mathbb{C}^{2N}$, or $X = X_1 \oplus X_2$, where $X_1 \equiv PX$ corresponds to modes $|n| \le N_1$ and X_2 to the others. Since JX' is a Fredholm operator of index 0, one may use a global Liapunov–Schmidt reduction: the equation $(I-P)JX' + (I-P)\nabla H(X) = 0$ is uniquely solvable for X_2 as a C^1-function of X_1, for N_1 large enough. In fact, the linearization at any X_0 in the ball has the property that

$$\|JX_2' + (I-P)D^2H(X_0)X_2\|_{L^2} \ge (1 - M/N_1)\|X_2\|_{H^1},$$

where M is a uniform bound for $\|D^2H(X_0)\|$, hence the global implicit function theorem may be applied. Furthermore, since $(\nabla H(X), AX) = 0$, one has that the scalar product $(P\nabla H(X_1 + X_2(X_1)), AX_1) = -((I-P)\nabla H, AX_2) = ((I-P)JX_2', AX_2) = 0$, hence, the reduced equation is orthogonal and the degree will be that of $JX_1' + P\nabla H(X_1 + X_2(X_1))$, in the finite dimensional space $PH^1(S^1)$, i.e. $\deg_\perp(Pf(X_1 + X_2(X_1)); P\Omega)$. Note that the second term inherits the gradient structure.

REMARK 6.1. After the research for this paper was completed, we were given the preprints of [6] and [19]. The first paper studies the non-autonomous case and its relation to Maslov's index. For the Hopf bifurcation, the change of the invariant in [6] is the sum of the changes of the Morse indices, given below (see also the different other Conley-like degrees mentionned in the bibliography of [6]). The second paper uses the finite dimensional reduction of Amann and Zendher and the orthogonal degree of [18] for S^1-actions, (there $\Gamma_0 = \{e\}$), and computes these indices at different stationary points (including infinity, provided there is no resonance there). See also [1] and [15].

In the case of $\Gamma_0 = \{e\}$, one should also compare to the results of [12, p. 120 and p. 135–147], where the existence of a first integral ($H(X(t))$ here), was used to add a parameter. It is clear that $\tau X' + (1-\tau)J\nabla H$ is orthogonal to $f(X)$ and that the new parameter corresponds to part of the construction given here.

Assume then that ΓX_0 is an isolated orbit of dimension k, of solutions of $f(X) = 0$, with $\ker Df(X_0)$ of real dimension k and generated by k among $AX_0 = X_0', A_1X_0, \ldots, A_nX_0$. Then, if H is the isotropy subgroup of X_0, one is in the position of applying Theorem 4, provided one identifies $\underline{H}$ and computes d_H, d_{H_i} and n_{K_i}.

Note that the hyperbolicity conditions (i.e. the conditions on $\ker Df(X_0)$) imply that $Df(X_0)$ cannot commute with J, unless $k = 0$. (This does not mean that pieces of $Df(X_0)$ can't commute with J). In fact, if J commutes with $Df(X_0)$, then if V belongs to the kernel so does JV, which has to be a real linear combination of X_0' and A_jX_0. On the nth mode X_n of X_0, one would have $\lambda_0 nX_n + \sum \lambda_j N_j X_n = nJX_n$, where $N_j = \operatorname{diag}(N_j^1, \ldots, N_j^N, N_j^1, \ldots, N_j^N)$, (just one N_j^s if J is multiplication by i on the pair of coordinates). This leads to $X_n = 0$ for $n \neq 0$, and the same argument for A_jX_0, gives that this vector has to be 0.

We shall consider three cases.

(a) Stationary solution. If X_0 is time stationary, then $\Gamma_{X_0} = H = S^1 \times H_0$ with $H_0 < \Gamma_0$ such that $\dim \Gamma_0/H_0 = k$ and $\underline{H} = S^1 \times T^{n-k}$ generated by $(\varphi, \varphi_j,\ j = k{+}1, \ldots, n)$. As before, we shall reparametrize T^n in such a way that the action on the first k complex non-zero variables of X_0 is of the form $e^{iN_j\varphi_j} z_j$ (and also on JX_0). Then, $V^{\underline{H}}$ is contained in $\mathbb{R}^{2N}$, the constant functions, $B \equiv Df(X_0)$ has the form $\operatorname{diag}(B^H, B_\perp)$, with $B_\perp = \operatorname{diag}(B_m^{\mathbb{R}}, B_l^{\mathbb{C}}, B_s^{\mathbb{C}})$, where, on each B_m, the group H acts as $\mathbb{Z}_2$, on the complex B_l as $\mathbb{Z}_p$ and on the complex B_s as S^1. Each of these matrices is self-adjoint, since $B = D^2H(X_0)$. The hyperbolicity condition means that $\ker B^H$ has dimension k, that $B_\perp$ is invertible and that, for $n > 0$, $inJ + B$ is invertible. Furthermore, from Lemma 4.1, B_s is complex self-adjoint and H-orthogonal. Note that since J commutes with Γ_0, J has also a diagonal structure $\operatorname{diag}(J_H, J_m, J_l, J_s))$. By looking at Fourier series (non-negative modes are enough), a straight application of Theorem 4.1 will give

