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SPATIALLY DISCRETE WAVE MAPS ON (1 + 2)-DIMENSIONAL SPACE-TIME

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Dedicated to Professor Jürgen Moser on the occasion of his 70th birthday

1. Introduction

Let N be a smooth, compact manifold without boundary of dimension k. By Nash's embedding theorem we may assume $N \subset \mathbb{R}^n$ isometrically for some n. A wave map $u = (u^1, \ldots, u^n) : \mathbb{R} \times \mathbb{R}^2 \to N \hookrightarrow \mathbb{R}^n$ by definition is a stationary point for the action integral

$$\mathcal{A}(u; Q) = \int_Q \mathcal{L}(u)\, dz, \quad Q \subset \mathbb{R} \times \mathbb{R}^2,$$

with Lagrangian

$$\mathcal{L}(u) = \frac{1}{2}(|\nabla u|^2 - |u_t|^2)$$

with respect to compactly supported variations u_ε satisfying the "target constraint" $u_\varepsilon(\mathbb{R} \times \mathbb{R}^2) \subset N$. Equivalently, a wave map is a solution to the equation

$$\Box u = u_{tt} - \Delta u = A(u)(Du, Du) \perp T_u N, \tag{1}$$

where A is the second fundamental form of N, $T_p N \subset T_p \mathbb{R}^n$ is the tangent space to N at a point $p \in N$, and "$\perp$" means orthogonal with respect to the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$.

We denote points on Minkowski space as $z = (t, x) = (x^\alpha)_{0 \le \alpha \le 2} \in \mathbb{R} \times \mathbb{R}^2$ and let $Du = (u_t, \nabla u) = (\partial_\alpha u)_{0 \le \alpha \le 2}$ denote the vector of space-time derivatives.

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Moreover, we raise and lower indeces with the Minkowski metric $\eta = (\eta_{\alpha\beta}) = (\eta^{\alpha\beta}) = \operatorname{diag}(-1, 1, 1)$. A summation convention is used; thus, $\Box u = -\partial^\alpha \partial_\alpha u$. Finally, we abbreviate

$$A(u)(Du, Du) = A(u)(\partial^\alpha u, \partial_\alpha u).$$

Recall that locally, near any point $p_0 \in N$, letting $\nu_{k+1}, \dots, \nu_n$ be a smooth orthonormal frame for the normal bundle $TN^\perp$ near p_0, that is, vector fields such that $(\nu_l(p))_{k<l\le n}$ is an orthonormal basis for the normal space $T_pN^\perp$ at any $p \in N$ near p_0, we have

$$A(p)(v, w) = A^l(p)(v, w)\nu_l(p)$$

at any such p, where

$$A^l(p)(v, w) = \langle v, d\nu_l(p)w\rangle$$

is the second fundamental form of N with respect to ν_l.

Given $u_0 : \mathbb{R}^2 \to N$, $u_1 : \mathbb{R}^2 \to \mathbb{R}^n$ satisfying the condition $u_1(x) \in T_{u_0(x)}N$ for all $x \in \mathbb{R}^2$, that is, $(u_0, u_1) : \mathbb{R}^2 \to TN$, we consider the Cauchy problem for wave maps u with initial data

$$(u, u_t)_{|t=0} = (u_0, u_1) : \mathbb{R}^2 \to TN \tag{2}$$

of finite energy

$$E_0 = \frac{1}{2}\int_{\mathbb{R}^2} (|u_1|^2 + |\nabla u_0|^2)\, dx.$$

Specifically, in the present paper we study the relation between solutions u of (1), (2) on $\mathbb{R} \times \mathbb{R}^2$ and their spatially discrete counterparts $u^h : \mathbb{R} \times M_h \to N \hookrightarrow \mathbb{R}^n$, where $\mathbb{R}^2$ is replaced by a uniform square lattice $M_h = (h\mathbb{Z})^2$ of mesh-size $h \to 0$.

In a previous paper [12], jointly with Vladimir Šverák, we studied the time-independent case and showed that a weakly convergent family of harmonic maps $u^h \in H^1(T_h; N)$ on a periodic lattice $T_h = (h\mathbb{Z})^2/\mathbb{Z}^2$ as $h \to 0$ accumulates at a harmonic map u on the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Here we extend this result to the time dependent case; see our main result Theorem 4.1 below. Since the Cauchy problem for wave maps on a spatially discrete domain is equivalent to an initial value problem for a system of ordinary differential equations which can be solved globally for any mesh-size h in view of the uniform energy bounds available, as a corollary we reobtain our existence result from [11] for global weak solutions to the Cauchy problem (1), (2) for wave maps on (1+2)-dimensional Minkowski space; see Theorem 5.1. The methods we use are similar to the methods of [12]. We essentially rely on our previous weak compactness results [7], [8] with Freire and exploit the equivalent formulation of (1) as a Hodge system as in [3] or [9] to which compensation techniques may be applied in a way similar to the work of Hélein [9], [10], Evans [5], and Bethuel [1] on weakly harmonic maps, that is, time independent solutions of (1). (See [7] for further references and a detailed comparison of the elliptic and hyperbolic cases.)

2. Technical framework

Whenever possible, we use the same notations as in [12] regarding difference calculus, discrete Hodge theory, interpolation and discretization. For the reader's convenience we recall the definition at each first appearance of a symbol.

2.1. Differential forms. For $h > 0$ with $h^{-1} \in \mathbb{N}$ let $M_h = (h\mathbb{Z})^2$, $T_h = (h\mathbb{Z}^2)/\mathbb{Z}^2$ with generic point $x = x_h = (x_h^1, x_h^2)$, and let $S^1 = \mathbb{R}/\mathbb{Z}$ with generic point $t = x^0 = x_h^0$. Differential forms on $\mathbb{R} \times M_h$ or $S^1 \times T_h$ may be most conveniently expressed in terms of the standard basis dx^α, $dx^\alpha \wedge dx^\beta$, $0 \le \alpha < \beta \le 2$, and $dt \wedge dx^1 \wedge dx^2 = dz$. In particular, for a 1-form φ^h we have $\varphi^h = \varphi^h_\alpha dx^\alpha$, and a 2-form b^h may be written in the standard form

$$b^h = b_0^h dx^1 \wedge dx^2 - b_1^h dx^0 \wedge dx^2 + b_2^h dx^0 \wedge dx^1 = b^h_{\widehat{\alpha\beta}} dx^\alpha \wedge dx^\beta$$

with real-valued functions φ^h_α, b^h_α.

The Hodge $*_g$-operator with respect to either the Euclidean metric $g = \mathrm{eucl}$ or the Minkowski metric $g = \eta$ in terms of this basis is defined as

$$\begin{aligned} *_g 1 &= dz, *_g dz = 1, \\ *_g \varphi^h &= g^{00}\varphi_0^h dx^1 \wedge dx^2 - \varphi_1^h dx^0 \wedge dx^2 + \varphi_2^h dx^0 \wedge dx^1, \\ *_g b^h &= g^{00} b_0^h dx^0 + b_1^h dx^1 + b_2^h dx^2, \end{aligned}$$

where $(g^{\alpha\beta}) = g^{-1} = \mathrm{diag}(\pm 1, 1, \dots, 1)$ and $\varphi^h = \varphi^h_\alpha dx^\alpha$, etc., as above.

From this definition we immediately deduce that $*_g \circ *_g = \mathrm{id}$ and, moreover,

$$\begin{aligned} \varphi^h \wedge *_g \varphi^h &= (*_g \varphi^h) \wedge \varphi^h = g^{\alpha\beta} \varphi^h_\alpha \varphi^h_\beta dz, \\ b^h \wedge *_g b^h &= (*_g b^h) \wedge b^h = g^{\alpha\beta} b^h_\alpha b^h_\beta dz \end{aligned}$$

for any 1-form φ^h or 2-form b^h as above.

Finally, two forms φ^h, ψ^h of the same degree may be contracted by letting

$$\varphi^h \cdot_g \psi^h dz = \varphi^h \wedge *_g \psi^h = g^{\alpha\beta} \varphi^h_\alpha \psi^h_\beta dz.$$

Spatially discrete differential and co-differential are defined as follows.

For $u^h : \mathbb{R} \times M_h \to \mathbb{R}, h \neq 0$, we let $d^h u^h = \partial^h_\alpha u^h dx^\alpha$ with components

$$\partial_0^h u^h = \partial_t^h u^h = \partial_t u^h = u_t^h, \quad \partial^h_\alpha u(z) = \frac{u(z + h\underline{e}_\alpha) - u(z)}{h}, \quad \alpha = 1, 2,$$

where $(\underline{e}_\alpha)_{1 \le \alpha \le 2}$ is the standard basis for $\mathbb{R}^2$. For a 1-form $\varphi^h = \varphi^h_\alpha dx^\alpha$ then

$$d^h \varphi^h = \partial^h_\alpha \varphi^h_\beta dx^\alpha \wedge dx^\beta$$

and for a 2-form b^h as above, $d^h b^h = \partial^h_\alpha b^h_\alpha \, dz$. The co-differential (with respect to g) is

$$\delta^h_g = - *_g \circ d^{-h} \circ *_g.$$

Explicitly, for $\varphi^h = \varphi^h_\alpha dx^\alpha$, $h \neq 0$, we have

$$\delta^h_g \varphi^h = -g^{\alpha\beta} \partial^{-h}_\alpha \varphi^h_\beta = -\partial^{-h}_2 \varphi^h_2 - \partial^{-h}_1 \varphi^h_1 - g^{00} \partial^{-h}_0 \varphi^h_0,$$

and similarly for forms of higher degree. Clearly, we have $d^h \circ d^h = 0, \delta^h \circ \delta^h = 0$ for all $h \neq 0$.

Finally, for $h > 0$, we let

$$\Box^h = \Box^{-h} = d^h \delta^h_\eta + \delta^h_\eta d^h = d^{-h} \delta^{-h}_\eta + \delta^{-h}_\eta d^{-h}$$

denote the spatially discrete wave operator, acting on forms on $\mathbb{R} \times M_h$. Explicitly, we have

$$\begin{aligned} \Box^h u^h = \delta^h_\eta d^h u^h &= (\partial^2_t - \Delta^h) u^h, & \Box^h (\varphi^h_\alpha dx^\alpha) &= (\Box^h \varphi^h_\alpha) \, dx^\alpha, \\ \Box^h (b^h_{\hat{\alpha\beta}} dx^\alpha \wedge dx^\beta) &= (\Box^h b^h_{\hat{\alpha\beta}}) dx^\alpha \wedge dx^\beta, & \Box^h (f^h dz) &= (\Box^h f^h) \, dz, \end{aligned}$$

where $\Delta^h = \Delta^{-h}$ is the discrete (5-point) Laplace operator on T_h; that is, $\Box^h$ acts as a diagonal operator with respect to the standard basis of forms.

