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TYPE II REGIONS BETWEEN CURVES OF THE FUČIK SPECTRUM AND CRITICAL GROUPS

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1. Introduction

The Fučík spectrum arises in the study of semilinear elliptic boundary value problems of the form

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ and $f(x,t)$ is a Carathéodory function on $\overline{\Omega} \times \mathbb{R}$ such that

$$\frac{f(x,t)}{t} \to \begin{cases} a & \text{a.e. as } t \to -\infty, \\ b & \text{a.e. as } t \to \infty. \end{cases} \tag{1.2}$$

When $|u(x)|$ is large, (1.1) approximates the equation

$$-\Delta u = bu^+ - au^-, \tag{1.3}$$

where $u^\pm(x) = \max\{\pm u(x), 0\}$. The set Σ of those points $(a,b) \in \mathbb{R}^2$ for which (1.3) (together with the zero boundary condition) has nontrivial solutions is called the Fučík spectrum of $-\Delta$.

It was shown in Schechter [7] that, if $0 < \lambda_1 < \lambda_2 < \ldots$ are the distinct Dirichlet eigenvalues of $-\Delta$, there are decreasing curves C_{l1}, C_{l2} (which may

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coincide) passing through the point (λ_l, λ_l) such that all points on the curves are in Σ, while points in the square $Q_l := (\lambda_{l-1}, \lambda_{l+1})^2$ that are above both curves or below both curves are not in Σ. When the curves do not coincide, the region between them is called a type II region. Points in a type II region may or may not belong to Σ.

From the variational point of view, the solutions of the asymptotic equation (1.3) are the critical points of the C^1 functional

$$I(u) = I(u, a, b) = \int_\Omega |\nabla u|^2 - a(u^-)^2 - b(u^+)^2, \quad u \in H = H_0^1(\Omega). \tag{1.4}$$

When $(a, b) \notin \Sigma$, $u = 0$ is the only critical point of I and it is well-known that for $\lambda_{l-1} < b < \lambda_l < a < \lambda_{l+1}$, the critical groups

$$C_q(I, 0) = 0 \quad \text{if } q < d_{l-1} \text{ or } q > d_l, \tag{1.5}$$

where d_l is the dimension of the subspace N_l spanned by the eigenfunctions corresponding to $\lambda_1, \dots, \lambda_l$ (see Dancer [2]). Some of the critical groups for $q = d_{l-1}$ and for $q = d_l$ were recently computed by Dancer [2]. In particular, if $(a, b) \in Q_l \setminus \Sigma$ lies in the type II region between C_{l1}, C_{l2} and $a > b$, then

$$a > t(b) := \inf\{a' > b : (a', b) \in \Sigma\}, \tag{1.6}$$

$$b < s(a) := \sup\{b' < a : (a, b') \in \Sigma\}, \tag{1.7}$$

and hence

$$C_{d_{l-1}}(I, 0) = C_{d_l}(I, 0) = 0 \tag{1.8}$$

(see Theorem 1 of Dancer [2]). Thus, when $(a, b) \in Q_l \setminus \Sigma$ is in a type II region,

$$C_q(I, 0) = 0 \quad \text{if } q \le d_{l-1} \text{ or } q \ge d_l \tag{1.9}$$

(interchanging a and b does not affect the critical groups). We will give a new proof of this fact based on some ideas developed in Schechter [7], [8].

Theorem1.1. *Let $(a, b) \in Q_l \setminus \Sigma$.*

(i) *If $q < d_{l-1}$ or $q > d_l$, then*

$$C_q(I, 0) = 0. \tag{1.10}$$

(ii) *If (a, b) lies below the lower curve C_{l1}, then*

$$C_q(I, 0) = \begin{cases} \mathbb{Z} & \text{for } q = d_{l-1}, \\ 0 & \text{for } q \ne d_{l-1}. \end{cases} \tag{1.11}$$

(iii) *If (a, b) lies above C_{l1}, then*

$$C_{d_{l-1}}(I, 0) = 0. \tag{1.12}$$

(iv) *If* (a,b) *lies above the upper curve* C_{l2}, *then*

$$C_q(I,0) = \begin{cases} \mathbb{Z} & \text{for } q = d_l, \\ 0 & \text{for } q \neq d_l. \end{cases} \tag{1.13}$$

(v) *If* (a,b) *lies below* C_{l2}, *then*

$$C_{d_l}(I,0) = 0. \tag{1.14}$$

Theorem 1.1 is proved in Section 2. When (a,b) lies between C_{l1} and C_{l2}, this theorem does not cover the critical groups $C_q(I,0)$ for $d_{l-1} < q < d_l$.

The following existence results for the problem (1.1) were obtained in Schechter [8] when $(a,b) \notin \Sigma$ is in a type II region. Assume that $f(x,t)$ is of the form

$$f(x,t) = bt^+ - at^- + p(x,t), \tag{1.15}$$

with

$$\frac{p(x,t)}{t} \to 0 \quad \text{a.e. as } t \to \pm\infty, \tag{1.16}$$

and define

$$F(x,t) = \int_0^t f(x,s)\,ds, \qquad P(x,t) = \int_0^t p(x,s)\,ds. \tag{1.17}$$

THEOREM 1.2. *Assume that* $(a,b) \in Q_l \setminus \Sigma$ *lies above* C_{l1} *and that*

$$f(x,t_1) - f(x,t_0) < \lambda_{l+1}(t_1 - t_0), \qquad x \in \Omega,\ t_0 < t_1, \tag{1.18}$$

$$2P(x,t) \geq -W(x) \in L^1(\Omega), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{1.19}$$

$$2F(x,t) \geq \lambda_{l-1} t^2, \qquad x \in \Omega,\ t \in \mathbb{R}, \tag{1.20}$$

$$2F(x,t) \leq \lambda_l t^2, \qquad |t| < \delta,\ \text{for some } \delta > 0. \tag{1.21}$$