Theorem 6.1. *For a stationary hyperbolic orbit, the orthogonal index is given by*

(a) $d_H = (-1)^{n_H}$, *with n_H the Morse index of B^H,*
(b) $d_{H_j} = d_H((-1)^{n_{H_j}} - 1)/2$, *with* $(-1)^{n_{H_j}} = \operatorname{Sign} \det B_j^{\mathbb{R}}$,
(c) *the Morse index of $inJ + \widetilde{B}$, where $\widetilde{B}$ is any of the matrices B^H (with $n > 0$), $B_m^{\mathbb{R}}$ (with $n > 0$), $B_l^{\mathbb{C}}$ (with $n > 0$) or $B_s^{\mathbb{C}}$ (with $n \geq 0$) for the mode n and the decomposition of $\mathbb{C}^{2N}$ (induced by that of $\mathbb{R}^{2N}$) in irreducible representations of H.*

Remark 6.2.

(a) If one has a family of hamiltonians $f(\lambda, X)$, with $f(\lambda, X_0) = 0$ and X_0 hyperbolic for λ_1 and λ_2 and if any of the above numbers change, then one has a global Hopf bifurcation in the interval from λ_1 to λ_2, in V^K, where $K < H$ is any of the isotropy subgroups for which d_K has changed. V^K can be characterized as in [14, Lemma 3.1(a)]. In particular, if there is no bifurcation in $V^{\underline{H}}$, then one has a bifurcation from a stationary k-torus ΓX_0 to a $(k+1)$-torus, either stationary if the Morse index of $B_s^{\mathbb{C}}$ has changed, or, if there is no bifurcation of stationary solutions, to a time-periodic solution, i.e. a pulsating k-torus.

(b) One may compute the Morse indices as in [12, p. 142].

(c) If J commutes with $B_j^{\mathbb{R}}$, then $d_{H_j} = 0$, since n_{H_j} is even. More generally, if J commutes with $\widetilde{B}$, then one may decompose the space into two-dimensional subspaces, $\langle X_k, JX_k\rangle$, corresponding to the eigenvalue λ_k of $\widetilde{B}$, orthogonal between them and invariant under J. The eigenvalues of $inJ + \widetilde{B}$, on this subspace, are $\lambda_k \pm n$ and the Morse index of $inJ + \widetilde{B}$ is $a(n) + a(-n)$, where $a(n)$ is half the number of eigenvalues of $\widetilde{B}$ which are less than n. This is also the case if J is multiplication by i, since we are considering the complex Morse index.

(d) For the system $X'' + \nabla V(X) = 0$, with $D^2V(X_0) = B$, then the Morse index of $-n^2 I + \widetilde{B}$ is $a(n^2)$. Note that for the system, $(X' = Y, Y' = -\nabla V(X))$, J commutes with $D^2(V(X) + \|Y\|^2/2)$ only if $B = I$.

(b) Reduction to the stationary case. Assume that X_0' is a linear combination of the $A_j X_0$. Then for each coordinate z_s of $\mathbb{R}^{2N}$, with a non-trivial action of T^n, there is at most one mode n_s such that X_0' is non-zero on that mode (n_s is the same for JX_0). As in [14, p. 387], consider the matrix $A(t) = \operatorname{diag}(\dots, e^{-in_s t}, \dots)$, written this way according to the action of Γ_0 (each exponential corresponds to a rotation for a pair of real coordinates of Y, and the same for the symmetric pair in Z, or to a single pair if J acts as i). If $Y(t) = A(t)X(t)$, then, $Y_0' = 0$ since $A'X_0 = -AX_0'$. Furthermore, $Y' = A'(0)Y + A(t)J\nabla H(A^{-1}(t)Y)$. Using the equivariance of ∇H with respect to Γ_0 (and the fact that $A(t)$ is defined that way) and the fact that J commutes with $A(t)$, one has that $JY' - JA'(0)Y + \nabla H(Y) = 0$ and a reduction to the previous case: the rotating wave X_0 has been frozen. Furthermore, from Proposition 3 (and the fact $A^T = A^{-1}$ as real matrices), both orthogonal degrees coincide.

For the case of $X'' + \nabla H(X) = 0$, then the above transformation gives $Y'' + A'(0)^2 Y - 2A'(0)Y' + \nabla H(Y) = 0$, which is also orthogonal.

