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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF 1D-BURGERS EQUATION WITH QUASI-PERIODIC FORCING

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Dedicated to Jürgen Moser

1. Introduction

We study in this paper the asymptotics of solutions of 1D-Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + \beta F'_\alpha(x) \tag{1}$$

as $t \to \infty$ for the case of quasi-periodic forcing, i.e.

$$F_\alpha(x) = \sum\nolimits_{n=(n_1,\dots,n_d)\in\mathbb{Z}^d} f_n \exp\{2ni(n, \alpha + x\omega)\}.$$

Here $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathrm{Tor}^d$, and $(\alpha + x\omega) \in \mathrm{Tor}^d$ is the orbit of the quasi-periodic flow $\{S^x\}$ on Tor^d, i.e. $S^x\alpha = \alpha + x\omega$, $-\infty < x < \infty$. We shall assume that ω is Diophantine, i.e. $|(\omega, n)| \geq K/|n|^\gamma$ for positive constants γ, K, $|n| = \sum_{i=1}^d |n_i| \neq 0$. The coefficients f_n decay so fast that $\sum |f_n| \cdot |n|^r < \infty$ for some $r > 1$. Then F_α, F'_α can be considered as values of continuous functions F, dF/dx on Tor^d along the orbit of $\{S^x\}$.

The case $d = 1$ was considered in [Si1]. It was shown there that any solution $u(x,t)$ for which $u(x,0)$, $\int_0^x u(y,0)\,dy$ are periodic functions of some period R

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converges as $t \to \infty$ to a limit $u_0(x)$ which does not depend on $u(x,0)$. The limiting solution satisfies the equation

$$\frac{du_0^2}{dx} = \nu \frac{d^2 u_0}{dx^2} + \beta F'_\alpha(x) \tag{2}$$

or

$$\nu \frac{du_0}{dx} - u_0^2 + \beta F_\alpha(x) + C = 0, \tag{3}$$

where C is a constant. This is a Riccati-type equation closely connected with the corresponding Schroedinger equation. To see this we put

$$v_0 = \exp\left\{ -1/\nu \int_0^x u_0(z)\,dz \right\}.$$

Then

$$-\nu v_0'' + \frac{\beta F_\alpha + C}{\nu} v_0 = 0. \tag{4}$$

For $d = 1$ all functions $u(x,t)$, $u_0(x)$, $v_0(x)$ are periodic functions of period 1. Then (4) shows that v_0 is a periodic eigen-function of the Schroedinger operator with periodic potential

$$L_\alpha \psi = -\nu \frac{d^2\psi}{dx^2} + \beta \frac{F_\alpha(x)}{\nu} \psi.$$

We are interested in periodic eigen-functions for which $v_0'/v_0 = -u_0/\nu$ is also a continuous periodic function. This is possible only if v_0 is positive, i.e. v_0 has to be the ground state of (4).

This argument gives the form of the limiting solution $u_0(x)$. The statement about the convergence to u_0 follows from the ergodic theorem for Markov chains with compact phase space. The convergence to the limit is actually exponential (see details in [Si1]).

In this paper we extend these results to the case of general quasi-periodic forcing, i.e. $d > 1$. The initial conditions $u(x,0)$ again are assumed to be periodic with some period $R > 0$ so that their primitives $\int_0^x u(y,0)\,dy$ are also periodic. Without this assumption, the asymptotic behavior can be quite different and not universal.

The same arguments as above can be applied to show that the limiting solution $u_0(x)$ can be expressed through the ground state of the same Schroedinger operator

$$L_\alpha = -\nu \frac{d^2}{dx^2} + \frac{\beta F_\alpha}{\nu} \tag{5}$$

considered on the whole line R^1. However, in this case the ground state exists only for small enough β while for large β we have Anderson localization and the absence of ground state (see [Si2], [FSW]). We shall consider related problems in another publication.

For small β we shall use the following theorem by S. M. Kozlov (see [K1]):

Kozlov Theorem. *Let $\mathcal{H}^r$ be the Sobolev space of periodic functions*

$$F(\alpha)=\sum f_n \exp\{2\pi i(n,\alpha)\} \quad \text{for which} \sum_{n\in\mathbb{Z}^d}|n|^r|f_n|<\infty.$$

Given Diophantine ω and $r\geq 1$ one can find $\beta_0>0$ and r_1 such that for any $F\in\mathcal{H}^{r_1}$ and $|\beta|\leq\beta_0$ the Schroedinger operator L_α has the ground state. This means that one can find positive $H\in\mathcal{H}^r$ and $\lambda\in R^1$ for which

$$L_\alpha H(S^x\alpha)=\lambda H(S^x\alpha).$$

The main result of this paper is the following theorem.