Then (1.1) *has a nontrivial solution.*

THEOREM 1.3. *Assume that* $(a,b) \in Q_l \setminus \Sigma$ *lies below* C_{l2} *and that*

$$f(x,t_1) - f(x,t_0) > \lambda_{l-1}(t_1 - t_0), \qquad x \in \Omega,\ t_0 < t_1, \tag{1.22}$$

$$2P(x,t) \leq W(x) \in L^1(\Omega), \quad x \in \Omega,\ t \in \mathbb{R}, \tag{1.23}$$

$$2F(x,t) \leq \lambda_{l+1} t^2, \qquad x \in \Omega,\ t \in \mathbb{R}, \tag{1.24}$$

$$2F(x,t) \geq \lambda_l t^2, \qquad |t| < \delta,\ \text{for some } \delta > 0. \tag{1.25}$$

Then (1.1) *has a nontrivial solution.*

We shall improve these theorems using a generalized notion of local linking introduced in Perera [5]. We assume that

$$|f(x,t)| \leq C(|t| + 1), \quad x \in \Omega,\ t \in \mathbb{R}. \tag{1.26}$$

Recall that the curves C_{l1} and C_{l2} were constructed in Schechter [7] as follows. Define

$$M_l(a,b) = \inf_{w\in M_l,\ \|w\|=1} \sup_{v\in N_l} I(v+w), \quad \text{where } M_l = N_l^{\perp}, \tag{1.27}$$

$$m_l(a,b) = \sup_{v\in N_l,\ \|v\|=1} \inf_{w\in M_l} I(v+w), \tag{1.28}$$

$$\nu_l(a) = \sup\{b : M_l(a,b) \geq 0\}, \tag{1.29}$$

$$\mu_l(a) = \inf\{b : m_l(a,b) \leq 0\}, \tag{1.30}$$

and let C_{l1} be the (lower) curve $b = \nu_{l-1}(a)$ and C_{l2} be the (upper) curve $b = \mu_l(a)$.

Theorem 1.4. *Assume that $(a,b) \in Q_l \setminus \Sigma$ lies above C_{l1},* (1.26) *holds, and for some $j \neq l+1$ and $a_0 \in [\lambda_{j-1}, \lambda_{j+1}]$,*

$$\lambda_{j-1}t^2 \leq 2F(x,t) \leq a_0(t^-)^2 + \nu_{j-1}(a_0)(t^+)^2, |t| < \delta, \quad \textit{for some } \delta > 0. \tag{1.31}$$

Then (1.1) *has a nontrivial solution.*

Theorem 1.5. *Assume that $(a,b) \in Q_l \setminus \Sigma$ lies below C_{l2},* (1.26) *holds, and for some $j \neq l-1$ and $a_0 \in [\lambda_{j-1}, \lambda_{j+1}]$,*

$$a_0(t^-)^2 + \mu_j(a_0)(t^+)^2 \leq 2F(x,t) \leq \lambda_{j+1}t^2, |t| < \delta, \quad \textit{for some } \delta > 0. \tag{1.32}$$

Then (1.1) *has a nontrivial solution.*

Note that, unlike in Theorems 1.2 and 1.3, the assumptions near zero and near infinity in Theorems 1.4 and 1.5 are not necessarily related to the same eigenvalues.

Corollary 1.6. *Assume that $(a,b) \in Q_l \setminus \Sigma$,* (1.26) *holds,*

$$f(x,t) = b_0t^+ - a_0t^- + p_0(x,t) \tag{1.33}$$

with $p_0(x,t) = o(t)$ as $t \to 0$, uniformly in $\overline{\Omega}$, and $(a_0,b_0) \in Q_l$. Then (1.1) *has a nontrivial solution in the following cases:*

(i) *(a,b) is above C_{l1} and (a_0,b_0) is either below C_{l1}, or on C_{l1} and*

$$P_0(x,t) := \int_0^t p_0(x,s)ds \leq 0, \quad |t| < \delta, \textit{ for some } \delta > 0, \tag{1.34}$$

(ii) *(a,b) is below C_{l2} and (a_0,b_0) is either above C_{l2}, or on C_{l2} and*

$$P_0(x,t) \geq 0, \quad |t| < \delta, \textit{ for some } \delta > 0. \tag{1.35}$$

As in Schechter [8], we search for solutions of (1.1) as critical points of

$$G(u) = \int_\Omega |\nabla u|^2 - 2F(x,u), \quad u \in H = H_0^1(\Omega). \tag{1.36}$$

It is well-known that G satisfies the Palais–Smale compactness condition (PS) when $(a,b) \notin \Sigma$. We obtain nontrivial critical points using the notion of homological local linking introduced in Perera [5], which we now recall.

Let G be a C^1 functional on a Hilbert space H and assume that G has only isolated critical points and satisfies (PS).

Definition 1.7. Assume that 0 is a critical point of G with $G(0) = 0$ and let q, β be positive integers. We say that G has a local (q,β)-linking near the origin if there exist a neighbourhood U of 0 and subsets A, S, B of U with $A \cap S = \emptyset$, $0 \notin A$, $A \subset B$ such that

(i) 0 is the only critical point in $G_0 \cap U$,
(ii) denoting by $i_{1*} : H_{q-1}(A) \to H_{q-1}(U \setminus S)$ and $i_{2*} : H_{q-1}(A) \to H_{q-1}(B)$ the embeddings of the homology groups induced by inclusions,

$$\operatorname{rank} i_{1*} - \operatorname{rank} i_{2*} \geq \beta,$$

(iii) $G \leq 0$ on B,
(iv) $G > 0$ on $S \setminus \{0\}$.

We will prove Theorems 1.4 and 1.5 using the following critical point theorem proved in Perera [5].