(c) Non-stationary solution. If $X_0', A_1X_0, \dots, A_{k-1}X_0$ are linearly independent, we may assume, from case (b), that $A_kX_0, \dots, A_nX_0$ are linear combinations of $A_1X_0, \dots, A_{k-1}X_0$ only. In particular, if $k = 1$, then $A_jX_0 = 0$ and X_0 belongs to V^{T^n}. In general, one may reparametrize T^n such that on

$V^{\underline{H}}$ one has $A_j X = 0$, for $j \geq k$. Here $H = \mathbb{Z}_p \times H_0$, with $\dim \Gamma_0/H_0 = k-1$, $\underline{H} = \underline{H}_0 = T^{n-k+1}$, if $X_0(t)$ is $2\pi/p$-time periodic. Then, $V^{\underline{H}} = \{X(t)$ in $(\mathbb{R}^{2N})^{\underline{H}_0} \equiv V_0\}$ and its complement is $\{X(t)$ in $V_0^{\perp}\}$. In fact $H = \{(\varphi, \Phi, L) : n\varphi + \langle N^j, \Phi\rangle + \langle K_j/M, L\rangle$ is in $\mathbb{Z}$, for each non-zero component X_n^j of $X_0\}$, as in [14, p. 386]. The fact that N_l^j is a linear combination of N_m^j for $l \geq k$ and $m < k$, allows to reparametrize T^n, as in the proof of Theorem 2, and eliminate from H the phases Φ_l, $l \geq k$. The fact that X_0' is linearly independent from $A_j X_0$, restricts Φ_m and φ to a discrete set, hence the claim on $\underline{H}$. From the compactness of Γ, there is a positive minimum φ_0, such that (φ_0, ψ_0, L_0) is in H. From the congruences, φ_0 (as well as each component of ψ_0) is a rational, of the form r/q. If $r > 1$, then there are integers k and a such that $kr + aq = 1$ and, changing φ_0 to $k\varphi_0$, one may take $\varphi_0 = 1/q$. Thus, $X_0(t) = \gamma_0 X_0(t + 2\pi/q)$, where γ_0 corresponds to (ψ_0, L_0). Now, any other element of H gives $X_0(t) = \gamma X_0(t + 2\pi\varphi)$. For such an element let k be such that $0 \leq \varphi - k\varphi_0 < \varphi_0$. Then, $X_0(t) = \gamma\overline{\gamma}_0^k X_0(t + 2\pi(\varphi - k\varphi_0))$ and $(\varphi - k\varphi_0, \psi - k\psi_0, L - kL_0)$ belongs to H, contradicting the minimality of φ_0, unless $\varphi = k\varphi_0$ and $\gamma = \gamma_0^k$.

Let $H_0 < \Gamma_0$ be the isotropy subgroup of the geometrical coordinates of $X_0(t)$. Then, since $\varphi_0 = 1/q$, one has that $\gamma_0^q \in H_0$ and $H = \{k(\varphi_0, \psi_0, L_0),\ k = 1, \ldots, q\} \cup \{(\psi, L) \in H_0\}$. Let q_0 be the smallest integer such that $\gamma_0^{q_0} \in H_0$. From the minimality $q = pq_0$ and one has $X_0(t) = \gamma_0 X_0(t + 2\pi/q)$, with $\gamma_0^{q_0} X_0 = X_0$ and $X_0(t)$ is $2\pi/p$-periodic.

Lemma 6.1. $V^H = \{X(t) \in V_0^{H_0},\ X(t) = \gamma_0 X(t + 2\pi/q)\}$.

Proof. On the component X_n^j the action of H is as

$$\exp 2\pi i(kn/q + k\langle N^j, \psi_0\rangle + k\langle K_j/M, \psi_0\rangle + \langle N^j, \psi\rangle + \langle K_j/M, L\rangle)$$

with (ψ, L) in H_0. Taking $k = 0$, one needs that (ψ, L) is in H_j, the isotropy of the jth coordinate, i.e. $H_0 < H_j$ and $X(t)$ is in $V_0^{H_0}$. In particular, $\gamma_0^{q_0}$ acts trivially on X_j. Hence, taking $k = q_0, n$ has to be a multiple of p. The converse is clear. □

Consider now K such that $H/K \cong \mathbb{Z}_2$. Since, $K = \bigcap H_{jn}$, the inclusions $K < H \cap H_{jn} < H$ imply that either $H < H_{jn}$ or $K = H \cap H_{jn}$. In the second case, one has that γ^2 is in H_{jn} for any $\gamma \in H$. In particular, for $\varphi = 0$ and $\widetilde{\gamma}$ in H_0, one needs $\widetilde{\gamma}^2 \in H_j$ and $H_0/H_0 \cap H_j$ has at most order 2. Let $K_0 = H_0 \cap H_j$, for all such j, then $K_0 = H_0$ or $H_0/K_0 \cong \mathbb{Z}_2$. In the second case, there is $\gamma_1 \in H_0$, with $\gamma_1^2 \in K_0$, i.e. γ_1 acts as Id on $V_0^{H_0}$ and as $-$Id on $V_0^{K_0} \cap (V_0^{H_0})^{\perp}$. Since $\gamma_0^{q_0} \in H_0$, one has $\gamma_0^{2q_0}$ acts as Id on $V_0^{K_0}$. Let $V_0^{\pm}$ be the subspaces of $V_0^{K_0}$ where $\gamma_0^{q_0}$ acts as $\pm$Id. Then $V_0^+ \supset V_0^{H_0}$.