Theorem 1. *We assume that $|\beta|\leq\beta_0$ where $\beta_0>0$ is so small that we can use* Kozlov Theorem *for $F\in\mathcal{H}^{r_1}$ with large enough r_1 and $r>1+2\gamma$. There exists a quasi-periodic function $u_0(x)=U_0(S^x\alpha)$, $U_0\in\mathcal{H}^r$ such that for every initial condition $u(x,0)$ for which $u(x,0)$, $\int_0^x u(y,0)\,dy$ are continuous periodic functions of some period R,*

$$\lim_{t\to\infty}u(x,t)=u_0(x) \quad \text{for every } x\in R^1.$$

Remarks. 1. The smoothness of u_0 is determined by the smoothness of $H(\alpha)$ in Kozlov's Theorem.

2. The statement of the theorem is actually true for a much wider class of initial conditions (see Section 2).

2. Proof of Theorem 1

We use the Hopf–Cole substitution $u=-\nu\varphi_x/\varphi$ (see [H], [C], [Si1]) which leads to the heat equation for φ.

$$\varphi_t=\nu\varphi_{xx}+\frac{\beta}{\nu}F_\alpha(x)\varphi. \tag{6}$$

Without any loss of generality, we may assume that $\int F(\alpha)\,d\alpha=0$. The Feynman–Kac formula (see [S]) enables one to write φ as the functional integral

$$\varphi(x,t)=\int_{-\infty}^{\infty}\exp\left\{-\frac{w(y)}{\nu}\right\}K(y,0;x,t)\,dy, \tag{7}$$

where $w(x)=\int_0^x u(z;0)\,dz$ and K is the partition function

$$K(y,s;x,t)=\int\exp\left\{\frac{\beta}{2\nu}\int_s^t F(\alpha+b(\tau)\omega)d\tau\right\}dW_{(x,t)}^{(y,s)}(b). \tag{8}$$

Here W is the (non-normed) Wiener measure on the Borel σ-algebra of Wiener trajectories $b(\tau)$, $s \le \tau \le t$, for which $b(t) = x$, $b(s) = y$. The diffusion constant of this measure is 2ν. The point (x,t) ($(y,0)$) is considered as the initial (end) point of b and time goes in the inverse direction.

The solution of the Burgers equation can be written as the ratio

$$u(x,t) = -\nu \frac{\frac{\partial}{\partial x}\int \exp\{-\frac{w(b(0))}{\nu} + \frac{\beta}{2\nu}\int_0^t F'(\alpha + b(\tau)\cdot\omega)\,d\tau\}\,dW_{(x,t)}(b)}{\int \exp\{-\frac{w(b(0))}{\nu} + \frac{\beta}{2\nu}\int_0^t F(\alpha + b(\tau)\omega)\,d\tau\}\,dW_{(x,t)}(b),} \tag{9}$$

where $W_{(x,t)}$ is the Wiener measure on trajectories $b(s), s \le t$ such that $b(t) = x$.

If β is small enough we can use Kozlov's Theorem which allows us to pass from the partition function $K(y,s;x,t)$ to the probability density $p(y,s;x,t)$ by putting

$$p(y,s;x,t) = \frac{K(y,s;x,t)H(\alpha + y\omega)}{\lambda^{t-s}H(\alpha + x\omega)}$$

and λ is the corresponding eigen-value. It is easy to check that p satisfies Chapman–Kolmogorov equation and thus determines a probability distribution of a diffusion process taking place on the orbit of the flow $\{S^x\alpha\}$. The probability density p satisfies Fokker–Plank–Kolmogorov equation

$$\frac{\partial p}{\partial s} = \nu\frac{\partial^2 p}{\partial y^2} - \frac{\partial}{\partial y}\left(\frac{H'(\alpha + y\omega)}{H(\alpha + y\omega)}\cdot p\right).$$

This equation shows also that the diffusion process considered on the whole torus Tor^d has an invariant measure given by the density

$$\frac{1}{H(\alpha)}\frac{dH(\alpha + t\omega)}{dt}\Big|_{t=0}.$$

Using p we can write another expression for the solution u:

$$u(x,t) = -\nu\frac{\frac{\partial}{\partial x}[H(\alpha + x\omega)\int_{-\infty}^{\infty}\exp\{-\frac{w(y)}{\nu}\}p(y,0;x,t)\,dy}{H(\alpha + x\omega)\int_{-\infty}^{\infty}\exp\{-\frac{w(y)}{\nu}\}p(y,0;x,t)\,dy}. \tag{10}$$

In Section 3 we show that our diffusion process satisfies local central limit theorem of probability theory which we shall formulate as a separate statement.