Theorem 1.8. *Assume that G has a local (q,β)-linking near the origin and a regular value $\alpha < 0$ such that*

$$\operatorname{rank} H_q(H, G_\alpha) < \beta. \tag{1.37}$$

Then G has a (*nontrivial*) *critical point u with either*

$$G(u) < 0, \quad C_{q-1}(G,u) \neq 0 \tag{1.38}$$

or

$$G(u) > 0, \quad C_{q+1}(G,u) \neq 0. \tag{1.39}$$

2. Critical group computations

Recall that the critical groups $C_q(I,0)$ are defined by

$$C_q(I,0) = H_q(I_0 \cap U, (I_0 \cap U) \setminus \{0\}), \tag{2.1}$$

where U is any neighbourhood of 0 such that 0 is the only critical point of I in $I_0 \cap U$. Since $(a,b) \notin \Sigma$, 0 is the only critical point of I in H and hence we may take $U = H$. Then

$$C_q(I,0) = H_q(I_0, I_0 \setminus \{0\}). \tag{2.2}$$

It was shown in Schechter [8] that there is a continuous map $\tau : N_l \to M_l$ such that

$$\tau(sv) = s\tau(v) \qquad \text{for } s \geq 0, \tag{2.3}$$

$$I(v + \tau(v)) = \inf_{w \in M_l} I(v + w) \quad \text{for } v \in N_l. \tag{2.4}$$

Let

$$S_{l1} = \{v + \tau(v) : v \in N_l\}. \tag{2.5}$$

Then, since $I(v+w)$ is convex in $w \in M_l$ and $I > 0$ on $M_l \setminus \{0\}$ for $(a,b) \in Q_l$, the mapping

$$I_0 \times [0,1] \to I_0, \quad (v + w, t) \mapsto v + (1-t)w + t\tau(v) \tag{2.6}$$

is a strong deformation retraction of the pair $(I_0, I_0 \setminus \{0\})$ onto the pair $(I_0 \cap S_{l1}, (I_0 \cap S_{l1}) \setminus \{0\})$. Hence

$$C_q(I,0) \cong H_q(I_0 \cap S_{l1}, (I_0 \cap S_{l1}) \setminus \{0\}). \tag{2.7}$$

Since I is positive homogeneous and S_{l1} is a radial manifold, the mapping

$$(I_0 \cap S_{l1}) \times [0,1] \to I_0 \cap S_{l1}, \quad (u,t) \mapsto (1-t)u \tag{2.8}$$

is a contraction of $I_0 \cap S_{l1}$ into 0, and hence

$$H_q(I_0 \cap S_{l1}) = \begin{cases} \mathbb{Z} & \text{for } q = 0, \\ 0 & \text{for } q \neq 0. \end{cases} \tag{2.9}$$

Also, the mapping $((I_0 \cap S_{l1}) \setminus \{0\}) \times [0,1] \to (I_0 \cap S_{l1}) \setminus \{0\}$, defined by

$$(u,t) \mapsto \left(1 - t + \frac{t}{\|P_l u\|}\right) u, \tag{2.10}$$

where P_l is the orthogonal projection onto N_l, is a strong deformation retraction of $(I_0 \cap S_{l1}) \setminus \{0\}$ onto $I_0 \cap \widehat{S}_{l1}$ where $\widehat{S}_{l1} = \{u \in S_{l1} : \|P_l u\| = 1\}$, and hence

$$H_q((I_0 \cap S_{l1}) \setminus \{0\}) \cong H_q(I_0 \cap \widehat{S}_{l1}). \tag{2.11}$$

Now consider the following portion of the exact sequence of the pair $(I_0 \cap S_{l1}, (I_0 \cap S_{l1}) \setminus \{0\})$.

$$\begin{aligned} H_q(I_0 \cap S_{l1}) &\to H_q(I_0 \cap S_{l1}, (I_0 \cap S_{l1}) \setminus \{0\}) \\ &\to H_{q-1}((I_0 \cap S_{l1}) \setminus \{0\}) \to H_{q-1}(I_0 \cap S_{l1}) \end{aligned} \tag{2.12}$$

It follows from (2.7), (2.9), the exactness of the sequence in (2.12), and (2.11) that

$$C_q(I,0) \cong H_{q-1}(I_0 \cap \widehat{S}_{l1}) \quad \text{if } q > 1. \tag{2.13}$$

But

$$H_{q-1}(I_0 \cap \widehat{S}_{l1}) = 0 \quad \text{if } q > d_l \tag{2.14}$$

since $\widehat{S}_{l1}$ is homeomorphic to the unit sphere in N_l, proving the second half of (i).

(iv) If (a,b) is above C_{l2}, then

$$m_l(a,b) \leq 0 \tag{2.15}$$

by (1.30), and hence

$$\inf_{w \in M_l} I(v+w) \leq 0 \quad \text{for all } v \in N_l,\ \|v\| = 1, \tag{2.16}$$

by (1.28). So $I \leq 0$ on S_{l1} by (2.3), (2.4) and hence, by (2.7),

$$C_q(I,0) \cong H_q(S_{l1}, S_{l1} \setminus \{0\}) = \begin{cases} \mathbb{Z} & \text{for } q = d_l, \\ 0 & \text{for } q \neq d_l, \end{cases} \tag{2.17}$$

(S_{l1} is contractible while $S_{l1} \setminus \{0\}$ is homotopic to the unit sphere in N_l).

(v) If (a,b) is below C_{l2}, then

$$m_l(a,b) > 0, \tag{2.18}$$

by (1.30), and hence there is a $v \in N_l$, $\|v\| = 1$ such that

$$\inf_{w \in M_l} I(v+w) > 0 \tag{2.19}$$

by (1.28). So $I_0 \cap \widehat{S}_{l1}$ is homeomorphic to a proper subset of the $(d_l - 1)$-dimensional sphere and hence, by (2.13),

$$C_{d_l}(I,0) \cong H_{d_l - 1}(I_0 \cap \widehat{S}_{l1}) = 0. \tag{2.20}$$

We prove the first half of (i), (ii) and (iii) by looking at I on another (infinite dimensional) manifold S_{l2} modeled on M_{l-1} and by using a finite dimensional approximation scheme.