Also note the product rule

$$\begin{aligned} \partial^h_\alpha (u^h v^h) &= \partial^h_\alpha u^h v^h + \tau^h_\alpha u^h \partial^h_\alpha v^h \\ &= \partial^h_\alpha u^h \tau^h_\alpha v^h + u^h \partial^h_\alpha v^h = \partial^h_\alpha u^h m^h_\alpha v^h + m^h_\alpha u^h \partial^h_\alpha v^h, \end{aligned} \tag{3}$$

and

$$\delta^{-h}_g (\varphi^h f^h) = -g^{\alpha\beta} \partial^h_\alpha (\varphi^h_\beta f^h) = -g^{\alpha\beta} [(\partial^h_\alpha \varphi^h_\beta) f^h + \tau^h_\alpha \varphi^h_\beta \partial^h_\alpha f^h];$$

in particular, we have

$$\delta^{-h}_g (\tau^{-h}_\alpha \varphi^h_\alpha dx^\alpha \cdot f^h) = -g^{\alpha\beta} [(\partial^{-h}_\alpha \varphi^h_\beta) f^h + \varphi^h_\beta \partial^h_\alpha f^h] = (\delta^h_g \varphi^h) f^h - \varphi^h \cdot_g d^h f.$$

Here and in the following we denote

$$\begin{aligned} \tau^{\pm h}_0 u^h &= m^{\pm h}_0 u^h = u^h, \tau^{\pm h}_\alpha u^h = u^h (\cdot \pm h \underline{e}_\alpha), \\ m^{\pm h}_\alpha u^h &= (u^h + \tau^{\pm h}_\alpha u^h)/2, \quad \alpha = 1, 2. \end{aligned}$$

2.2. Dirichlet's integral. For $u^h : \mathbb{R} \times M_h \to \mathbb{R}$ we let

$$e_h(u^h) = \frac{1}{4} \sum_{0 \le \alpha \le 2} (|\partial^h_\alpha u^h|^2 + |\partial^{-h}_\alpha u^h|^2) \tag{4}$$

be the energy density and let

$$E_h(u^h(t)) = \int_{M_h} e_h(u^h(t)) := h^2 \sum_{x_h \in M_h} e_h(u^h(t, x_h))$$

be the energy of u^h at any time t. If $h^{-1} \in \mathbb{N}$ and if u^h has period one in each variable, we regard u^h as a map $u^h : S^1 \times T_h \to \mathbb{R}$. Then we define

$$D_h(u^h) = \int_{S^1 \times T_h} e_h(u^h) := \int_0^1 h^2 \sum_{x_h \in T_h} e_h(u^h)(t, x_h)\, dt,$$

and similarly for forms of degree ≥ 1.

Note that the first variation of D_h at u^h in direction v^h is given by

$$\begin{aligned}\langle dD_h(u^h), v^h\rangle &= \frac{d}{d\varepsilon} D_h(u^h + \varepsilon v^h)|_{\varepsilon=0} \\ &= \frac{1}{2}\sum_\alpha \int_{S^1 \times T_h} (\partial_\alpha^h u^h \partial_\alpha^h v^h + \partial_\alpha^{-h} u^h \partial_\alpha^{-h} v^h) \\ &= \sum_\alpha \int_{S^1 \times T_h} \partial_\alpha^h u^h \partial_\alpha^h v^h = -\int_{S^1 \times T_h} \Delta_3^h u^h v^h,\end{aligned}$$

where $-\Delta_3^h = \delta_{\mathrm{eucl}}^h d^h + d^h \delta_{\mathrm{eucl}}^h = -\partial_t^2 - \Delta^h$ is the spatially discrete Laplace operator, acting on forms on $S^1 \times T_h$.

Similarly, for $u^h : \mathbb{R} \times M_h \to \mathbb{R}^n$ the spatially discrete Lagrangian of u^h is

$$\mathcal{L}_h(u^h) = \frac{1}{4}\eta^{\alpha\beta}(\langle \partial_\alpha^h u^h, \partial_\beta^h u^h\rangle + \langle \partial_\alpha^{-h} u^h, \partial_\beta^{-h} u^h\rangle).$$

The action integral over any spatially discrete domain $Q \subset \mathbb{R} \times M_h$ then is

$$\mathcal{A}_h(u^h; Q) = \int_Q \mathcal{L}_h(u^h),$$

and u^h is stationary for $\mathcal{A}_h$ with respect to compactly supported variations if and only if

$$\begin{aligned}\langle d\mathcal{A}_h(u^h), v^h\rangle &= \frac{d}{d\varepsilon}\mathcal{A}_h(u^h + \varepsilon v^h)|_{\varepsilon=0} \\ &= \int_{\mathbb{R} \times M_h} \eta^{\alpha\beta}\langle \partial_\alpha^h u^h, \partial_\beta^h v^h\rangle = \int_{\mathbb{R} \times M_h} \Box^h u^h v^h = 0\end{aligned} \tag{5}$$

for any $v^h \in C_0^\infty(\mathbb{R} \times M_h)$; that is, if and only if $\Box^h u^h = 0$.

2.3. Hodge decomposition. Analogous to the continuous case or the case of a planar lattice, we have the following result on Hodge decomposition of forms on $S^1 \times T_h$.

Proposition 2.1. *Any* 1*-form* $\varphi^h = \varphi_\alpha^h dx^\alpha$ *on* $S^1 \times T_h$ *may be decomposed uniquely as*

$$\varphi^h = d^h a^h + \delta_{\mathrm{eucl}}^h b^h + c^h \tag{6}$$

where a^h *and* b^h *are normalized to satisfy*

$$\int_{S^1 \times T_h} a^h = \int_{S^1 \times T_h} b_{\widehat{\alpha\beta}}^h = 0 \quad \text{for } 0 \leq \alpha < \beta \leq 2,\ d^h b^h = 0, \tag{7}$$

and $d^h c^h = 0$, $\delta^h_{\mathrm{eucl}} c^h = 0$.

PROOF. Let a^h, b^h be the unique solutions to the equations

$$-\Delta^h_3 a^h = \delta^h_{\mathrm{eucl}}\varphi^h, \quad -\Delta^h_3 b^h = d^h\varphi^h,$$

normalized by (7), obtained, for instance, by minimizing the integral

$$F_h(a^h) = \int_{S^1\times T_h} \{e_h(a^h) - a^h\delta^h_{\mathrm{eucl}}\varphi^h\}$$

among functions $a^h : S^1 \times T_h \to \mathbb{R}$ satisfying (7), and similarly for b^h. The remainder $c^h = \varphi^h - d^h a^h - \delta^h_{\mathrm{eucl}} b^h$ then satisfies

$$d^h c^h = d^h\varphi^h + \Delta^h_3 b^h = 0, \quad \delta^h_{\mathrm{eucl}} c^h = \delta^h_{\mathrm{eucl}}\varphi^h + \Delta^h_3 a^h = 0,$$

as desired. □

Via the Euclidean Hodge $*$-operator, we obtain an analogous decomposition of 2-forms. Observe that the decomposition (6) is L^2-orthogonal and hence we have

$$\int_{S^1\times T_h} |\varphi^h|^2 = \int_{S^1\times T_h} (|d^h a^h|^2 + |\delta^h_{\mathrm{eucl}} b^h|^2 + |c^h|^2). \tag{8}$$

2.4. Discretization and interpolation. We discretize a map $u : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ by letting, for each $t \in \mathbb{R}$,

$$u^h(t, x_h) = h^{-2}\int_{Q^+_h(x_h)} u(t,x)\,dx, \quad x_h \in M_h,$$

where for $l \in \mathbb{N}$ the set

$$Q^+_{lh}(x_h) = \{x = (x^1, x^2) \in \mathbb{R}^2 \,:\, x^\alpha_h \le x^\alpha < x^\alpha_h + lh,\ \alpha = 1, 2\}$$

is a square with lower left corner x_h of size lh, and similarly for periodic maps $u : T^3 = S^1 \times T \to \mathbb{R}$, assuming $h^{-1} \in \mathbb{N}$.

Conversely, we interpolate a map $u^h : \mathbb{R} \times M_h \to \mathbb{R}$ either trivially, by letting

$$u^h(t,x) = u^h(t,x_h) \quad \text{for } x \in Q^+_h(x_h),\ x_h \in M_h,$$

or bilinearly, by letting

$$\overline{u}^h(t,x) = u^h(t,x_h) + \sum_{\alpha=1,2} \xi^\alpha \partial^h_\alpha u^h(t,x_h) + \xi^1\xi^2\partial^h_1\partial^h_2 u^h(t,x_h)$$

whenever $x = x_h + \xi \in Q^+_h(x_h)$, $x_h \in M_h$, and similarly for maps $u^h : S^1 \times T_h \to \mathbb{R}$.

Observe that

$$\partial^{\pm h}_\alpha u^h(t,x) = \partial^{\pm h}_\alpha u^h(t,x_h)$$

for all $t \in S^1$, $x \in Q^+_h(x_h)$, $x_h \in T_h$; moreover,

$$\partial_1 \overline{u}^h(t, x_h + h\xi) = (1-\xi_2)\partial^h_1 u^h(t,x_h) + \xi_2\partial^h_1 u^h(t, x_h + h\underline{e}_2)$$

for $t \in S^1$, $x_h \in T_h$, $\xi \in Q_1^+(0)$, and similarly with x^1- and x^2-directions exchanged.

From this identity the following result is immediate.

PROPOSITION 2.2. *For $u^h : \mathbb{R} \times M_h \to \mathbb{R}$ with $\sup_t E_h(u^h(t)) < \infty$ we have $\overline{u}^h \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^2)) \cap C^0(\mathbb{R} \times \mathbb{R}^2)$, and with a uniform constant C for all $t \in \mathbb{R}$ there holds*

(i) $||(\overline{u}^h - u^h)(t)||^2_{L^\infty(Q_h^+(x_h))} \le C \int_{Q_{2h}^+(x_h)} e_h(u^h(t))$ *for all $x_h \in M_h$;*
(ii) $||(\overline{u}^h - u^h)(t)||^2_{L^2(\mathbb{R}^2)} \le Ch^2 E_h(u^h(t))$*;*
(iii) $C^{-1} E_h(u^h(t)) \le E(\overline{u}^h(t)) \le C E_h(u^h(t))$.

Moreover, by comparing u^h and $\overline{u}^h$, using Proposition 2.2(i), it is clear that the Poincaré inequality

$$||(u^h - u^h_{r,x_0}(t)||^2_{L^2(Q_r(x_0))} \le Cr^2 E_h(u^h(t); Q_{r+h}(x_0))$$

holds for every $(t, x_0) \in \mathbb{R} \times M_h$, any $r = kh$, $k \in \mathbb{N}$, where

$$Q_r(x_0) = \{x = (x^1, x^2) \, : \, |x^\alpha - x_0^\alpha| < r, \; \alpha = 1, 2\}$$

and where

$$u^h_{r,x_0}(t) = \fint_{Q_r(x_0)} u^h(t, x)$$

is the mean value.

Similar results hold true if we also take time dependence into account.

For $z_0 = (x_0^\alpha)_{0 \le \alpha \le 2}$, $r > 0$, let

$$P_r(z_0) = \prod_{\alpha=0}^{2}]x_0^\alpha - r, x_0^\alpha + r[$$

and let $u^h : \mathbb{R} \times M_h \to \mathbb{R}$ with locally finite energy as above. For $z \in \mathbb{R} \times M_h$, $r = kh$, $k \in \mathbb{N}$, we also let

$$u^h_{r,z} = \fint_{P_r(z)} u^h$$

denote the average of u^h on $P_r(z)$.