Lemma 6.2. *V^K consists of all 2π-periodic functions $X(t)$ in $V_0^{K_0}$ of the form $X(t) = X_+(t) + X_-(t)$, with $X_\pm(t) = \pm\gamma_0 X_\pm(t + 2\pi/q)$. In particular, if q is odd, then $X_\pm(t)$ is in $V_0^\pm$ and both are $2\pi/p$-periodic. If q is even and p is odd, then $X(t)$ is in V_0^+ and it is $2\pi/p$-periodic. The components of $X_+(t)$ in V_0^+ are $2\pi/p$-periodic and those in V_0^- are $2\pi/p$-antiperiodic. The behavior of the components of $X_-(t)$ differ by a factor $(-1)^{q_0}$.*

Proof. For the coordinate X_j, we know that $2q_0(\langle N^j, \psi_0\rangle + \langle K_j/M, L_0\rangle) = a_j$, where a_j is an integer, even if X_j is in V_0^+ and odd if X_j is in V_0^-. Since $(2\varphi_0, 2\psi_0, 2L_0)$ fixes X_n^j, one has that $2n/q + a_j/q_0 = b$ is an integer. From $n = bq/2 - a_j p/2$, one has that, if q is odd, then b has the parity of a_j, while if q is even and a_j is odd, then p has to be even. Even b will give $X_+(t)$ and odd b give $X_-(t)$. There are minimum $n_j^\pm$ such that the modes of $X_\pm^j$ are of the form $n^\pm = n_j^\pm + cq$, for any integer c. The numbers $n_j^\pm$ are multiples of p, except if p is even and, for X_+^j, a_j is odd or, for X_-^j, a_j and q_0 have opposite parities, in which case $n_j^\pm$ are odd multiples of $p/2$. The converse is clear. $\square$

It remains to identify the irreducible representations of H in $V_0^\perp$. Since the action of H on X_n^j is

$$\exp \pi i(ns/q + s(\langle N^j, \psi_0\rangle + \langle K_j/M, L_0\rangle) + \langle N^j, \widetilde{\psi}\rangle + \langle K_j/M, \widetilde{L}\rangle + \langle N^j, \psi\rangle),$$

where $s = 0, \dots, q$, $(\widetilde{\psi}, \widetilde{L})$ gives an element of H_0 and $\langle N^j, \psi\rangle = \sum_k^n N_l^j \psi_l$ is non-trivial, then one has the same action for different (n, j) if the following happens: taking $s = 0$ and $(\widetilde{\psi}, \widetilde{L}) = 0$, then N_l^j has to be the same for all j, for $l = k, \dots, n$. Taking $s = 0$ and $\psi = 0$, one needs the same action for all $(\widetilde{\psi}, \widetilde{L})$. Hence, the different X^j are in the same irreducible representation of H_0 in $V_0^\perp$. If $\alpha_j = \langle N^j, \psi_0\rangle + \langle K_j/M, L_0\rangle$ gives the action of γ_0, then, since $\gamma_0^{q_0}$ is in H_0, one needs that $q_0(\alpha_j - \alpha_l)$ is an integer a_{jl}. Then, for $X_j^{n_j}$ and $X_l^{n_l}$, one has that $(n_j - n_l)/q + a_{jl}/q_0$ is an integer b_j. One has proved the following result.

Lemma 6.3. *Assume $X_0, \dots, X_r$ are the coordinates of an irreducible representation of H_0 in $V_0^\perp$. Then, for each $n_0 = 0, \dots, [q/2]$, there is a different irreducible representation of H in $(V^{\underline{H}})^\perp$ given by functions of the form $X(t) = \mathrm{Re}(X_{n_0}(t)Y(t))$, where $Y(t)$ is $2\pi/q$-periodic and the j-component of $X_{n_0}(t)$, $j = 0, \dots, r$, is $\exp(in_j^0 t)$, and n_j^0 is the minimum positive integer n_j such that $n_j = n_0 - a_{j_0}p + b_j q = n_j^0 + c_j q$ for any integer c_j.*

Note that the facts that all integers c_j are possible and that $X(t)$ has to be real will couple the modes corresponding to n_0 and to $q - n_0$, as real representations. Note also that for $q = 1$, then $n_j^0 = 0$ and $V^K = \{Y(t), 2\pi\text{-periodic in } V_0^\perp\}$.