Since $(a,b) \notin \Sigma$, 0 is the only critical point of I and hence

$$C_q(I,0) \cong H_q(H, I_\alpha) \tag{2.21}$$

for any $\alpha < 0$. Since I has no critical values less than 0, the negative gradient flow of I defines a strong deformation retraction of $\{u \in H : I(u) < 0\}$ onto I_α. Hence

$$C_q(I,0) \cong H_q(H, H \setminus I^0), \tag{2.22}$$

where $I^0 = \{u \in H : I(u) \geq 0\}$. Since $\bigcup_k N_k$ is dense in H, by a theorem of Palais [4],

$$H_q(H, H \setminus I^0) \cong \varinjlim H_q(N_k, N_k \setminus I^0), \tag{2.23}$$

the inductive limit of the sequence of groups $\{H_q(N_k, N_k \setminus I^0)\}$ (under the homomorphisms $H_q(N_k, N_k \setminus I^0) \to H_q(N_{k+1}, N_{k+1} \setminus I^0)$ induced by the inclusions $(N_k, N_k \setminus I^0) \hookrightarrow (N_{k+1}, N_{k+1} \setminus I^0)$). Since I is positive homogeneous, the pair $(N_k, N_k \setminus I^0)$ is homotopic to the pair $(B_k, S_k \setminus I^0)$, where

$$B_k = \{u \in N_k : \|u\| \leq 1\} \quad \text{and} \quad S_k = \partial B_k,$$

and hence

$$H_q(N_k, N_k \setminus I^0) \cong H_q(B_k, S_k \setminus I^0). \tag{2.24}$$

Consider

$$H_{q+1}(B_k, S_k)\text{‘}H_q(S_k, S_k \setminus I^0) \to H_q(B_k, S_k \setminus I^0) \to H_q(B_k, S_k), \tag{2.25}$$

part of the exact sequence of the triple $(B_k, S_k, S_k \setminus I^0)$. Since

$$H_q(B_k, S_k) = \begin{cases} \mathbb{Z} & \text{for } q = d_k, \\ 0 & \text{for } q \neq d_k, \end{cases} \tag{2.26}$$

it follows that

$$H_q(B_k, S_k \setminus I^0) \cong H_q(S_k, S_k \setminus I^0) \quad \text{if } d_k > q + 1. \tag{2.27}$$

By the Alexander duality theorem

$$H_q(S_k, S_k \setminus I^0) \cong H^{d_k - q - 1}(I^0 \cap S_k), \tag{2.28}$$

where H^* is the Alexander cohomology (see, e.g., Greenberg [4]). But, again by the positive homogeneity of I, $I^0 \cap S_k$ is a strong deformation retract of $(I^0 \cap N_k) \setminus \{0\}$ and hence

$$H^{d_k - q - 1}(I^0 \cap S_k) \cong H^{d_k - q - 1}((I^0 \cap N_k) \setminus \{0\}). \tag{2.29}$$

Combining (2.22)–(2.24) and (2.27)–(2.29), we have

$$C_q(I, 0) \cong \varinjlim H^{d_k - q - 1}((I^0 \cap N_k) \setminus \{0\}). \tag{2.30}$$

Now we recall from Schechter [8] that there is a continuous map $\theta : M_{l-1} \to N_{l-1}$ such that

$$\begin{aligned} \theta(sw) &= s\theta(w), & s &\geq 0, \\ I(\theta(w) + w) &= \sup_{v \in N_{l-1}} I(v + w), & w &\in M_{l-1}, \end{aligned} \tag{2.31}$$

and we let

$$S_{l2} = \{\theta(w) + w : w \in M_{l-1}\}. \tag{2.33}$$

Since $I(v+w)$ is concave in $v \in N_{l-1}$ and $I < 0$ on $N_{l-1} \setminus \{0\}$ for $(a,b) \in Q_l$, the mapping $(I^0 \cap N_k) \setminus \{0\} \times [0,1] \to (I^0 \cap N_k) \setminus \{0\}$, defined by

$$(v+w,t) \mapsto (1-t)v + t\theta(w) + w \tag{2.34}$$

is a strong deformation retraction of $(I^0 \cap N_k) \setminus \{0\}$ onto $(I^0 \cap S_{l2} \cap N_k) \setminus \{0\}$ and hence

$$C_q(I,0) \cong \varinjlim H^{d_k-q-1}((I^0 \cap S_{l2} \cap N_k) \setminus \{0\}). \tag{2.35}$$

But, for $k \geq l$, $(I^0 \cap S_{l2} \cap N_k) \setminus \{0\}$ is homotopic to a subset of the $((d_k - d_{l-1} - 1)$-dimensional) unit sphere in $M_{l-1} \cap N_k$ and hence

$$H^{d_k-q-1}((I^0 \cap S_{l2} \cap N_k) \setminus \{0\}) = 0 \quad \text{if } q < d_{l-1}, \tag{2.36}$$

completing the proof of (i).

(ii) If (a,b) is below C_{l1}, then

$$M_{l-1}(a,b) \geq 0, \tag{2.37}$$

by (1.29), and hence

$$\sup_{v \in N_{l-1}} I(v+w) \geq 0 \quad \text{for all } w \in M_{l-1}, \tag{2.38}$$

by (1.27). So $I \geq 0$ on S_{l2} and hence

$$\begin{aligned} H^{d_k-q-1}((I^0 \cap S_{l2} \cap N_k) \setminus \{0\}) &= H^{d_k-q-1}((S_{l2} \cap N_k) \setminus \{0\}) \\ &= \begin{cases} \mathbb{Z} & \text{for } q = d_{l-1}, \\ 0 & \text{for } q \neq d_{l-1}, \end{cases} \end{aligned} \tag{2.39}$$

for $k \geq l$ and $d_k > q+1$ ($(S_{l2} \cap N_k) \setminus \{0\}$ is homotopic to the $(d_k - d_{l-1} - 1)$-dimensional sphere).