PROPOSITION 2.3. *For any $z = (t, x) \in \mathbb{R} \times M_h$, $0 < h \le r = kh$, $k \in \mathbb{N}$, $\alpha \in \{1, 2\}$, with an absolute constant C there holds*

(i) $|(\tau_\alpha^h u^h - u^h)(z)|^2 \le Ch^{-1} \int_{P_{2h}(z)} e_h(u^h)$,
(ii) $||u^h - u^h_{r,z}||^2_{L^2(P_r(z))} \le Cr^2 \int_{P_{r+h}(z)} e_h(u^h)$.

PROOF. (i) Integrating in time, for any $s \in]t-h, t+h[$ we obtain

$$|(\tau_\alpha^h u^h - u^h)(t, x)| \le |(\tau_\alpha^h u^h - u^h)(s, x)| + \int_{t-h}^{t+h} \left(|\partial_t(\tau_\alpha^h u^h)| + |\partial_t u^h|\right) ds.$$

Squaring and averaging with respect to s, in view of Proposition 2.2(i) we find

$$|(\tau_\alpha^h u^h - u^h)(z)|^2 \le h^{-1}\int_{t-h}^{t+h} |(\tau_\alpha^h u^h - u^h)(s,x)|^2\,ds + Ch\int_{t-h}^{t+h} (|\partial_t \tau_\alpha^h u^h|^2 + |\partial_t u^h|^2)\,ds \le Ch^{-1}\int_{P_{2h}(z)} e_h(u^h).$$

(ii) The asserted inequality is immediate from Proposition 2.2(i) and the usual Poincaré inequality, applied to the function $\overline{u}^h$. □

If we consider the trivial extensions of a function $u^h : \mathbb{R}\times M_h \to \mathbb{R}$ and its energy density $e_h(u^h)$ to $\mathbb{R}\times\mathbb{R}^2$, Proposition 2.3(ii) remains valid for all $z\in\mathbb{R}\times\mathbb{R}^2$ and $0<h\le r$.

Regarding a function $u^h : S^1\times T_h \to \mathbb{R}$ as a periodic function on $\mathbb{R}\times M_h$, the above results also hold for $u^h : S^1\times T_h\to\mathbb{R}$. In addition, by integrating in time, from Proposition 2.2(iii) we obtain the following result.

Proposition 2.4. *For $u^h : S^1\times T_h\to\mathbb{R}$ with $D_h(u^h)<\infty$ we have $\overline{u}^h\in H^1(T^3)$ and with a uniform constant C there holds*

$$C^{-1}D_h(u^h)\le D(\overline{u}^h) = \frac{1}{2}\int_{T^3}(|u_t^h|^2 + |\nabla u^h|^2)\,dz \le CD_h(u^h).$$

In view of Proposition 2.4 we will say that $u^h \rightharpoonup u$ weakly in $H^1(T^3)$ as $h\to 0$, if $\overline{u}^h\rightharpoonup u$ weakly in $H^1(T^3)$, or, equivalently, if $u^h\rightharpoonup u$ and $d^h u^h \rightharpoonup du$ weakly in $L^2(T^3)$, where u^h, $d^h u^h$ denote the trivial extensions of u^h, $d^h u^h$ to T^3, defined above.

3. Spatially discrete wave maps

In analogy with the continuous case a map $u^h : \mathbb{R}\times M_h \to N \hookrightarrow \mathbb{R}^n$ is a spatially discrete wave map if and only if u^h is stationary for $\mathcal{A}_h$ among maps $u_\varepsilon^h : \mathbb{R}\times M_h\to N$ such that $u_\varepsilon^h = u^h$ at $\varepsilon = 0$ and outside some compact set $Q\subset\mathbb{R}\times M_h$; in particular, then

$$\frac{d}{d\varepsilon}\mathcal{A}_h(\pi_N(u^h+\varepsilon v^h))|_{\varepsilon=0} = 0$$

for all $v^h\in C_0^\infty(\mathbb{R}\times M_h;\mathbb{R}^n)$, where $\pi_N : U_\delta(N)\to N$ is the smooth map projecting a point p in a tubular neighbourhood of N of sufficiently small width $\delta>0$ to its nearest neighbour $\pi_N(p)\in N$.

Computing the first variation using (5), we deduce that u^h satisfies the equation

$$d\pi_N(u^h)\Box^h u^h = 0;$$

that is,

$$\Box^h u^h \perp T_{u^h}N. \tag{9}$$

Hence, letting $\nu_{k+1}, \dots, \nu_n$ be a local frame for $TN^\perp$ as above, we have

$$\Box^h u^h = \lambda^l \nu_l \circ u^h,$$

where λ^l may be computed as

$$\lambda^l = \langle \Box^h u^h, \nu_l \circ u^h \rangle = -\eta^{\alpha\beta}\partial^h_\beta \langle \partial^{-h}_\alpha u^h, \nu_l \circ u^h \rangle + \eta^{\alpha\beta} \langle \partial^h_\alpha u^h, \partial^h_\beta(\nu_l \circ u^h)\rangle. \tag{10}$$

Observe that for $\alpha = 0$, $\beta = 0$ the first term vanishes because $\langle \partial_t u^h, \nu_l \circ u^h \rangle = 0$.

In view of this representation of (9), for $h > 0$ equation (9) is equivalent to a system of ordinary differential equations of the form

$$U^h_{tt} = F(U^h, U^h_t) \tag{11}$$

for $U^h(t) = (u^h(t, x_h))_{x_h \in M_h}$, with coupling involving only neighbouring lattice sites.

Given $(u^h_0, u^h_1) : M_h \to TN$ with finite energy

$$E_h(u^h(0)) := \frac{1}{2}\int_{M_h} (|u^h_1|^2 + |d^h u^h_0|^2), \tag{12}$$

we therefore expect to obtain a unique global solution u^h of the initial value problem for (9) with initial data

$$(u^h, u^h_t)|_{t=0} = (u^h_0, u^h_1). \tag{13}$$

In fact, we have the following result.

Theorem 3.1. *For any $h > 0$, any $(u^h_0, u^h_1) : M_h \to N$ with $E_h(u^h(0)) < \infty$ there exists a unique global solution $u^h : \mathbb{R} \times M_h \to N$ of the Cauchy problem* (9), (13), *and $E_h(u^h(t)) = E_h(u^h(0))$ for all t.*

The proof is achieved by combining the local existence and uniqueness results for systems of ordinary differential equations with the a priori bounds on solutions resulting from the following energy inequality.

3.1. Energy inequality. For $u^h : \mathbb{R} \times M_h \to N$ let $e_h(u^h)$ be the energy density defined in (4), and for $\alpha = 1, 2$ let

$$g^{\pm h}_\alpha(u^h) = \langle \partial^{\pm h}_\alpha u^h, u^h_t \rangle$$

be the momentum of u^h in direction α.

For a solution of (9) then we have

$$0 = \langle \Box^h u^h, u^h_t \rangle = \frac{d}{dt} e_h(u^h) - \frac{1}{2}\sum_{\alpha=1,2} (\partial^h_\alpha g^{-h}_\alpha(u^h) + \partial^{-h}_\alpha g^h_\alpha(u^h)). \tag{14}$$

In particular, the total energy is conserved; that is,

$$E_h(u^h(t)) = \int_{M_h} e_h(u^h(t)) = E_h(u^h(0)) \quad \text{for all } t. \tag{15}$$

For the proof of Theorem 3.1 and for our later purposes, we also need a local version of this result. Observe that in the discrete case (9) cannot exhibit finite propagation speed. However, as $h \to 0$ equation (9) approximates a system of wave equations. Therefore we expect the (essential) domains of influence and dependence of any given point to approach the light cone through that point; in particular, in the limit $h \to 0$, on any bounded region of space-time the discrete evolution should essentially be determined by the data on a finite region of the hyperplane $t = 0$.

Below we verify this behavior in detail. Because in the discrete case we are working on a quadratic lattice, we prove the local energy inequality on squares, not on circles.

3.2. Local energy inequality. For any function φ, upon multiplying (14) by the discretized function φ^h we obtain

$$\begin{aligned} 0 = \frac{d}{dt}(e_h(u^h)\varphi^h) - \frac{1}{2}\sum_{\alpha=1,2} [\partial_\alpha^h(g_\alpha^{-h}(u^h)\varphi^h) + \partial_\alpha^{-h}(g_\alpha^h(u^h)\varphi^h)] - e_h(u^h)\partial_t\varphi^h \\ + \frac{1}{2}\sum_{\alpha=1,2} [(g_\alpha^{-h}(u^h)\partial_\alpha^{-h}\varphi^h)(\,\cdot\,+he_\alpha) + (g_\alpha^h(u^h)\partial_\alpha^h\varphi^h)(\,\cdot\,-he_\alpha)]. \end{aligned}$$

Now let $\psi : \mathbb{R} \to \mathbb{R}$ be given by

$$\psi(s) = \begin{cases} e^{-h^{-1/3}s} & s \geq 0, \\ 2 - e^{h^{-1/3}s} & s < 0, \end{cases}$$

and choose

$$\varphi(t,x) = \inf_{1\leq\alpha\leq 2} \psi(|x^\alpha| + t) = \psi(\sup_\alpha |x^\alpha| + t), \tag{16}$$

satisfying

$$(\partial_t\varphi^h + \max_\alpha\{|\partial_\alpha^h\varphi^h|, |\partial_\alpha^{-h}\varphi^h|\})(t,x_h) \leq (\psi'(s) + \max\{|\partial^h\psi(s)|, |\partial^{-h}\psi(s)|\}),$$

for $x_h \in M_h$, where $s = \sup_\alpha |x_h^\alpha| + t$.

Integrating in spatial direction and shifting coordinates in the last two terms, we then find that

$$\begin{aligned} &\frac{d}{dt}\left(\int_{M_h} e_h(u^h)\varphi^h\right) \\ &\quad \leq \int_{M_h}\left(e_h(u^h)\partial_t\varphi^h + \frac{1}{2}\sum_{\alpha=1,2}(|g_\alpha^{-h}(u^h)||\partial_\alpha^{-h}\varphi^h| + |g_\alpha^h(u^h)||\partial_\alpha^h\varphi^h|)\right) \\ &\quad \leq \int_{M_h} e_h(u^h)(\partial_t\varphi^h + \max_{1\leq\alpha\leq 2}\{|\partial_\alpha^h\varphi^h|, |\partial_\alpha^{-h}\varphi^h|\}). \end{aligned}$$

Remark that at any point (t, x_h) at most two of the terms $\partial_\alpha^{\pm h}\varphi^h \neq 0$; hence in the Cauchy–Schwarz inequality we may replace the Euclidean norm of $\partial_\alpha^{\pm h}\varphi^h$ by the maximum norm. Let

$$\rho(s) = \psi'(s) + \max\{|\partial^h\psi(s)|, |\partial^{-h}\psi(s)|\}.$$

We distinguish the cases $s \leq -h$, $s \geq h$, $-h \leq s \leq 0$, and $0 \leq s \leq h$.