Let $B(t) = D\nabla H(X_0(t))$, which is symmetric, $2\pi/p$-periodic and H_0-equivariant. Hence, since $\gamma_0^{q_0}$ and γ_1 are in H_0, one has a diagonal structure for $B(t) = \mathrm{diag}\,(B_0, B_+^j, B_-^j, \dots, B_{K_0}, \dots)$, where B_0 corresponds to $V_0^{H_0}$, $B_\pm^j$ correspond to $(V_0^{K_j})^\pm \cap (V_0^{H_0})^\perp$ with $H_0/K_j \cong \mathbb{Z}_2$ and $\gamma_0^{q_0}$ acts as $\pm\mathrm{Id}$ on $(V_0^{K_j})^\pm$, and B_{K_0} is on an irreducible representation of H_0 in $V_0^\perp$.

Lemma 6.4. *The fact that $X_0(t)$ is in V^H implies a further decomposition of each of the components of $B(t)$ as $B_1(t)+B_2(t)$, where $B_1(t)$ is $2\pi/q$-periodic and in block-diagonal form on coordinates with the same action of γ_0 and $B_2(t)$, which is non-zero only if q_0 is even, is $4\pi/q$-periodic, $e^{-qt/2}B_2(t)$ is $2\pi/q$-periodic and $B_2(t)$ has a block-diagonal form $\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$ on $\begin{pmatrix} X \\ Y \end{pmatrix}$ where γ_0 has the same action on X and the opposite on Y.*

Proof. Since $X_0(t) = \gamma_0 X_0(t+2\pi/q)$ and $Df(\gamma X)\gamma = \gamma Df(X)$, one has $\gamma_0 B(t+2\pi/q) = B(t)\gamma_0$. Hence, for $B(t) = \sum B_n e^{int}$ and if γ_0 acts on the jth coordinate as $\exp(2\pi i\alpha_j)$, then $\exp 2\pi i(\alpha_j - \alpha_l + n/q)B_n^{jl} = B_n^{jl}$, for the entries of B_n. Hence, whenever $B_n^{jl} \neq 0$, one has that $\alpha_j - \alpha_l + n/q$ is an integer. In particular, if, for some l, $B_n^{ll} \neq 0$, then n is a multiple of q and for all (j,k) with $B_n^{jk} \neq 0$ one has $\alpha_j - \alpha_k$ is an integer. Thus, B_n will contribute to $B_1(t)$. On the other hand, if $B_n^{ll} = 0$, for all l and $B_n^{jl} \neq 0$, then, since B_n is symmetric, $B_n^{lj} \neq 0$ and $2n/q$ is an integer. If $2n$ is an even multiple of q, we are back to the previous situation, while if $2n$ is an odd multiple of q (hence q is even), then $2(\alpha_j - \alpha_l)$ is an odd integer, giving opposite actions of γ_0 on X_j and X_l, if $B_n^{jl} \neq 0$. Thus, B_n contributes to $B_2(t)$. Finally, since $X_0(t)$ is $2\pi/p$-periodic, one has $\gamma_0^{q_0} B(t+2\pi/p) = \gamma_0^{q_0} B(t) = B(t)\gamma_0^{q_0}$. Thus, $q_0(\alpha_j - \alpha_l)$ is an integer, which implies, for $B_2(t)$, that q_0 is even. □

Now, recall that $LX = JX' + B(t)X$ is a bounded Fredholm operator of index 0, from $H^1(S^1)$ into $L^2(S^1)$ and self-adjoint on $L^2(S^1)$, with kernel generated by $\{X_0', A_1X_0, \ldots, A_{k-1}X_0\}$. Hence, one has the decompositions $H^1(S^1) = \ker L \oplus \operatorname{Range} L \cap H^1$, $L^2(S^1) = \ker L \oplus \operatorname{Range} L$ (orthogonal in L^2) and one has a bounded pseudo-inverse K from $\operatorname{Range} L$ onto $\operatorname{Range} L \cap H^1$.

Furthermore, the reduction to finite dimensions, on V_{N_1} generated by all modes less or equal to N_1, was done by using the implicit function theorem on the higher modes to solve the equation $J\widetilde{X}_{N_1}' + (I - P_{N_1})\nabla H(X_{N_1} + \widetilde{X}_{N_1}) = 0$ for $\widetilde{X}_{N_1}$ in $V_{N_1}^\perp$ and reduce to $JX_{N_1}' + P_{N_1}\nabla H(X_{N_1} + \widetilde{X}_{N_1}(X_{N_1})) = 0$, which is the problem which we have studied. It is then not difficult to prove that the linearization of this last equation is of the form

$$L_{N_1}X_{N_1} = JX_{N_1}' + P_{N_1}B(t)(X_{N_1} + \widetilde{X}_{N_1}),$$

where $\widetilde{X}_{N_1}$ in $V_{N_1}^\perp$ is the unique solution of the equation