(iii) If (a,b) is above C_{l1}, then

$$M_{l-1}(a,b) < 0, \tag{2.40}$$

by (1.29), and hence

$$\sup_{v \in N_{l-1}} I(v+w) < 0 \tag{2.41}$$

for some $w \in M_{l-1}$ by (1.27). So $(I^0 \cap S_{l2} \cap N_k) \setminus \{0\}$ is homotopic to a proper subset of the $(d_k - d_{l-1} - 1)$- dimensional sphere and hence

$$H^{d_k-d_{l-1}-1}((I^0 \cap S_{l2} \cap N_k) \setminus \{0\}) = 0. \tag{2.42}$$

This completes the proof of Theorem 1.1.

3. Local estimates

Assume that the origin is an isolated critical point of G.

Lemma 3.1.

(i) *If* $\lambda_{j-1}t^2 \le 2F(x,t)$, $|t| < \delta$, *for some* $\delta > 0$, *then*

$$G(v) \le 0, \quad v \in N_{j-1},\ \|v\| \le r,\ r > 0 \text{ small.} \tag{3.1}$$

(ii) *If* $a_0 \in [\lambda_{j-1}, \lambda_{j+1}]$ *and* $2F(x,t) \le a_0(t^-)^2 + \nu_{j-1}(a_0)(t^+)^2$, $|t| < \delta$, *for some* $\delta > 0$, *then*

$$G(\theta(w) + w) > 0, \quad w \in M_{j-1},\ 0 < \|w\| \le r,\ r > 0 \text{ small.} \tag{3.2}$$

(iii) *If* $a_0 \in [\lambda_{j-1}, \lambda_{j+1}]$ *and* $a_0(t^-)^2 + \mu_j(a_0)(t^+)^2 \le 2F(x,t)$, $|t| < \delta$, *for some* $\delta > 0$, *then*

$$G(v + \tau(v)) \le 0, \quad v \in N_j,\ \|v\| \le r,\ r > 0 \text{ small.} \tag{3.3}$$

(iv) *If* $2F(x,t) \le \lambda_{j+1}t^2$, $|t| < \delta$, *for some* $\delta > 0$, *then*

$$G(w) > 0, \quad w \in M_j,\ 0 < \|w\| \le r,\ r > 0 \text{ small.} \tag{3.4}$$

Proof. Proof of (i) is routine.

(ii) *Case 1.* If

$$2F(x,t) \equiv a_0(t^-)^2 + b_0(t^+)^2, \quad |t| < \delta, \tag{3.5}$$

where $b_0 = \nu_{j-1}(a_0)$, then

$$f(x,t) \equiv b_0t^+ - a_0t^-, \quad |t| < \delta, \tag{3.6}$$

and (1.1) has a nontrivial solution since $(a_0, b_0) \in \Sigma$.

Case 2. If

$$2F(x,t) \not\equiv a_0(t^-)^2 + b_0(t^+)^2, \quad |t| < \delta, \tag{3.7}$$

let K_0 denote the set of nontrivial critical points of $I_0 = I(\,\cdot\,, a_0, b_0)$. First we note that

$$|u|_\infty \le C\|u\|, \quad u \in K_0. \tag{3.8}$$

To see this we note that by the Sobolev imbedding, for $1/q_k = 1/q_{k-1} - 2/n$,

$$|u|_{q_k} \le C|\Delta u|_{q_{k-1}} = C|b_0u^+ - a_0u^-|_{q_{k-1}} \le C|u|_{q_{k-1}}, \quad u \in K_0. \tag{3.9}$$

Taking $q_0 = 2$ and iterating until $k > n/4$ gives

$$|u|_\infty \le C|u|_2 \le C\|u\|. \tag{3.10}$$

Thus there is a $\rho > 0$ such that

$$0 < \|u\| < \rho \Rightarrow |u|_\infty < \delta, \quad u \in K_0. \tag{3.11}$$

Then

$$\int_\Omega a_0(u^-)^2 + b_0(u^+)^2 - 2F(x,u) > 0, \quad u \in K_0 \cap B_\rho \setminus \{0\} \tag{3.12}$$

since functions in $K_0 \setminus \{0\}$ are continuous and nonzero a.e. Thus given $u \in K_0 \cap \partial B_\rho$, there is a neighbourhood $\theta(u)$ such that (3.12) holds for all $v \in B_\rho \setminus \{0\}$ with $\rho v/\|v\| \in \theta(u)$. Let $\theta = \bigcup_{u \in K_0} \theta(u)$ and $V = \{u \in H \setminus \{0\} : \rho u/\|u\| \in \theta\}$. Then (3.12) holds for all $u \in V \cap B_\rho$.

Next let

$$\begin{aligned}\widetilde{M}_{j-1}(a_0) = \inf\{I_0(u) : u = \theta(y+w) + y + w,\\ y \in E(\lambda_j),\ w \in M_j,\ \|w\| = 1,\ u \notin V\},\end{aligned} \tag{3.13}$$

where $E(\lambda_j)$ is the eigenspace of λ_j. Since $(a_0, b_0) \in C_{j1}$,

$$I_0(\theta(v) + v) \geq 0, \quad v \in M_{j-1}, \tag{3.14}$$

by Lemma 3.6 of Schechter [6], so $\widetilde{M}_{j-1}(a_0) \geq 0$. We claim that $\widetilde{M}_{j-1}(a_0) > 0$. If not, there is a sequence $u_k = \theta(y_k + w_k) + y_k + w_k \notin V$ such that $\|w_k\| = 1$ and $I_0(u_k) \to 0$.