If $s \leq -h$, we have

$$\begin{aligned}\rho(s) &= \Big(- h^{-1/3} + \max\Big\{\frac{e^{h^{2/3}} - 1}{h}, \frac{1 - e^{-h^{2/3}}}{h}\Big\}\Big)e^{h^{-1/3}s} \\ &= h^{-1/3}e^{h^{-1/3}s}\Big(\frac{e^{h^{2/3}} - 1}{h^{2/3}} - 1\Big).\end{aligned}$$

By Taylor's formula

$$\frac{e^{h^{2/3}} - 1}{h^{2/3}} - 1 = \frac{1}{2}h^{2/3} + O(h^{4/3}) \leq h^{2/3}$$

for $h \leq h_0$. Hence for such h and s we conclude

$$\rho(s) \leq h^{1/3}e^{h^{-1/3}s} \leq h^{1/3} \leq h^{1/3}\psi(s).$$

Similarly, if $s \geq h$, for $h \leq h_0$ we find

$$\rho(s) = h^{-1/3}e^{-h^{-1/3}s}\Big(\frac{e^{h^{2/3}} - 1}{h^{2/3}} - 1\Big) \leq h^{1/3}e^{-h^{-1/3}s} = h^{1/3}\psi(s).$$

If $-h \leq s \leq 0$ we only need to check that

$$\begin{aligned}\psi'(s) + |\partial^h\psi(s)| &\leq -h^{-1/3}e^{h^{-1/3}s} + \frac{2 - e^{h^{-1/3}s} - e^{-h^{-1/3}(s+h)}}{h} \\ &\leq h^{-1/3}e^{h^{-1/3}s}\Big(- 1 + \frac{2e^{-h^{-1/3}s} - 1 - e^{-h^{2/3}}e^{-2h^{-1/3}s}}{h^{2/3}}\Big) \\ &\leq Ch^{1/3}e^{h^{-1/3}s} \leq Ch^{1/3} \leq Ch^{1/3}\psi(s)\end{aligned}$$

with an absolute constant C, if $h \leq h_0$. The estimate $\psi'(s) + |\partial^{-h}\psi(s)| \leq h^{1/3}\psi(s)$ for $h \leq h_0$ is obtained as in the case $s \leq -h$.

Similarly, for $0 \leq s \leq h \leq h_0$, we have

$$\psi'(s) + |\partial^{-h}\psi(s)| \leq Ch^{1/3}\psi(s).$$

The remaining estimate

$$\psi'(s) + |\partial^h\psi(s)| \leq h^{1/3}\psi(s), h \leq h_0,$$

is obtained as in the case $s \geq h$.

Thus, we conclude that with the above choice of φ for $h \leq h_0$ there holds

$$\partial_t\varphi^h + \max_\alpha\{|\partial_\alpha^h\varphi^h|, |\partial_\alpha^{-h}\varphi^h|\} \leq Ch^{1/3}\varphi^h$$

with an absolute constant C, and hence also

$$\frac{d}{dt}\int_{M_h} e_h(u^h)\varphi^h \le Ch^{1/3}\int_{M_h} e_h(u^h)\varphi^h.$$

We may shift the argument of φ by an arbitrary vector (t_0, x_0) and integrate in time to obtain the following result.

LEMMA 3.2. *There exist constants $h_0 > 0$, C such that for any $h \le h_0$, any solution u^h of* (9)*, any $z_0 = (t_0, x_0) \in \mathbb{R} \times M_h$, if $0 \le t \le t_0$ there holds*

$$\int_{\{t\}\times M_h} e_h(u^h)\varphi^h_{z_0} \le e^{Ch^{1/3}t}\int_{\{0\}\times M_h} e_h(u^h)\varphi^h_{z_0},$$

where $\varphi_{z_0}(t,x) = \varphi(t-t_0, x-x_0)$ is given by (16).

PROOF OF THEOREM 3.1. We first consider initial data $(u_0^h, u_1^h) : M_h \to TN$ having compact support in the sense that $u_0^h \equiv$ const, $u_1^h \equiv 0$ outside some compact set. Then for sufficiently large $K \in \mathbb{N}$ the support of $d^{\pm h}u_0^h$, u_1^h is strictly contained in the square of edge-length $2Kh$ centered at $(0,0)$. Extending u_0^h, u_1^h periodically with period $2Kh$ in the x^1- and x^2-directions, we may regard u_0^h, u_1^h alternatively as maps $(u_0^h, u_1^h) : M_h/(2Kh\mathbb{Z})^2 =: M_{h,K} \to TN$ or as periodic maps on M_h.

The Cauchy problem for equation (9) now reduces to an initial value problem for a finite-dimensional system (11) of ordinary differential equations, which in view of the uniform a-priori bound on the energy

$$E_{h,K}(u_K^h(t)) = \int_{M_{h,K}} e_h(u_K^h(t)) \equiv E_{h,K}(u_K^h(0)) = E_h(u^h(0)) \tag{17}$$

of a solution u_K^h, which results from integrating (14) over $M_{h,K}$, can be solved uniquely for all time.

Moreover, regarding $u_K^h : \mathbb{R} \times M_h \to N$ as spatially periodic solutions of (9), in view of these uniform energy bounds a subsequence $u_K^h \to u^h$, $\partial_t u_K^h \to \partial_t u^h$ locally uniformly on $\mathbb{R} \times M_h$ as $K \to \infty$, where u^h satisfies (9). Combining (17), Lemma 3.2, and (15) we conclude that $E_h(u^h(t)) \equiv$ const. Indeed, given $t > 0$, $z_0 = (t_0, x_0)$, by exponential decay of φ there are constants K_0, $C_1 = e^{Ch^{1/3}t}$ such that for $L \ge K \ge K_0$ there holds

$$\begin{aligned} 2C_1\int_{M_h} e_h(u^h(0))\varphi^h_{z_0}(0) &\ge C_1\int_{M_h} e_h(u_L^h(0))\varphi^h_{z_0}(0) \ge \int_{M_h} e_h(u_L^h(t))\varphi^h_{z_0}(t) \\ &\ge \int_{\{x_h\in M_h;|x_h^\alpha|\le Kh\}} e_h(u_L^h(t))\varphi^h_{z_0}(t). \end{aligned}$$

Fixing K and letting $L \to \infty$, from locally uniform convergence $u_L^h \to u^h$, $d^h u_L^h \to d^h u^h$ we conclude that

$$\int_{\{x_h \in M_h; |x_h^\alpha| \le Kh\}} e_h(u^h(t)) \varphi_{z_0}^h(t) \le 4C_1 E_h(u^h(0)).$$

Letting $K \to \infty$ and then $t_0 \to \infty$, we deduce that

$$E_h(u^h(t)) \le 2C_1 E_h(u^h(0)) < \infty$$

locally uniformly in time and therefore, in fact, $E_h(u^h(t)) = E_h(u^h(0))$ for all t, by (15).

Uniqueness of u^h is obtained as follows. Let u^h, $v^h : \mathbb{R} \times M_h \to N$ be solutions to (9) with $u^h(0, \cdot\,) = v^h(0, \cdot\,) = u_0^h$, $u_t^h(0, \cdot\,) = v_t^h(0, \cdot\,) = u_1^h$ and such that $E_h(u^h(t)) + E_h(v^h(t)) \le C$, uniformly in t. Observe that this also implies that

$$|u_t^h(t, x_h)|^2 + |v_t^h(t, x_h)|^2 \le Ch^{-2},$$

uniformly in $\mathbb{R} \times M_h$.

Expanding (9) and (10), we deduce that $w^h = u^h - v^h$ satisfies

$$\begin{aligned}
|\Box^h w^h| &\le C \sum_{\alpha=1,2} \left(|\partial_\alpha^{-h} \partial_\alpha^h w^h| + h^{-1}|\partial_\alpha^{\pm h} w^h| + h^{-2}|w^h(\,\cdot\, \pm he_\alpha)| + h^{-2}|w^h|\right) \\
&\quad + C(|u_t^h| + |v_t^h|)|w_t^h| + C(|u_t^h|^2 + |v_t^h|^2)|w^h| \\
&\le Ch^{-2}\left(\sum_{\alpha=1,2} |w^h(\,\cdot\, \pm he_\alpha)| + |w^h|\right) + Ch^{-1}|w_t^h|.
\end{aligned}$$

Multiplying by w_t^h and integrating over M_h, we obtain

$$\begin{aligned}
\frac{d}{dt} E_h(w^h(t)) &\le C(1 + h^{-2}) \int_{M_h} (|w^h(t)|^2 + |w_t^h(t)|^2) \\
&\le C(1 + h^{-2}) \int_{M_h} |w^h(t)|^2 + C(1 + h^{-2}) E_h(w^h(t)).
\end{aligned} \tag{18}$$

Moreover, by Hölder's inequality, for any $t \ge 0$, any $x \in M_h$ we have

$$|w^h(t, x)|^2 = \left(\int_0^t w_t^h(s, x)\, ds\right)^2 \le t \int_0^t |w_t^h(s, x)|^2\, ds.$$

Hence for $0 \le t \le T$ we can estimate

$$\int_{M_h} |w^h(t)|^2 \le 2t \int_0^t E_h(w^h(s))\, ds \le 2T^2 \sup_{0 \le s \le T} E_h(w^h(s)).$$

Given $T > 0$, we fix $t \in [0, T]$ such that

$$E_h(w^h(t)) = \sup_{0 \le s \le T} E_h(w^h(s)).$$

We may assume that $T \le 1$. Integrating (18) from 0 to t, it then follows that

$$E_h(w^h(t)) = \sup_{0\le s\le T} E_h(w^h(s)) \le CT(1+h^{-2}) \sup_{0\le s\le T} E_h(w^h(s)). \tag{19}$$

Choosing $T > 0$ sufficiently small, we conclude that $w^h \equiv 0$ on $[0,T]\times M_h$. By iteration therefore $w^h \equiv 0$ on $\mathbb{R}\times M_h$.

Finally, we may use (18) to remove the assumption that $d^h u_0^h$, u_1^h have compact support. Indeed, given data $(u_0^h, u_1^h) : M_h \to TN$ of finite energy we may approximate (u_0^h, u_1^h) by data $(u_{0,l}^h, u_{1,l}^h) : M_h \to TN$, $l \in \mathbb{N}$, such that $d^h u_{0,l}^h$, $u_{1,l}^h$ have compact support for any l and such that

$$\int_{M_h} (|d^h(u_{0,l}^h - u_0^h)|^2 + |u_{1,l}^h - u_1^h|^2) \to 0$$

as $l \to \infty$. (The proof of this density result is analogous to the proof that maps $u \in H^1(\mathbb{R}^2; N)$ with $\operatorname{supp}(\nabla u) \subset \mathbb{R}^2$ are H^1-dense in this space; see for instance [13].) Letting $(u_l^h)_{l\in\mathbb{N}}$ be the solutions to (9) with data $(u_l^h, \partial_t u_l^h)|_{t=0} = (u_{0,l}^h, u_{1,l}^h)$, from (18), applied to $w^h = u_l^h - u_m^h$ for large $l, m \in \mathbb{N}$, we obtain convergence of (u_l^h) to the unique solution u of (9), (13). ☐

4. Passing to the limit $h \to 0$

Our aim in this section is to prove the following weak convergence result.