$$J\widetilde{X}_{N_1}' + (I - P_{N_1})B(X_{N_1} + \widetilde{X}_{N_1}) = 0.$$

Then, $\|\widetilde{X}_{N_1}\|_1 \le C\|X_N\|_0$ and $\|\widetilde{X}_{N_1}\|_0 \le \|\widetilde{X}_{N_1}\|_1/N_1$. Furthermore, $\ker L_{N_1} = P_{N_1}(\ker L)$ and has also dimension k, if N_1 is large enough, and L_{N_1} is self-adjoint. In fact, one may use the gradient structure of the linearization of the reduction, or see directly that

$$\begin{aligned}(L_{N_1}X_{N_1}, Z_{N_1})_{L^2} - (X_{N_1}, L_{N_1}Z_{N_1})_{L^2} &= (B\widetilde{X}_{N_1}, Z_{N_1}) - (X_{N_1}, B\widetilde{Z}_{N_1})\\ &= (X, BZ_{N_1}) - (X_{N_1}, BZ),\end{aligned}$$

using the symmetry of B. But, since $J\widetilde{X}'_{N_1} = -(I - P_{N_1})BX$, then

$$(J\widetilde{X}'_{N_1}, \widetilde{Z}_{N_1}) = -(BX, \widetilde{Z}_{N_1}) = (\widetilde{X}_{N_1}, J\widetilde{Z}'_{N_1}) = -(BZ, \widetilde{X}_{N_1}),$$

hence the above difference is $(Z, BX) - (X, BZ) = 0$. The constant C depends only on $\sup|B(t)|$. Furthermore, if $L_{N_1}X_{N_1} = Z_{N_1}$, then $L(X_{N_1} + \widetilde{X}_{N_1}) = Z_{N_1} + 0$, i.e. $\operatorname{Range} L_{N_1} = \operatorname{Range} L \cap V_{N_1}$ and since $LKZ = Z$, for $Z = Z_{N_1}$ in V_{N_1}, one has that K_{N_1}, from $\operatorname{Range} L_{N_1}$ onto $\operatorname{Range} L_{N_1} \cap H^1$, the pseudo-inverse of L_{N_1} is $P_{N_1}KP_{N_1}$, in particular, as operator from L^2 into H^1, one has $\|K_{N_1}\| \le \|K\|$.

Finally, if P is the projection onto $\ker L$ and $I - P$ that on $\operatorname{Range} L$, one has that $P_{N_1}PP_{N_1}$ will project on $\ker L_{N_1}$ while $P_{N_1}(I-P)P_{N_1}$ will project onto $\operatorname{Range} L_{N_1}$ and one has $L_{N_1}P = PL_{N_1} = 0$.

Recall that $\sigma(L)$, the spectrum of L, is discrete, since $L - \lambda I$ is also a Fredholm operator of index 0 (the inclusion of H^1 in L^2 is compact) and self-adjoint in L^2 and K, as an operator from L^2 into L^2, is compact. Furthermore, if λ is not in $\sigma(L)$, then, since $(L-\lambda)(X_{N_1} + \widetilde{X}_{N_1}) = (L_N - \lambda)X_{N_1} - \lambda\widetilde{X}_{N_1}$ and $\|(L-\lambda)X\|_0 \ge \|K_\lambda\|^{-1}\|X\|_1$, with K_λ the inverse of $L - \lambda$, one has

$$\|(L_\lambda - \lambda)X_{N_1}\|_0 \ge \|K_\lambda\|^{-1}\|X_{N_1}\|_0 - |\lambda|\|\widetilde{X}_{N_1}\|_0 \ge (\|K_\lambda\|^{-1} - C|\lambda|/N_1)\|X_{N_1}\|_0.$$

Hence, for N_1 large enough, λ is not in $\sigma(L_{N_1})$. Thus, if $\mathcal{K}$ is a compact subset of $\mathbb{R}$, with $\mathcal{K} \cap \sigma(L) = \phi$, then for N_1 large enough (depending on $\mathcal{K}$), one has that $\mathcal{K}\sigma(L_{N_1}) = \phi$.

Conversely, if $\lambda_0 \in \sigma(L)$, then treating $(L_{N_1} - \lambda)X_{N_1} = (L - \lambda_0)(X_{N_1} + \widetilde{X}_{N_1}) + (\lambda_0 - \lambda)X_{N_1} + \lambda_0\widetilde{X}_{N_1}$ as a bifurcation problem by projecting on $\ker(L - \lambda_0)$ and $\operatorname{Range}(L - \lambda_0)$, one obtains

$$\begin{aligned}(L_{N_1} - \lambda)X_{N_1} = {}&(L-\lambda)((I - P_0)(X_{N_1} + \widetilde{X}_{N_1} + K_{\lambda_0}[(\lambda_0 - \lambda)(I - P_0)X_{N_1}\\ &+ \lambda_0(I - P_0)\widetilde{X}_{N_1}] \oplus (\lambda_0 - \lambda)P_0X_{N_1} + \lambda_0P_0\widetilde{X}_{N_1},\end{aligned}$$

where P_0 projects on $\ker(L - \lambda_0)$ and $I - P_0$ on *Range*$(L - \lambda_0)$. Then, see [10], $\ker(K - \lambda_0)$ will give d eigenvalues for L_{N_1}, close to λ_0, with $d = \dim\ker(L - \lambda_0) \le 2N$.