Case 1. If $\rho_k = \|y_k\| \to \infty$, let $\widetilde{y}_k = y_k/\rho_k$ and $\widetilde{w}_k = w_k/\rho_k$. Then there is a renamed subsequence such that $\widetilde{y}_k \to \widetilde{y} \neq 0$ and $\widetilde{w}_k \to 0$, so $\widetilde{u}_k = u_k/\rho_k \to \widetilde{u} = \theta(\widetilde{y}) + \widetilde{y} \neq 0$ and $I_0(\widetilde{u}) = 0$. So $I_0(\widetilde{u}) = \inf_{v \in M_{j-1}} I_0(\theta(v) + v)$ and hence $I_0'(\widetilde{u}) \perp M_{j-1}$. Since $\widetilde{u} \in S_{j2}$, $I_0'(\widetilde{u}) \perp N_{j-1}$ also, so $I_0'(\widetilde{u}) = 0$ and hence $\widetilde{u} \in K_0$. But $\widetilde{u} \notin V$ since $\widetilde{u}_k \notin V$, a contradiction.

Case 2. If ρ_k is bounded, there is a renamed subsequence such that $y_k \to y$ in $E(\lambda_j)$ and $w_k \to w$ weakly in H (so $w \in M_j$), strongly in $L^2(\Omega)$, and a.e. in Ω. Let $u = \theta(y+w) + y + w$. Then for any $v \in N_{j-1}$,

$$I_0(v + y + w) \leq \liminf I_0(v + y_k + w_k) \leq \liminf I_0(u_k) = 0, \tag{3.15}$$

so

$$I_0(u) = \sup_{v \in N_{j-1}} I_0(v + y + w) \leq 0. \tag{3.16}$$

Combining this with (3.14), we see that $I_0(u) = 0$ and hence $\|u_k\| \to \|u\|$, so $u_k \to u$ strongly in H. So $u \neq 0$ and $u \in K_0$ as before. But this is impossible since $u \notin V$.

Next we note that for each $r > 0$ sufficiently small there is an $\varepsilon > 0$ such that

$$G(u) \geq \varepsilon \|w\|^2, \quad u = \theta(y+w) + y + w, \\ y \in E(\lambda_j),\ w \in M_j,\ \|u\| \leq r,\ u \notin V. \tag{3.17}$$

To see this we note that since N_j is finite dimensional, there is an $r > 0$ such that

$$\|v\| \leq r \Rightarrow |v|_\infty \leq \delta/2, \quad v \in N_j. \tag{3.18}$$

Let $v = \theta(y+w) + y$ and $u = v + w$. If $\|u\| \leq r$ and $|u(x)| \geq \delta$, then

$$\delta \leq |v(x)| + |w(x)| \leq \delta/2 + |w(x)|, \tag{3.19}$$

so

$$|v(x)| \leq \delta/2 \leq |w(x)| \tag{3.20}$$

and hence

$$|u(x)| \leq 2|w(x)|. \tag{3.21}$$

By (1.26), (3.13) and (3.21), $\sigma > 0$, we have

$$\begin{aligned} G(u) &\geq \int_\Omega |\nabla u|^2 - \int_{|u|<\delta} a_0(u^-)^2 + b_0(u^+)^2 - \int_{|u|\geq\delta} 2F(x,u) \\ &\geq I_0(u) - C \int_{|u|\geq\delta} (u^2 + |u|) \\ &\geq \widetilde{M}_{j-1}(a_0)\|w\|^2 - C \int_{2|w|\geq\delta} |w|^{2+\sigma} \\ &\geq [\widetilde{M}_{j-1}(a_0) - Cr^\sigma]\|w\|^2. \end{aligned} \tag{3.22}$$

If we take r sufficiently small, this implies (3.17) since $\widetilde{M}_{j-1}(a_0) > 0$.

From this it follows that for each $r > 0$ sufficiently small there is an $\varepsilon > 0$ such that

$$G(u) \geq \varepsilon, \quad u \in S_{j2} \cap \partial B_r \setminus V. \tag{3.23}$$

For otherwise there would be a sequence $u_k = \theta(y_k + w_k) + y_k + w_k \in S_{j2} \cap \partial B_r \setminus V$ such that $G(u_k) \to 0$. We see from (3.17) that $w_k \to 0$ in H. Since $\|y_k\| \leq r$ and $E(\lambda_j)$ is finite dimensional, there is a renamed subsequence such that $y_k \to y$ in $E(\lambda_j)$. Then $u_k \to v = \theta(y) + y$, so $\|v\| = r$ and $G(v) = 0$. But then (3.18) implies

$$2F(x, v(x)) \leq a_0(v(x)^-)^2 + b_0(v(x)^+)^2, \tag{3.24}$$

so

$$0 = G(v) \geq I_0(v) \geq 0 \tag{3.25}$$

by (3.14). From (3.24) and (3.25) we see that

$$2F(x,v(x)) \equiv a_0(v(x)^-)^2 + b_0(v(x)^+)^2. \tag{3.26}$$

Let $\zeta(x)$ be any function in $C_0^\infty(\Omega)$. Then for $t>0$ sufficiently small,

$$\begin{aligned}\frac{2[F(x,v+t\zeta)-F(x,v)]}{t} \\ \leq \frac{a_0[((v+t\zeta)^-)^2-(v^-)^2]+b_0[((v+t\zeta)^+)^2-(v^+)^2]}{t}.\end{aligned} \tag{3.27}$$

Taking the limit as $t\to 0$, we have

$$f(x,v(x))\zeta(x) \leq (b_0v(x)^+ - a_0v(x)^-)\zeta(x). \tag{3.28}$$

From this we conclude that

$$f(x,v(x)) \equiv b_0v(x)^+ - a_0v(x)^-. \tag{3.29}$$

Hence there is a $v\in N_j$ satisfying

$$-\Delta v = b_0v^+ - a_0v^- = f(x,v), \quad \|v\| = r, \tag{3.30}$$

for each $r>0$ sufficiently small, which is a contradiction since the origin is an isolated critical point of G by assumption.