Theorem 4.1. *Let $u^h : \mathbb{R}\times M_h \to N \hookrightarrow \mathbb{R}^n$, $h > 0$, be spatially discrete wave maps such that*

$$E_h(u^h(t)) \le C \text{ uniformly in } h > 0,\ t \in \mathbb{R}. \tag{20}$$

Then a subsequence $u^h \to u$ locally in $L^2(\mathbb{R}^{1+2})$, $d^h u^h \rightharpoonup Du$ weakly-$*$ in $L^\infty(\mathbb{R}; L^2(\mathbb{R}^2))$ as $h \to 0$ where $u : \mathbb{R}\times\mathbb{R}^2 \to N \hookrightarrow \mathbb{R}^n$ is a weak solution of (1) *with*

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^2} |Du(t)|^2\,dx \le \limsup_{h\to 0} E_h(u^h(t)) \le C$$

uniformly in $t \in \mathbb{R}$.

The proof of Theorem 4.1 uses certain compensation properties of Jacobians exhibited by the first order equations equivalent to (1), (9), respectively, as in [7], [8], [12].

To derive these equations we proceed as in [3] or [9]. First suppose that TN is parallelizable and let $\overline{e}_1, \dots, \overline{e}_k$ be a smooth orthonormal frame field. For any $h > 0$ and any $R^h : \mathbb{R}\times M_h \to SO(k)$ then

$$e_i^h = R_{ij}^h(\overline{e}_j \circ u^h), \quad 1 \le i \le k,$$

is a frame field for $(u^h)^{-1}TN$.

4.1. First order equations. Let

$$\theta^h_{i,0} = \langle \partial_t u^h, e^h_i \rangle, \quad \theta^h_{i,\alpha} = \langle \partial^h_\alpha u^h, e^h_i(\,\cdot\, + h\underline{e}_\alpha)\rangle, \quad \alpha = 1,2;$$

observe that the shift is arranged so that the functions

$$\theta^{-h}_{i,\alpha} = \theta^h_{i,\alpha}(\,\cdot\, - h\underline{e}_\alpha) = \langle \partial^{-h}_\alpha u^h, e^h_i \rangle, \quad \alpha = 1,2,$$

are the coefficients of the representation of $d^{-h}u^h$ in terms of the frame (e^h_i). Also let

$$\omega^{\pm h}_{ij,0} = \langle \partial_t e^h_i, e^h_j \rangle, \quad \omega^{\pm h}_{ij,\alpha} = \langle \partial^{\pm h}_\alpha e^h_i, m^{\pm h}_\alpha e^h_j \rangle, \quad \alpha = 1,2.$$

Clearly, the ω^h_{ij} are a discrete approximation of the connection 1-forms $\omega_{ij} = \langle de_i, e_j \rangle$ of a frame (e_i) in the continuum limit $h = 0$. The definition is made to insure anti-symmetry $\omega^h_{ij} = -\omega^h_{ji}$ also in the discrete case.

Letting $\partial^{\pm h}_t := \partial_t$, $\underline{e}_0 = 0$, $m^{\pm h}_0 = \mathrm{id}$, we have

$$\theta^h_{i,\alpha} = \langle \partial^h_\alpha u^h, e^h_i(\,\cdot\, + h\underline{e}_\alpha)\rangle, \quad \theta^{-h}_{i,\alpha} = \langle \partial^{-h}_\alpha u^h, e^h_i \rangle, \quad \omega^{\pm h}_{ij,\alpha} = \langle \partial^{\pm h}_\alpha e^h_i, m^{\pm h}_\alpha e^h_j \rangle$$

for all α. Then

$$\delta^h_\eta \theta^h_i = -\eta^{\alpha\beta}\partial^{-h}_\alpha \theta^h_{i,\beta} = -\eta^{\alpha\beta}\partial^h_\alpha \theta^{-h}_{i,\beta} = -\langle \Box^h u^h, e^h_i \rangle - \eta^{\alpha\beta}\langle \partial^h_\alpha u^h, \partial^h_\beta e^h_i \rangle.$$

That is, $u^h : \mathbb{R} \times M_h \to N$ solves (9) if and only if

$$\delta^h_\eta \theta^h_i = -\eta^{\alpha\beta}\langle \partial^h_\alpha u^h, \partial^h_\beta e^h_i \rangle = -\eta^{\alpha\beta}\theta^h_{j,\alpha}\cdot \omega^h_{ij,\beta} + \tau^h_{1i}, \tag{21}$$

where

$$\begin{aligned}\tau^h_{1i} = &- \eta^{\alpha\beta}\Bigg[\theta^h_{j,\alpha}\left\langle \frac{e^h_j(\,\cdot\, + h\underline{e}_\alpha) - e^h_j}{2}, \partial^h_\beta e^h_i \right\rangle \\ &+ \langle \partial^h_\alpha u^h, \nu_l \circ u^h(\,\cdot\, + h\underline{e}_\alpha)\rangle \langle \nu_l \circ u^h(\,\cdot\, + h\underline{e}_\alpha), \partial^h_\beta e^h_i \rangle\Bigg].\end{aligned}$$

Observe that there exists a constant $C = C(N)$ such that for $p, q \in N$ there holds $|\langle p - q, \nu_l(p)\rangle| \le C|p-q|^2$. It follows that

$$|\langle \partial^h_\alpha u^h, \nu_l \circ u^h(\,\cdot\, + h\underline{e}_\alpha)\rangle| \le Ch^{-1}|u^h(\,\cdot\, + h\underline{e}_\alpha) - u^h|^2 = Ch|\partial^h_\alpha u^h|^2.$$

Moreover, remark that

$$|\eta^{\alpha\beta}\theta^h_{j,\alpha}\langle (e^h_j(\,\cdot\, + h\underline{e}_\alpha) - e^h_j), \partial^h_\beta e_j \rangle| \le h|\theta^h_{j,\alpha}||\partial^h_\alpha e^h_j|^2 \le |u^h(\,\cdot\, + h\underline{e}_\alpha) - u^h||\partial^h_\alpha e^h_j|^2.$$

Thus, we may estimate the error term

$$|\tau^h_{1i}| \le C \sum_{\alpha=1,2} |u^h(\,\cdot\, + h\underline{e}_\alpha) - u^h| \Bigg(|\partial^h_\alpha u^h|^2 + \sum_j |\partial^h_\alpha e^h_j|^2 \Bigg).$$

Our aim is to pass to the distributional limit in (9) or, equivalently, (21) for a suitable sequence $h \to 0$. As in [7], [8] we may convert this convergence problem into a problem on a compact domain, as follows. Given $\varphi \in C^\infty_0(\mathbb{R} \times \mathbb{R}^2)$, let Q

be a cube centered at $(0,0)$ containing the support of φ. Scaling the coordinates suitably, we may assume that $Q = [-1/4, 1/4]^3$; moreover, we may suppose that $1/4h \in \mathbb{N}$. We then extend u^h by even reflection in the faces of Q to periodic functions v^h on $\mathbb{R} \times M_h$ of period 1 in each variable, satisfying (9) on the support of φ.

Given a frame (e_i) for $(v^h)^{-1}TN$, then also (21) will hold on the support of φ. Regarding v^h as maps $v^h : S^1 \times T_h \to N$ on the compact spatially discrete 3-torus, moreover, following Hélein [9], we may choose a frame (e_i) which is in minimal Coulomb gauge, defined as follows.

4.2. Gauge condition. Choose $R^h = (R^h_{ij}) \in H^1(S^1 \times T_h; SO(k))$ such that

$$D_h(R^h(\overline{e} \circ u^h)) = \frac{1}{4}\int_{S^1\times T_h} \sum_{\alpha,i}(|\partial^h_\alpha e^h_i|^2 + |\partial^{-h}_\alpha e^h_i|^2) = \inf_R D_h(R(\overline{e} \circ u^h)),$$

and let $e^h_i = R^h_{ij}(\overline{e}_j \circ u^h), 1 \le i \le k$. Observe that

$$D_h(e^h_i) \le C\int_{S^1\times T_h} e_h(u^h) \le C D_h(u^h). \tag{22}$$

Moreover, minimality implies

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon} D_h((id + \varepsilon S)e^h)|_{\varepsilon=0} \\ &= \frac{1}{2}\int_{S^1\times T_h} (\langle \partial^h_\alpha e^h_i, \partial^h_\alpha(S_{ij}e^h_j)\rangle + \langle \partial^{-h}_\alpha e^h_i, \partial^{-h}_\alpha(S_{ij}e^h_j)\rangle) \\ &= -\frac{1}{2}\int_{S^1\times T_h} \{\partial^h_\alpha\langle \partial^{-h}_\alpha e^h_i, m^{-h}_\alpha e^h_j\rangle + \partial^{-h}_\alpha\langle \partial^h_\alpha e^h_i, m^h_\alpha e^h_j\rangle\}S_{ij} \\ &= -\int_{S^1\times T_h} \partial^{-h}_\alpha \omega^h_{ij,\alpha} S_{ij} \end{aligned}$$

for all $S_{ij} \in SO(k)$, where we also used anti-symmetry of S and the discrete product rule (3) to derive the second identity.

Since $\omega^h_{ij,\alpha} = -\omega^h_{ji,\alpha}$ we conclude

$$\partial^{-h}_\alpha \omega^h_{ij,\alpha} = \delta^h_{\mathrm{eucl}}\omega^h_{ij} = \delta^{-h}_{\mathrm{eucl}}\omega^{-h}_{ij} = 0.$$

In view of (22) we may assume that, as $h \to 0$ suitably,

$$\begin{aligned} e^h_i &\rightharpoonup e_i \ \text{ weakly in } H^1(T^3), \\ \theta^h_i &\rightharpoonup \theta_i \ \text{ weakly in } L^2(T^3), \\ \omega^h_{ij} &\rightharpoonup \omega_{ij} \text{ weakly in } L^2(T^3), \end{aligned}$$

where e_i is a frame for $u^{-1}TN$ and $\theta_i = \langle du, e_i\rangle$, $\omega_{ij} = \langle de_i, e_j\rangle$.

Our aim is to show that

$$\int_Q (\theta_i \cdot_\eta d\varphi + \omega_{ij} \cdot_\eta \theta_j\varphi)\, dz = 0,$$

where $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$ with $\operatorname{supp}(\varphi) \subset Q$ is the testing function that we chose above.

In fact, we will show that

$$\delta_\eta \theta_i + \omega_{ij} \cdot_\eta \theta_j = 0 \quad \text{in } \mathcal{D}'(Q), \tag{23}$$

where we extend u periodically as above and regard Q as part of a fundamental domain for $T^3 = \mathbb{R}^3/\mathbb{Z}^3$. In view of the equations (21), that is,

$$\delta_\eta^h \theta_i^h + \omega_{ij}^h \cdot_\eta \theta_j^h = \tau_{1i}^h \quad \text{in } Q,$$

and distributional convergence $\delta_\eta^h \theta_i^h \rightharpoonup \delta_\eta \theta_i$ in $\mathcal{D}'(T^3)$, it will suffice to show that

$$\omega_{ij}^h \cdot_\eta \theta_j^h - \tau_{1i}^h \rightharpoonup \omega_{ij} \cdot_\eta \theta_i \quad \text{in } \mathcal{D}'(T^3) \tag{24}$$

as $h \to 0$ suitably.