Note also that $\|L_{N_1}X_{N_1} - P_{N_1}LX_{N_1}\|_0 = \|P_{N_1}B\widetilde{X}_{N_1}\|_0 \le C\|X_{N_1}\|_0/N_1$, hence the spectra of the matrices L_{N_1} and $P_{N_1}LP_{N_1}$ are close, for N_1 large.

Theorem 6.2. *The orthogonal index of $P_{N_1}X_0$ is given by, for N_1 large enough,*

(1) $d_H = (-1)^{n_H}$, *where n_H is the real Morse number of L_{N_1} restricted to V^H, where $\gamma_0^{q_0} = \mathrm{Id}$ and $X(t) = \gamma_0 X(t + 2\pi/q)$. In particular, d_H is independent of N_1, for N_1 large enough.*

(2) $d_{H_j} = d_H((-1)^{n_{H_j}} - 1)/2$, *where n_{H_j} is the real Morse number of L_{N_1} restricted to $V^{H_j} \cap (V^H)^\perp$, where $\gamma_0^{2q_0} = \mathrm{Id}$ and $X(t)$ has the decomposition given in Lemma* 6.2. *In particular, d_{H_j} is independent of N_1, for N_1 large enough.*

(3) $n_K^{N_1}$ *the complex Morse number of L_{N_1} restricted to one of the q different irreducible representations of H in $(V^{\underline{H}})^\perp$, based on V_1 an irreducible representation of H_0 in $V_0^\perp$ and with functions given in Lemma* 6.3, *of the form $X(t) = \mathrm{Re}(X_{n_0}(t)Y(t))$, with $Y(t)$ of period $2\pi/q$. One has that $n_K^{N_1+q} = n_K^{N_1} + \dim V_1$, $(\dim V_1$ is even$)$.*

(4) *The relations of Theorem* 4.

Proof. From Theorem 4, the only thing to study is how the spectrum of L_{N_2} is related to that of L_{N_1}, where N_2 is the next integer after N_1 where one has to consider new modes. From the composition of the spaces one may take $N_2 = N_1 + q$, with $X_{N_2} = X_{N_1} \oplus Y_{N_1}$, where Y_{N_1} has two conjugate modes based on an even dimensional (because of J) space V_1. Then,

$$
\begin{aligned}
L_{N_2}X_{N_2} = {} & L_{N_1}X_{N_1} + P_{N_1}B(\widetilde{X}_{N_2} - \widetilde{X}_{N_1} + Y_{N_1}) \\
& \oplus JY'_{N_1} + (P_{N_2} - P_{N_1})B(X_{N_1} + Y_{N_1} + \widetilde{X}_{N_2}).
\end{aligned}
$$

But, since $\widetilde{X}_{N_1} = \widetilde{X}_{N_2} \oplus \widetilde{Y}_{N_1}$, with $J\widetilde{Y}'_{N_1} + (P_{N_2} - P_{N_1})B(X_{N_1} + \widetilde{X}_{N_1}) = 0$, one has

$$
\begin{aligned}
L_{N_2}X_{N_2} = {} & L_{N_1}X_{N_1} + P_{N_1}B(Y_{N_1} - \widetilde{Y}_{N_1}) \oplus J(Y'_{N_1} - \widetilde{Y}'_{N_1}) \\
& + (P_{N_2} - P_{N_1})B(Y_{N_1} - \widetilde{Y}_{N_1}).
\end{aligned}
$$

Now, since L_{N_2} and $L_{N_1}X_{N_1} \oplus JY'_{N_1}$ are self-adjoint, this is also the case for the linear deformation

$$
\begin{aligned}
L^\tau_{N_2}X_{N_2} = {} & L_{N_1}[(I - P)X_{N_1} + \tau K_{N_1}(I - P)P_{N_1}B(Y_{N_1} - \widetilde{Y}_{N_1})] \\
& \oplus \tau P P_{N_1}B(Y_{N_1} - \widetilde{Y}_{N_1}) \oplus JY'_{N_1} - \tau J\widetilde{Y}'_{N_1} \\
& + \tau(P_{N_2} - P_{N_1})B(Y_{N_1} - \widetilde{Y}_{N_1}),
\end{aligned}
$$

where we have used the decomposition of the space on $\ker L_{N_2} \oplus \mathrm{Range}\, L_{N_2}$ induced by that for L. Then, if $L^\tau_{N_2}X_{N_2} = 0$, one may solve uniquely the first and last terms in function of PX_{N_1}, with $\|Y_{N_1}\|_0 \le C\|X_{N_1}\|_0/N_1$, $\|(I - P)X_{N_1}\|_1 \le C\|PX_{N_1}\|_0/N_1$ and hence $\|Y_{N_1}\|_0 \le C\|PX_{N_1}\|_0/N_1$, where the constant C is independent of N_1. In particular if $X_{N_1} + Y_{N_1}$ is in $\mathrm{Range}\, L_{N_2} =$