It follows from (3.23) that there is an $r\leq\rho$ such that

$$G(u) > 0, \quad u\in S_{j2}\cap B_r\setminus(V\cup\{0\}). \tag{3.31}$$

From (3.12) and (3.14) it follows that

$$G(u) > I_0(u) \geq 0, \quad u\in S_{j2}\cap V\cap B_\rho. \tag{3.32}$$

Combining (3.31) and (3.32) we get (3.2).

(iii) Since N_j is finite dimensional and

$$\|\tau(v)\| \leq C\|v\|, \tag{3.33}$$

there is an $r>0$ such that

$$\|v\|\leq r \Rightarrow |v+\tau(v)|_\infty < \delta, \quad v\in N_j. \tag{3.34}$$

Then for $u=v+\tau(v)$, $v\in N_j$, $\|v\|\leq r$,

$$G(u) \leq \int_\Omega |\nabla u|^2 - a_0(u^-)^2 - \mu_j(a_0)(u^+)^2 \leq 0, \tag{3.35}$$

by Lemma 3.7 of Schechter [6] since $(a_0,\mu_j(a_0))\in C_{j2}$.

For the proof of (iv) see Theorem 1.3 in Schechter [9]. □

Lemma 3.2. *If* (1.31) *holds for some* $a_0 \in [\lambda_{j-1}, \lambda_{j+1}]$, *then* G *has a local* $(d_{j-1}, 1)$*-linking near the origin.*

Proof. We take

$$U = \{v + \theta(w) + w : v \in N_{j-1},\ w \in M_{j-1},\ \|v\| \le r,\ \|w\| \le r\} \tag{3.36}$$

with $r > 0$ sufficiently small,

$$A = \partial U \cap N_{j-1}, \quad S = U \cap S_{j2}, \quad \text{and} \quad B = U \cap N_{j-1}. \tag{3.37}$$

The mapping $(U \setminus S) \times [0,1] \to U \setminus S$, defined by

$$(v + \theta(w) + w, t) \mapsto \begin{cases} v + (1-2t)(\theta(w) + w), & \text{for } 0 \le t \le 1/2, \\ (2 - 2t + (2t-1)r/\|v\|)v, & \text{for } 1/2 < t \le 1, \end{cases}$$

is a strong deformation retraction of $U \setminus S$ onto A and hence the rank of $i_{1*} : H_{d_{j-1}-1}(A) \to H_{d_{j-1}-1}(U \setminus S)$ is $1 + \delta_{d_{j-1},1}$ since A is the sphere in N_{j-1}. On the other hand, the rank of $i_{2*} : H_{d_{j-1}-1}(A) \to H_{d_{j-1}-1}(B)$ is $\delta_{d_{j-1},1}$ since B is the disk in N_{j-1}.

By (i) and (ii) of Lemma 3.1, $G \le 0$ on B and $G > 0$ on $S \setminus \{0\}$ for r sufficiently small. □

Lemma 3.3. *If* (1.32) *holds for some* $a_0 \in [\lambda_{j-1}, \lambda_{j+1}]$, *then* G *has a local* $(d_j, 1)$*-linking near the origin.*

Proof. Take

$$U = \{v + \tau(v) + w : v \in N_j,\ w \in M_j,\ \|v\| \le r,\ \|w\| \le r\} \tag{3.38}$$

with $r > 0$ sufficiently small,

$$A = \partial U \cap S_{j1}, \quad S = U \cap M_j, \quad \text{and} \quad B = U \cap S_{j1}. \tag{3.39}$$

The mapping $(U \setminus S) \times [0,1] \to U \setminus S$, defined by

$$(v + \tau(v) + w, t) \mapsto \begin{cases} v + \tau(v) + (1-2t)w & \text{for } 0 \le t \le 1/2, \\ (2 - 2t + (2t-1)r/\|v\|)(v + \tau(v)) & \text{for } 1/2 < t \le 1, \end{cases}$$

is a strong deformation retraction of $U \setminus S$ onto A and hence the rank of $i_{1*} : H_{d_j-1}(A) \to H_{d_j-1}(U \setminus S)$ is $1 + \delta_{d_j,1}$ since A is homeomorphic to the sphere in N_j, while the rank of $i_{2*} : H_{d_j-1}(A) \to H_{d_j-1}(B)$ is $\delta_{d_j,1}$ since B is contractible.

By (iii) and (iv) of Lemma 3.1, $G \le 0$ on B and $G > 0$ on $S \setminus \{0\}$ for r sufficiently small. □

4. Estimates at infinity

For $\alpha \in \mathbb{R}$, we denote by G_α the sublevel set $\{u \in H : G(u) \leq \alpha\}$.

Lemma 4.1. *If $(a,b) \in Q_l \setminus \Sigma$ lies above C_{l1} and*

$$2|P(x,t)| \leq W(x) \in L^1(\Omega), \quad x \in \Omega, \ t \in \mathbb{R}, \tag{4.1}$$

then for any $j \neq l+1$,

$$H_{d_{j-1}}(H, G_\alpha) = 0 \quad \textit{for all } \alpha < 0, \ |\alpha| \textit{ sufficiently large.} \tag{4.2}$$

Proof. By (the proof of) Lemma 5.1, the set of critical values of G is bounded. Choose $\alpha < 0$ with $|\alpha|$ so large that G has no critical values in $(-\infty, \alpha]$, and set $\alpha_i = \alpha - i|W|_1$ for $i = 1, 2, 3$. Then it is easily seen that

$$I_{\alpha_3} \subset G_{\alpha_2} \subset I_{\alpha_1} \subset G_\alpha. \tag{4.3}$$