Let

$$*_\eta \theta_i^{-h} = d^h a_i^h + \delta_{\text{eucl}}^h b_i^h + c_i^h \tag{25}$$

be the Hodge decomposition of $*_\eta \theta_i^{-h}$ on $S^1 \times T_h$ as determined in Proposition 2.1. We may assume that as $h \to 0$ suitably

$$a_i^h \rightharpoonup a_i, \quad b_i^h \rightharpoonup b_i \text{ weakly in } H^1(T^3),$$

and $c_i^h \to c_i$ smoothly. Observe that the harmonic forms c_i^h, c_i are constant linear combinations of the basis $dx^\alpha \wedge dx^\beta$, $0 \le \alpha < \beta \le 2$.

Using this decomposition, we may write

$$\omega_{ij}^{-h} \cdot_\eta \theta_j^{-h} dz = \omega_{ij}^{-h} \wedge *_\eta \theta_j^{-h} = \omega_{ij}^{-h} \wedge d^h a_j^h + \omega_{ij}^{-h} \wedge \delta_{\text{eucl}}^h b_j^h + \omega_{ij}^{-h} \wedge c_j^h.$$

Since $c_j^h \to c_j$ smoothly, passing to the desired limit in the last term is no problem. To show convergence of the second last term, for convenience denote $- *_{\text{eucl}} b_j^h = \beta_j^h$. Observe that β_j^h is a scalar function and $\beta_j^h \rightharpoonup \beta_j = - *_{\text{eucl}} b_j$ weakly in $H^1(T^3)$, whence strongly in $L^2(T^3)$ by the Rellich-Kondrakov theorem. Then

$$\begin{aligned} \omega_{ij}^{-h} \wedge \delta_{\text{eucl}}^h b_j^h &= \omega_{ij}^{-h} \wedge *_{\text{eucl}} d^{-h} \beta_j^h = \omega_{ij}^{-h} \cdot_{\text{eucl}} d^{-h} \beta_j^h dz \\ &= (*_{\text{eucl}} \omega_{ij}^{-h}) \wedge d^{-h} \beta_j^h = d^{-h}(*_{\text{eucl}} \omega_{ij}^h \beta_j^h), \end{aligned}$$

as $(\delta_{\text{eucl}}^h \omega_{ij}^h) \beta_j^h = 0$ on account of the Coulomb gauge condition. (In coordinates, $\omega_{ij}^{-h} \cdot_{\text{eucl}} d^{-h} \beta_j^h = \omega_{ij,\alpha}^{-h} \partial_\alpha^{-h} \beta_j^h = \partial_\alpha^{-h}(\omega_{ij,\alpha}^h \beta_j^h) - (\partial_\alpha^{-h} \omega_{ij,\alpha}^h) \beta_j^h$.)

Since $\omega_{ij}^h \rightharpoonup \omega_{ij}$ weakly in L^2, while $\beta_j^h \to \beta_j$ strongly in L^2, we conclude that

$$\omega_{ij}^{-h} \wedge \delta_{\text{eucl}}^h b_j^h \rightharpoonup \omega_{ij} \wedge \delta_{\text{eucl}} b_j \quad \text{in } \mathcal{D}'.$$

For the remaining term by the discrete product rule we have

$$\begin{aligned}\omega_{ij}^{-h}\wedge d^h a_j^h &= \omega_{ij,\alpha}^{-h}\partial_\beta^h a_{j,\gamma}^h dx^\alpha\wedge dx^\beta\wedge dx^\gamma\\ &= [\partial_\beta^h(\omega_{ij,\alpha}^{-h}(\,\cdot\,-h\underline{e}_\beta)a_{j,\gamma}^h)-\partial_\beta^{-h}\omega_{ij,\alpha}^{-h}a_{j,\gamma}^h]dx^\alpha\wedge dx^\beta\wedge dx^\gamma,\\ &= d^{-h}\omega_{ij}^{-h}\wedge a_j^h+\partial_\beta^{-h}(\omega_{ij,\alpha}^{-h}\tau_\beta^h a_{j,\gamma}^h)dx^\alpha\wedge dx^\beta\wedge dx^\gamma.\end{aligned}$$

Since we also have that $\tau_\beta^h a_j^h \rightharpoonup a_j$ weakly in $H^1(T^3)$ and hence strongly in L^2, as $h\to 0$ the last term converges to $\partial_\beta(\omega_{ij,\alpha}a_{j,\gamma})dx^\alpha\wedge dx^\beta\wedge dx^\gamma = -d(\omega_{ij}\wedge a_j)$ in $\mathcal{D}'$.

Thus we have shown distributional convergence

$$\omega_{ij}^{-h}\cdot_\eta \theta_j^{-h}dz - d^{-h}\omega_{ij}^{-h}\wedge a_j^h \rightharpoonup \omega_{ij}\cdot_\eta\theta_j dz - d\omega_{ij}\wedge a_j \tag{26}$$

as $h\to 0$, and it remains to prove that

$$d^{-h}\omega_{ij}^{-h}\wedge a_j^h - \tau_{1i}^h \rightharpoonup d\omega_{ij}\wedge a_j \quad \text{in } \mathcal{D}'. \tag{27}$$

The proof of (27) will be accomplished by adapting the ideas of [8] to the spatially discrete case.

Passing to a further subsequence, if necessary, we may assume that, as $h\to 0$,

$$e_h(u^h)+e_h(e^h)\overset{*}{\rightharpoonup}\mu \quad \text{in } \mathcal{M}(T^3)$$

as Radon measures. Theorem 4.1 then will be a consequence of the following Proposition.

Proposition 4.2. *There exists a Radon measure ν such that, as $h\to 0$ suitably,*

$$d^{-h}\omega_{ij}^{-h}\wedge a_j^h-\tau_{1i}^h\rightharpoonup d\omega_{ij}\wedge a_j-\nu \quad \textit{in } \mathcal{D}'(Q),$$

where

$$\operatorname{supp}(\nu)\subset\Sigma=\{z=(t,x)\,:\,\limsup_{R\to 0}(R^{-1}\mu(P_R(z)))>0\}$$

has finite 1-dimensional Hausdorff measure.

Proof of Theorem 4.1. Combining Proposition 4.2 and (26), we conclude that, as $h\to 0$,

$$0=\delta_\eta^h\theta_i^h+\omega_{ij}^h\cdot_\eta\theta_j^h-\tau_{1i}^h\rightharpoonup\delta_\eta\theta_i+\omega_{ij}\cdot_\eta\theta_j-\nu$$

in $\mathcal{D}'(Q)$. Hence

$$\nu=\delta_\eta\theta_i+\omega_{ij}\cdot_\eta\theta_j\in H^{-1}+L^1.$$

But since the support of ν is contained in a set of finite 1-dimensional Hausdorff measure, as in [8], Proof of Theorem 1.3, we conclude that, in fact, $\nu=0$ and

$$\delta_\eta\theta_i+\omega_{ij}\cdot_\eta\theta_j=0 \quad \text{in } \mathcal{D}'(Q),$$

as claimed. □

4.3 Proof of Proposition 4.2. We proceed as in [8]. The key ingredients in the proof are the duality between the Hardy space $\mathcal{H}^1$ and BMO (due to Fefferman and Stein [6]), the $\mathcal{H}^1$ estimates for Jacobians of Coifman, Lions, Meyer and Semmes [4] (see Lemma 4.4 below for the discrete setting), and a characterization of concentration points in the spirit of concentration compactness for sequences of products whose factors are bounded in $\mathcal{H}^1$ and BMO, respectively (see [8], Lemma 3.7). To obtain the BMO estimate (see Lemma 4.3 below) we exploit the energy inequality and apply Campanato theory and Poincaré's inequality. For elliptic problems similar arguments were used by Hélein [9], [10], Evans [5], Bethuel [1], and others.

Fix a function $\varphi \in C_0^\infty(B_1(0))$ with $\int_{\mathbb{R}^3} \varphi \, dz = 1$. For $f \in L^1(T^3)$ then let

$$(\mathcal{M}_\varphi f)(z_0) = \sup_{0<r<1} \left| \int_{T^3} r^{-3} \varphi\left(\frac{z - z_0}{r}\right) f(z) \, dz \right|$$

be the regularized maximal function of f. The Hardy space on T^3 then is the space

$$\mathcal{H}^1(T^3) = \left\{ f \in L^1(T^3); \int_{T^3} f \, dz = 0, \ \mathcal{M}_\varphi(f) \in L^1(T^3) \right\}$$

with norm

$$||f||_{\mathcal{H}^1} := ||\mathcal{M}_\varphi(f)||_{L^1}.$$

Also let $\mathrm{BMO}(T^3)$ be the space of functions $f \in L^1(T^3)$ such that

$$[f]_{\mathrm{BMO}(T^3)} = \sup_{0<r<1/2} \sup_{z_0 \in T^3} \fint_{P_r(z_0)} |f - f_{r,z_0}| \, dz < \infty$$

with norm

$$||f||_{\mathrm{BMO}(T^3)} = \left| \int_{T^3} f \, dz \right| + [f]_{\mathrm{BMO}(T^3)},$$

where $P_r(z_0)$ and f_{r,z_0} are defined as in Section 2.

By [6], $\mathrm{BMO}(T^3)$ is the dual space of $\mathcal{H}^1(T^3)$, and for $g \in \mathcal{H}^1(T^3)$, $f \in \mathrm{BMO}(T^3)$ there holds

$$\langle f, g \rangle_{\mathrm{BMO} \times \mathcal{H}^1} \leq C[f]_{\mathrm{BMO}(T^3)} ||g||_{\mathcal{H}^1}.$$

Moreover, for any $\varphi \in C^\infty(T^3)$, $f \in \mathrm{BMO}(T^3)$ the function $f\varphi \in \mathrm{BMO}(T^3)$ and

$$[f\varphi]_{\mathrm{BMO}} \leq C||f||_{\mathrm{BMO}} ||\varphi||_{C^1};$$

see for instance [8], Proposition 3.8. In particular, for any $f \in \mathrm{BMO}(T^3)$, $g \in \mathcal{H}^1(T^3)$ the product $T = fg$ is defined as a distribution in T^3 by letting

$$\langle T, \varphi \rangle_{\mathcal{D}' \times \mathcal{D}} := \langle f\varphi, g \rangle_{\mathrm{BMO} \times \mathcal{H}^1}$$

for any $\varphi \in C^\infty(T^3)$. Finally, for $0 \le \lambda \le 3$, $f \in L^2(T^3)$ let

$$[f]^2_{\mathcal{L}^{2,\lambda}} = \sup_{0<r<1/2} \sup_{z_0} r^{-\lambda} \fint_{P_r(z_0)} |f - f_{r,z_0}|^2 \, dz$$

and for $0 \le \lambda < 3$ denote

$$||f||^2_{L^{2,\lambda}} = \sup_{0<r<1/2} \sup_{z_0} r^{-\lambda} \fint_{P_r(z_0)} |f|^2 \, dz.$$

Define the Morrey–Companato spaces

$$\begin{aligned} \mathcal{L}^{2,\lambda}(T^3) &= \{f \in L^2(T^3) \,:\, [f]_{\mathcal{L}^{2,\lambda}} < \infty\}, \\ L^{2,\lambda}(T^3) &= \{f \in L^2(T^3) \,:\, ||f||_{L^{2,\lambda}} < \infty\} \end{aligned}$$

with norms $||\cdot||_{L^{2,\lambda}}$ and $||f||_{\mathcal{L}^{2,\lambda}} = ||f||_{L^2} + [f]_{\mathcal{L}^{2,\lambda}}$, respectively. Recall that $L^{2,\lambda} \cong \mathcal{L}^{2,\lambda}$ for $0 \le \lambda < 3$ and $\mathcal{L}^{2,3} \cong \mathrm{BMO}$ with equivalent norms.