Range $L \cap V_{N_2}$, then $PX_{N_1} + PY_{N_1} = 0$ and one has that $Y_{N_1} = 0 = X_{N_1}$, i.e. Range $L_{N_2} \cap \ker L^{\tau}_{N_2} = \{0\}$. Hence, the non-zero eigenvalues of L_{N_2} don't cross over 0. (One could also prove this fact by taking λ_0 a mid-point between 0 and the first negative eigenvalue of L. Then as seen above, $L_{N_1} - \lambda_0$ and $L_{N_2} - \lambda_0$ are invertible, for N_1 large enough, with inverses bounded independently of N_1. then, it is not difficult to show that $L^{\tau}_{N_2} - \lambda_0 I$ is also invertible, for N_1 large). Thus, $n(L_{N_2}) = n(L_{N_1}) + n(JY'_{N_1})$.

Now, if $JY'_{N_1} = \lambda Y_{N_1}$, then, since $Y_{N_1} = (X_M, X_{-M} = \overline{X}_M)$, one has $iMJX_M = \lambda X_M$, with $X_M = (X, Y)$ in $\mathbb{C}^{2r}$, where $2r = \dim V_i$, $V_i = V_0^{H_0}$, or $V_0^{K_0}$, or V_1. Then, $\lambda = \pm M$, each with an eigenspace isomorphic to $\mathbb{C}^r$, hence taking into account X_{-M} or writing $Y_{N_1} = \cos MtX + \sin MtY$, with X and Y in $\mathbb{R}^{2r}$, one obtains that $n(JY'_{N_1}) = 2r$. $\square$

Remark 6.3. For the case of $-X'' + \nabla H(X)$, the linearization $LX = -X'' + B(t)X$ is an elliptic operator and hence has a spectrum bounded from below. The numbers $n(H)$, $n(H_j)$, $n(K)$ are those for LX.

Remark 6.4. If $J\widetilde{B} = \widetilde{B}J$ for some block in B, then let $\Phi(t)$ be the fundamental matrix for $X' = J\widetilde{B}X$, with $\Phi(0) = I$. If $JX' + \widetilde{B}X = \lambda X$, then $X(t) = e^{-\lambda Jt}\Phi(t)X(0)$ and $X(2\pi) = X(0)$ if and only if $X(0)$ is in $\ker(I - e^{-\lambda 2\pi J}\Phi(2\pi))$. Note that, since $\Phi' = J\widetilde{B}\Phi = \widetilde{B}J\Phi$, then $J\Phi$ and ΦJ are also fundamental matrices and, being equal for $t = 0$, one has that J and Φ commute. Since $\Phi^T J\Phi = J$ (by differentiating the left hand side), one has that Φ is an orthogonal matrix and hence with spectrum on the unit disc. Furthermore $e^{\lambda Jt}$ preserves the generalized eigenspaces of $\Phi(t)$. Thus, if $\Phi(2\pi)W = \mu W$, one has $(I - e^{-\lambda 2\pi J}\Phi(2\pi))W = 0$ if and only if $e^{\lambda 2\pi J}W = \mu W = (\cos\lambda 2\pi I + \sin\lambda 2\pi J)W$, that is $\mu = e^{\pm i\lambda 2\pi}$.

Note also that if $JX' + BX = \lambda X$ then $Y(t) = e^{-Jt}X(t)$ satisfies $JY' + BY = (\lambda + 1)Y$ and is 2π-periodic if $X(t)$ is 2π-periodic. Similarly, if $X(t)$ belongs to V^H or V^{H_j} or V^K, then $Y(t) = e^{-qJt}X(t)$ belongs to the same space. From these last observations (with the fact that if X is in $\ker(\widetilde{L} - \lambda I)$ also JX is in the same kernel), one has that $d_{H_j} = 0$ for these subspaces and that $n(K)$ is even and the spectrum of $\widetilde{L}$ is completely determined by its restriction to $(-q, 0]$. Note finally, that one may relate the spectrum of $\Phi(2\pi/q)$ to that of $\Phi(2\pi)$ as in [14, p. 390] or as in [5].

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Manuscript received July 29, 1998

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