Since G has no critical values in $[\alpha_2, \alpha]$, the negative gradient flow of G defines a strong deformation retraction of G_α onto G_{α_2}. Similarly, the negative gradient flow of I defines a strong deformation retraction of I_{α_1} onto I_{α_3}. Composing them we obtain a strong deformation retraction of G_α onto I_{α_3} and hence

$$H_{d_{j-1}}(H, G_\alpha) \cong H_{d_{j-1}}(H, I_{\alpha_3}). \tag{4.4}$$

But, since $(a,b) \notin \Sigma$, 0 is the only critical point of I and hence

$$H_{d_{j-1}}(H, I_{\alpha_3}) \cong C_{d_{j-1}}(I, 0) = 0,$$

for $j \neq l+1$, by (i) and (ii) of Theorem 1.1. □

Lemma 4.2. *If $(a,b) \in Q_l \setminus \Sigma$ lies below C_{l2} and* (4.2) *holds, then for any $j \neq l-1$,*

$$H_{d_j}(H, G_\alpha) = 0 \quad \textit{for all } \alpha < 0, \ |\alpha| \textit{ sufficiently large.} \tag{4.6}$$

Proof. As in the proof of Lemma 4.1,

$$H_{d_j}(H, G_\alpha) \cong H_{d_j}(H, I_{\alpha_3}) \cong C_{d_j}(I, 0) = 0, \tag{4.7}$$

for $j \neq l-1$ by (i) and (ii) of Theorem 1.1. □

5. Proofs of Theorems 1.4 and 1.5

For $(a,b) \notin \Sigma$, there is an a priori estimate for the solution by Lemma 5.1 below, and hence we may assume (4.1). Solutions of the modified problem will still be solutions of the original problem. Then Theorem 1.4 (resp. 1.5) follows from Theorem 1.8 and Lemmas 3.2 and 4.1 (resp. 3.3 and 4.2).

Lemma 5.1. *If* $(a,b) \notin \Sigma$ *and* (1.26) *holds, then there is a constant* C *such that*

$$|u(x)| \le C, \tag{5.1}$$

for all solutions u *of* (1.1).

Proof. By a standard iteration argument, it suffices to obtain an a priori estimate in H. So suppose $\rho_k = \|u_k\| \to \infty$ for a sequence $\{u_k\}$ of solutions of (1.1), and let $\widetilde{u}_k = u_k/\rho_k$. Then $\|\widetilde{u}_k\| = 1$, and there is a renamed subsequence such that $\widetilde{u}_k \to \widetilde{u}$ weakly in H, strongly in $L^2(\Omega)$, and a.e. in Ω. Now

$$0 = \frac{(G'(u_k), u_k)}{2\rho_k^2} = 1 - \int_\Omega \frac{f(x,u_k)\widetilde{u}_k}{\rho_k}. \tag{5.2}$$

By (1.26),

$$\frac{|f(x,u_k)\widetilde{u}_k|}{\rho_k} \le C\bigg(\widetilde{u}_k^2 + \frac{|\widetilde{u}_k|}{\rho_k}\bigg) \tag{5.3}$$

and the right hand side converges to $C\widetilde{u}^2$ in $L^1(\Omega)$. Since

$$\frac{f(x,u_k)\widetilde{u}_k}{\rho_k} \to a(\widetilde{u}^-)^2 + b(\widetilde{u}^+)^2 \quad \text{a.e.}, \tag{5.4}$$

it follows that

$$\int_\Omega \frac{f(x,u_k)\widetilde{u}_k}{\rho_k} \to \int_\Omega a(\widetilde{u}^-)^2 + b(\widetilde{u}^+)^2. \tag{5.5}$$

Hence

$$\int_\Omega a(\widetilde{u}^-)^2 + b(\widetilde{u}^+)^2 = 1, \tag{5.6}$$

so $\widetilde{u} \not\equiv 0$. Also,

$$0 = \frac{(G'(u_k), v)}{2\rho_k} = \int_\Omega \nabla\widetilde{u}_k \cdot \nabla v - \frac{f(x,u_k)v}{\rho_k}, \quad v \in H. \tag{5.7}$$

Again, by (1.26),

$$\frac{|f(x,u_k)|}{\rho_k} \le C\bigg(|\widetilde{u}_k| + \frac{1}{\rho_k}\bigg) \tag{5.8}$$

and the right hand side converges to $C|\widetilde{u}|$ in $L^2(\Omega)$. Since

$$\frac{f(x,u_k)}{\rho_k} \to b\widetilde{u}^+ - a\widetilde{u}^- \quad \text{a.e.}, \tag{5.9}$$

we have

$$\int_\Omega \nabla\widetilde{u} \cdot \nabla v - (b\widetilde{u}^+ - a\widetilde{u}^-)v = 0, \quad v \in H, \tag{5.10}$$

i.e., $\widetilde{u}$ satisfies (1.3). Since $(a,b) \notin \Sigma$, we must have $\widetilde{u} \equiv 0$, contradicting the conclusion reached earlier. □

References

[1] E. N. Dancer, *Remarks on jumping nonlinearities*, preprint.

[2] ———, *Multiple solutions of asymptotically homogeneous problems*, Ann. Mat. Pura Appl. (4) **152** (1988), 63–78.

[3] M. J. Greenberg, *Lectures on Algebraic Topology*, W. A. Benjamin, Inc., New York–Amsterdam, 1967.

[4] R. S. Palais, *Homotopy theory of infinite dimensional manifolds*, Topology **5** (1966), 1–16.

[5] K. Perera, *Homological local linking*, Abstr. Appl. Anal. **3** (1998), 181–189.

[6] M. Schechter., *Resonance problems with respect to the Fučík spectrum*, preprint.

[7] ———, *The Fučík spectrum*, Indiana Univ. Math. J. **43** (1994), 1139–1157.

[8] ———, *Type II regions between curves of the Fučík spectrum*, NoDEA Nonlinear Differential Equations Appl. **4** (1997), 459–476.

[9] ———, *New linking theorems*, Rend. Sem. Mat. Univ. Padova **99** (1998), 1–15.

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