For an open set $U \subset T^3$ define the local BMO-seminorm by letting

$$[f]_{\mathrm{BMO}(U)} = \sup \left\{ \fint_{B_r(z_0)} |f - f_{r,z_0}|^2 \, dz \,:\, B_r(z_0) \subset U \right\}.$$

LEMMA 4.3. *For any $h > 0$ we have $a^h_j \in \mathrm{BMO}(T^3)$ with $d^h a^h_j \in L^{2,1}(T^3)$ and*

$$||a^h_j||^2_{\mathrm{BMO}} \le C||d^h a^h_j||^2_{L^{2,1}} \le CE_h(u^h) \le C$$

independently of h. Moreover, for any $0 < h \le r \le R < 1/2$, any $z_0 \in T^3$ there holds

$$[a^h_j]_{\mathrm{BMO}(P_r(z_0))} + [d^h a^h_j]_{L^{2,1}(P_r(z_0))} \le C\left(\frac{r}{R} ||a^h_j||_{\mathrm{BMO}(P_R(z_0))} + ||\theta^{-h}_j||_{L^{2,1}(P_R(z_0))} \right).$$

PROOF. A global bound for a^h_j follows from (8). From (25) we obtain the equation

$$-\Delta^h_3 a^h_j = \delta^h_{\mathrm{eucl}} d^h a^h_j = \delta^h_{\mathrm{eucl}} *_\eta \theta^{-h}_j = D^h \theta^{-h}_j,$$

where D^h is a discrete first order differential operator with constant coefficients. The proof now proceeds as the proof [8], Lemma 3.11, in the case $h = 0$. Omitting the index j for brevity, given $0 < h \le r < R = Kh < 1/2$, $z_0 \in S^1 \times T_h$, we split $a^h = a^h_1 + a^h_2$ on $P_R(z_0)$, where

$$-\Delta^h_3 a^h_1 = 0 \qquad \text{in } P_R(z_0), \qquad a^h_1 = a^h \quad \text{on } \partial P_R(z_0),$$

and

$$-\Delta^h_3 a^h_2 = D^h \theta^{-h} \quad \text{in } P_R(z_0), \qquad a^h_2 = 0 \quad \text{on } \partial P_R(z_0).$$

Standard estimates yield that

$$||e_h(a^h_1)||_{L^\infty(P_{R/2}(z_0))} \le CR^{-2} \fint_{P_R(z_0)} |a^h_1 - (a^h_1)_{R,z_0}|^2.$$

Hence, from Proposition 2.3(ii), for any $r = kh$, $z \in S^1 \times T_h$ such that $P_{r+h}(z) \subset P_{R/2}(z_0)$ we conclude

$$\begin{aligned}\fint_{P_r(z)} |a_1^h - (a_1^h)_{r,z}|^2 &\le Cr^{-1} \int_{P_{r+h}(z)} e_h(a_1^h) \le Cr^2 ||e_h(a_1^h)||_{L^\infty(P_{R/2}(z_0))} \\ &\le C\left(\frac{r}{R}\right)^2 \fint_{P_R(z_0)} |a_1^h - (a_1^h)_{R,z_0}|^2 \\ &\le C\left(\frac{r}{R}\right)^2 [a_1^h]^2_{\mathrm{BMO}(P_R(z_0))}.\end{aligned}$$

Clearly, these estimates remain valid for any $r > h$ and any $z \in T^3$ with $P_{r+h}(z) \subset P_{R/2}(z_0)$ if we extend a^h as the spatially piecewise constant function

$$a^h(t,x) = a^h(t,x_h), \quad \text{for } x \in Q_h(x_h).$$

Moreover, for $0 < r < h$, if we compare a_1^h to its bilinearly interpolated function $\overline{a}_1^h$, for any $z_1 = (t_1, x_1) \in T^3$ with $P_r(z_1) \subset P_{2h}(z_h) \subset P_{R/2}(z_0)$ for some $z_h = (t, x_h) \in S^1 \times T_h$, from Proposition 2.2 (i), (iii) and the (standard) Poincaré inequality applied to $\overline{a}_1^h$ we obtain

$$\begin{aligned}&\fint_{P_r(z_1)} |a_1^h - (a_1^h)_{r,z_1}|^2\, dz + r^{-1} \int_{P_r(z_1)} |d^h a_1^h|^2\, dz \\ &\quad \le C \fint_{t_1-r}^{t_1+r} ||(a_1^h - \overline{a}_1^h)(t)||^2_{L^\infty(Q_r(x_1))}\, dt \\ &\qquad + C \fint_{P_r(z_1)} |\overline{a}_1^h - (\overline{a}_1^h)_{r,z_1}|^2\, dz + r^{-1} \int_{P_r(z_1)} |d^h a_1^h|^2\, dz \\ &\quad \le C \fint_{t_1-r}^{t_1+r} \int_{Q_{2h}(x_h)} (e_h(a_1^h(t)) + |d\overline{a}_1^h(t)|^2)\, dx\, dt \le C \fint_{t_1-r}^{t_1+r} \int_{Q_{2h}(x_h)} e_h(a_1^h(t)) \\ &\quad \le Ch^2 ||e_h(a_1^h)||_{L^\infty(P_{R/2}(z_0))} \le C\left(\frac{h}{R}\right)^2 [a_1^h]_{\mathrm{BMO}(P_R(z_0))}.\end{aligned}$$

It follows that for $r \ge h$ there holds

$$\begin{aligned}[a_1^h]_{\mathrm{BMO}(P_r(z_0))} + ||d^h a_1^h||_{L^{2,1}(P_r(z_0))} &\le C\frac{r}{R} [a_1^h]_{\mathrm{BMO}(P_R(z_0))} \\ &\le C\left(\frac{r}{R}[a^h]_{\mathrm{BMO}(P_R(z_0))} + [a_2^h]_{\mathrm{BMO}(P_R(z_0))}\right).\end{aligned}$$

The analogous estimate

$$[a_2^h]_{\mathrm{BMO}(P_R(z_0))} \le C||d^h a_2^h||_{L^{2,1}(P_R(z_0))} \le C||\theta^{-h}||_{L^{2,1}(P_R(z_0))}$$

is obtained exactly as in the continuous case from [2], Teorema 16.I, and Poincaré's inequality. $\square$

Observe that the local energy inequality Lemma 3.2 implies that

$$\limsup_{h\to 0} ||\theta_j^{-h}||^2_{L^{2,1}(P_R(z_0))} \le CR^{-1}\mu(\overline{P_{3R}(z_0)}). \tag{28}$$

Indeed, for any $r < R$, any $z_1 = (t_1, x_1)$ such that $P_r(z_1) \subset P_R(z_0)$, if $3r < R$ by Lemma 3.2 we have

$$\begin{aligned}
(4r)^{-1}||\theta_j^{-h}||^2_{L^2(P_r(z_1))} &\le \sup_{|t-t_1|<r} \int_{Q_r(x_1)} e_h(u^h(t)) \\
&\le \int_{Q_{4r}(x_1)} e_h(u^h(t_1 - r)) + o(1) \\
&\le \int_{Q_{2R}(x_0)} e_h(u^h(t_1 - r)) + o(1) \\
&\le R^{-1} \int_{t_1-r-R}^{t_1-r} \int_{Q_{3R}(x_0)} e_h(u^h(t))\,dt + o(1) \\
&\le R^{-1} \int_{P_{3R}(z_0)} e_h(u^h(t)) + o(1) \le R^{-1}\mu(\overline{P_{3R}(z_0)}) + o(1)
\end{aligned}$$

where $o(1) \to 0$ as $h \to 0$.

If $R/3 \le r \le R$, clearly

$$(3r)^{-1}||\theta_j^{-h}||_{L^2(P_r(z_1))} \le R^{-1}||\theta_j^{-h}||_{L^2(P_R(z_0))} \le R^{-1}\mu(P_{3R}(z_0)) + o(1),$$

where $o(1) \to 0$ as $h \to 0$.

Regarding ω_{ij}^h, we now introduce the bilinearly interpolated frame to split

$$\omega_{ij,\alpha}^h = \langle \partial_\alpha^h e_i^h, \overline{e}_j^h \rangle + \langle \partial_\alpha^h e_i^h, m_\alpha^h e_j^h - \overline{e}_j^h \rangle. \tag{29}$$

Lemma 4.4. *For any $h > 0$ there holds $d^h\langle d^h e_i^h, \overline{e}_j^h\rangle \in \mathcal{H}^1(T^3)$ and*

$$d^h\langle d^h e_i^h, \overline{e}_j^h \rangle \overset{*}{\rightharpoonup} d\langle de_i, e_j\rangle = d\omega_{ij}$$

in $\mathcal{H}^1(T^3)$ as $h \to 0$ suitably.

Proof. In view of the identity $d^h \circ d^h = 0$, we have

$$\begin{aligned}
d^h\langle d^h e_i^h, \overline{e}_j^h\rangle &= \partial_\alpha^h \langle \partial_\beta^h e_i^h, \overline{e}_j^h\rangle dx^\alpha \wedge dx^\beta \\
&= \langle \partial_\beta e_i^h(\,\cdot\, + h\underline{e}_\alpha), \partial_\alpha^h \overline{e}_j^h\rangle dx^\alpha \wedge dx^\beta = d^h\langle d^h e_i^h, (\overline{e}_j^h - q)\rangle
\end{aligned}$$

for any $q \in \mathbb{R}^n$. Exactly, as in [4], Theorem 2.1, we may therefore show that

$$d^h\langle d^h e_i^h, \overline{e}_j^h\rangle \in \mathcal{H}^1(T^3)$$

with

$$||d^h\langle d^h e_i^h, \overline{e}_j^h\rangle||_{\mathcal{H}^1} \le CE_h(e^h) \le C,$$

where we also used Proposition 2.2(iii). Since the space VMO(T^3), the pre-dual of $\mathcal{H}^1(T^3)$ is separable, we conclude that $(d^h\langle d^h e_i^h, \overline{e}_j^h\rangle)_{h>0}$ is relatively weakly-$*$ sequentially compact. But, as $h \to 0$ suitably, $d^h\langle d^h e_i^h, \overline{e}_j^h\rangle \rightharpoonup d\omega_{ij}$ in the sense of distributions. By density of $C^\infty(T^3)$ in $\mathrm{VMO}(T^3)$, therefore we also have weak-$*$ convergence

$$d^h\langle d^h e_i^h, \overline{e}_j^h\rangle \overset{*}{\rightharpoonup} d\omega_{ij}$$

in $\mathcal{H}^1(T^3)$, as claimed. $\square$

From Lemma 4.3 and [8], Theorem 3.7, we hence conclude that, as $h \to 0$,

$$d^{-h}\langle d^{-h} e_i^h, \overline{e}_j^h\rangle \wedge a_j^h \rightharpoonup d\omega_{ij} \wedge a_j + \nu_1 \quad \text{in } \mathcal{D}', \tag{30}$$

where ν_1 is a Radon measure with

$$\mathrm{supp}(\nu_1) \subset \left\{ z \,:\, \lim_{r\to 0}\limsup_{h\to 0}[a^h]_{\mathrm{BMO}(P_r(z_0))} > 0 \right\}.$$

But by Lemma 4.3, for $r \geq h$ we have

$$[a^h]_{\mathrm{BMO}(P_r(z_0))} \leq C\left(\frac{r}{R}||a^h||_{\mathrm{BMO}(P_R(z_0))} + ||\theta^{-h}||_{L^{2,1}(P_R(z_0))}\right).$$

Fixing $R > 0$, from (28) we conclude that

$$\lim_{r\to 0}\limsup_{h\to 0}[a^h]^2_{\mathrm{BMO}(P_r(z_0))} \leq C\limsup_{h\to 0}||\theta^{-h}||_{L^{2,1}(P_R(z_0))} \leq C(R^{-1}\mu(\overline{P_{3R}(z_0)})).$$

Since $R > 0$ is arbitrary, therefore $\mathrm{supp}(\nu_1) \subset \sum$, as defined in Proposition 4.2.

The contribution to (27) from the second term in (29), after shifting in directions α and β, is

$$\begin{aligned}\partial_\alpha^h\langle\partial_\beta e_i^h, m_\beta^h e_j^h - \overline{e}_j^h\rangle\tau_\alpha^h\tau_\beta^h a_{j,\gamma}^h dx^\alpha \wedge dx^\beta \wedge dx^\gamma &= \{\partial_\alpha^h(\langle\partial_\beta^h e_i^h, m_\beta^h e_j^h - \overline{e}_j^h\rangle\tau_\beta^h a_{j,\gamma}^h) \\ &- \langle\partial_\beta^h e_i^h, m_\beta^h e_j^h - \overline{e}_j^h\rangle\partial_\alpha^h\tau_\beta^h a_{j,\gamma}^h\}dx^\alpha \wedge dx^\beta \wedge dx^\gamma =: I^h + II^h.\end{aligned}$$

Since, as $h \to 0$ suitably, $\tau_\beta^h a_{j,\gamma}^h \to a_{j,\gamma}$ while $m_\beta^h e_j^h, \overline{e}_j^h \to e_j$ in $L^p(T^3)$ for any $p < \infty$, and since $(\partial_\beta^h e_i^h)$ is bounded in $L^2(T^3)$, the first term $I^h \rightharpoonup 0$ in $\mathcal{D}'(T^3)$. Observing that for any $t \in S^1$, $x_h \in T_h$, $x = x_h + \xi \in T$, $\xi \in Q_h^+(0)$, we have

$$(m_\beta^h e_j^h - \overline{e}_j^h)(t,x) = \frac{1}{2}(\tau_\beta^h e_j^h - e_j^h)(t,x_h) - \sum_{\alpha=1,2}\xi^\alpha\partial_\alpha^h e_j^h(t,x_h) - \xi^1\xi^2\partial_1^h\partial_2^h e_j^h(t,x_h),$$

moreover, we can estimate

$$|(m_\beta^h e_j^h - \overline{e}_j^h)(t,x)| \leq Ch\sum_{\alpha=1,2}|\partial_\alpha^h e_j^h(t,x_h)|.$$

Thus, the second term above may be bounded

$$\begin{aligned}|II^h| &\leq |\langle\partial_\beta^h e_i^h, m_\beta^h e_j^h - \overline{e}_j^h\rangle\partial_\alpha^h\tau_\beta^h a_{j,\gamma}^h| \\ &\leq Ch|d^h e^h|^2|\partial_\alpha^h\tau_\beta^h a^h| \leq C|\tau_\alpha^h(\tau_\beta^h a^h) - \tau_\beta^h a^h||d^h e^h|^2.\end{aligned}$$

Shifting back, from (29) and (30) we thus obtain that

$$d^{-h}\omega_{ij}^{-h} \wedge a_j^h - d\omega_{ij} \wedge a_j = \tau_{1i}^h + \tau_{2i}^h + \nu_1 + o(1),$$

where $o(1) \to 0$ in $\mathcal{D}'(T^3)$ and where

$$|\tau^h_{2i}| \le C \sum_\alpha |a^h - a^h(\,\cdot\, - h\underline{e}_\alpha)| e_h(e^h).$$

Lemma 4.5. *$\tau^h_{1i} + \tau^h_{2i} \rightharpoonup \nu_2$ in $\mathcal{M}(T^3)$, where ν_2 is a Radon measure with $\operatorname{supp}(\nu_2) \subset \sum$, as defined in Proposition 4.2.*

Proof. For any $\varphi \in C^0(T^3)$ we can estimate

$$\begin{aligned}
&\bigg|\int_{T^3} \tau^h_{1i}\varphi\,dz\bigg| + \bigg|\int_{T^3} \tau^h_{2i}\varphi\,dz\bigg| = \bigg|\int_{S^1\times T_h} \tau^h_{1i}\varphi^h\bigg| + \bigg|\int_{S^1\times T_h} \tau^h_{2i}\varphi^h\bigg| \\
&\le C\int_{S^1\times T_h} \sum_\alpha (|u^h(\cdot \pm h\underline{e}_\alpha) - u^h| + |a^h(\,\cdot\, \pm h\underline{e}_\alpha) - a^h|)\cdot(e_h(u^h) + e_h(e^h))|\varphi^h|.
\end{aligned}$$

Now by Proposition 2.2(i) and Lemma 3.2, for any $z = (t, x_h) \in S^1 \times T_h$, any $0 < h \le 3h \le r < 1/2$ we have

$$|(u^h(\,\cdot\, \pm h\underline{e}_\alpha) - u^h)(t, x_h)|^2 \le E_h(u^h(t); Q^+_{2h}(x_h)) \le Cr^{-1}\int_{P_r(z)} e_h(u^h) + o(1)$$

where $o(1) \to 0$ as $h \to 0$.

Similarly, for any $z = (t, x_h) \in S^1 \times T_h$, any $0 < h \le 2h \le r < R \le 1/2$, by Proposition 2.3 (i) we can estimate

$$|(a^h_j(\,\cdot\, \pm h\underline{e}_\alpha) - a^h_j)(t, x_h)|^2 \le Ch^{-1}\int_{P_{2h}(z)} e_h(a^h_j) \le C[d^h a^h_j]^2_{L^{2,1}(P_r(z))}.$$

Hence by Lemma 4.3 we obtain

$$|(a^h_j(\,\cdot\, \pm h\underline{e}_\alpha) - a^h_j)(z)| \le C\bigg(\frac{r}{R}||a^h_j||_{\mathrm{BMO}(P_R(z))} + ||\theta^h_j||_{L^{2,1}(P_R(z))}\bigg).$$

It follows that $\tau^h_{1i} + \tau^h_{2i} \rightharpoonup \nu_2$ in $\mathcal{M}(T^3)$ as $h \to 0$, where ν_2 is absolutely continuous with respect to μ with density

$$\begin{aligned}
\frac{d\nu_2}{d\mu}(z) &= \lim_{r\to 0}\frac{\nu_2(P_r(z))}{\mu(P_r(z))} \\
&\le C\lim_{r\to 0}\limsup_{h\to 0}\bigg(\frac{r}{R}||a^h||_{\mathrm{BMO}(P_R(z))} + ||d^{\pm h}u^h||_{L^{2,1}(P_{3R}(z))}\bigg) \\
&\le CR^{-1}\mu(P_R(z))
\end{aligned}$$

for any $z \in T^3$. Since $R > 0$ is arbitrary, the asserted characterization of $\operatorname{supp}(\nu_2)$ follows. $\square$

This completes the proof of Theorem 4.1 if TN is parallelizable. In the general case, by the results of [3] and [9] we may embed N as a totally geodesic submanifold of another manifold $\widetilde{N}$ with this property. As above, we now obtain weak convergence of a subsequence $u^h \rightharpoonup u$, where $u : \mathbb{R} \times \mathbb{R}^2 \to N \hookrightarrow \widetilde{N}$ is a weak wave map into $\widetilde{N}$. But then as in [11], p. 255 f., it follows that u also is a weak wave map into N.

5. Global existence of wave maps

Theorems 3.1 and 4.1 easily give rise to the following existence result, previously established in [11] by a different method.

Theorem 5.1. *For any* $(u_0, u_1) \in H^1 \times L^2(\mathbb{R}^2; TN)$ *there exists a global weak solution* u *of the Cauchy problem* (1), (2) *satisfying the energy inequality*

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^2} |Du(t)|^2\,dx \le E_0 = \frac{1}{2}\int_{\mathbb{R}^2} (|u_1|^2 + |\nabla u_0|^2)\,dx$$

for all t *and which continuously attains the initial data in* $H^1 \times L^2$.

Proof. Let u_0^h, u_1^h be the maps u_0, u_1, discretized as in Section 2.4. Note that

$$\begin{aligned}\operatorname{dist}^2(u_0^h(x), N) &\le \fint_{Q_h^+(x)} |u_0^h(x) - u_0(y)|^2\,dy \\ &\le \fint_{Q_h^+(x)}\fint_{Q_h^+(x)} |u_0(y) - u_0(y')|^2\,dy\,dy' \\ &\le C\int_{Q_h^+(x)} |\nabla u_0|^2\,dy \to 0\end{aligned}$$

as $h \to 0$. Hence for $0 < h \le h_0$ the range of u_0^h lies in a sufficiently small tubular neighbourhood of N and we may project to obtain spatially discrete data $(\widetilde{u}_0^h = \pi_N \circ u_0^h, \widetilde{u}_1^h = u_1^h) : M_h \to TN$ such that

$$\widetilde{E}_h := \frac{1}{2}\int_{M_h} (|\widetilde{u}_1^h|^2 + |d^h\widetilde{u}_0^h|^2) < \infty$$

and such that

$$(\widetilde{u}_0^h, \widetilde{u}_1^h) \to (u_0, u_1) \quad \text{in } H^1 \times L^2$$

as $h \to \infty$. In particular $\widetilde{E}_h \to E_0$ as $h \to 0$.

By Theorem 3.1 now, for any $h > 0$ there exists a unique global solution $\widetilde{u}^h$ of (9) with data $(\widetilde{u}^h, \widetilde{u}_t^h)_{|t=0} = (\widetilde{u}_0^h, \widetilde{u}_1^h)$, satisfying the energy identity $E_h(\widetilde{u}^h(t)) = \widetilde{E}_h$ for all t.

By Theorem 4.1 a subsequence $(\widetilde{u}^h)$ as $h \to 0$ weakly converges to a weak solution u of (1), (2) with

$$E(u(t)) \le \liminf_{h\to 0} E_h(\widetilde{u}^h(t)) = E_0$$

for all t. In particular,

$$\limsup_{t\to 0} \frac{1}{2}\int_{\mathbb{R}^2} |Du(t)|^2\,dx = \limsup_{t\to 0} E(u(t)) \le E_0$$

and we conclude that $Du(t) \to Du(0)$ strongly in $L^2(\mathbb{R}^2)$ as $t \to 0$, showing that the initial data are attained continuously in $H^1 \times L^2$. $\square$

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