THEOREM 2. *For $t \to \infty$ the function p has the following asymptotic representation*

$$(x,t;y,s) = \frac{1}{\sqrt{2\pi\sigma(t-s)}}\exp\left\{-\frac{1}{2}\frac{(y-x)^2}{\sigma(t-s)}\right\}(1 + \delta(x,t;y,s)),$$

where the remainder $\delta(x,t;y,s)$ satisfies the inequalities:

(i) *$\delta(x,t;y,s)$, $\partial\delta(x,t;y,s)/\partial x$ tend to zero uniformly in y in any interval of the form $(-C\sqrt{t-s},\, C\sqrt{t-s})$; C is an arbitrary constant;*

(ii) $|p(x,t;y,s)| \le h\left(\frac{y-x}{\sqrt{t-s}}\right) \cdot \frac{1}{\sqrt{t-s}}$,
$\left|\frac{\partial p(x,t;t,s)}{\partial x}\right| \le h\left(\frac{y-x}{\sqrt{t-s}}\right) \cdot \frac{1}{\sqrt{t-s}}$, *where* $h \in L^1(R^1)$.

Using Theorem 2 we can easily finish the proof of Theorem 1. Indeed, from Theorem 2 we conclude that

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{w(y)}{\nu}\right\} \frac{p(y,0;x,t)}{H(\alpha+y\omega)}\, dy = a + \varepsilon(x,t),$$

where a is a constant, $\varepsilon(x,t)$, $\partial\varepsilon(x,t)/\partial x \to 0$ as $t \to \infty$ for any x. In view of (10) it implies

$$\lim_{t\to\infty} u(x,t) = -\nu \frac{H'(\alpha+x\omega)}{H(\alpha+x\omega)}.$$

The last expression also gives the form of the limiting solution which is consistent with our arguments at the beginning of Section 1.

Proof of Theorem 2

The central limit theorem for quasi-periodic diffusion processes was proven by S. M. Kozlov (see [K2]). We shall describe here a different approach based on the theory of Levy excursions and some ideas from [Si3].

We consider the torus $\mathrm{Tor}^{d-1} = \{\alpha \mid \alpha_d = 0\}$ and the induced map $T = S^{1/\omega_d}$ on Tor^{d-1} corresponding to our flow $\{S^x\}$. It is clear that for $\alpha = (\alpha_1, \dots, \alpha_{d-1}, 0)$ its image $T\alpha = (\alpha_1 + \omega_1/\omega_d, \dots, \alpha_{d-1} + \omega_{d-1}/\omega_d, 0)$. In the arguments in this section, we assume that the initial point α is given. Without any loss of generality we may assume that $\alpha \in \mathrm{Tor}^{d-1}$. In what follows, we use the notation $\alpha^{(0)}$ for α and put $\alpha^{(m)} = T^m\alpha^{(0)}$, $-\infty < m < \infty$. We can represent also the sequence $\{\alpha^{(m)}\}$ as the one-dimensional lattice $\{m/\omega_d\} \subset R^1$. Having a trajectory $b(s)$ of our diffusion process, we can consider it also as a diffusion on R^1. This means that the x-coordinate on R^1 corresponds to $S^x\alpha^{(0)}$. We shall use the notation $b(s)$ for such trajectories.

We take a point $\alpha^{(0)} \in \mathrm{Tor}^{d-1}$ and consider the set of trajectories $b(s)$ for which $b(0) = \alpha^{(0)}$ and $b(s)$ reaches $\alpha^{(1)}$ earlier than $\alpha^{(-1)}$. The probability of this set is denoted by $p(\alpha^{(0)})$, $1 - p(\alpha^{(0)})$ is the probability of those trajectories which go out of $\alpha^{(0)}$ and reach $\alpha^{(-1)}$ earlier than $\alpha^{(1)}$. In this way we get a simple random walk on Tor^{d-1} in the sense of [Si3]. I owe the following lemma to M. Aizenman.

Lemma 1. *This random walk is symmetric, i.e.*

$$\int_{\mathrm{Tor}^{d-1}} \ell n p(\alpha)\, d\alpha = \int_{\mathrm{Tor}^{d-1}} \ell n(1-p(\alpha))\, d\alpha.$$

PROOF. We have the following symmetry of the partition function

$$K(y,s;x,t) = K(y,0;x,t-s) = K(x,0;y,t-s)$$

which follows from the map $\{b(\tau)\} \to \{b((t-s)-\tau)\}$ preserving the statistical weight $\exp\{\beta/2\nu \int_0^{t-s} F(\alpha + b(\tau)\omega)\, d\tau\}$. Therefore, for probabilities p, we have

$$\text{const} \le \frac{p(y,s;x,t)}{p(x,s;y,t)} \le \text{const}$$

which shows that the mean drift of our diffusion process is zero. Lemma is proven. □

We take any point $\alpha^{(0)} \in \mathrm{Tor}^{d-1}$ and consider trajectories $\{b(s)\}$ which go out of $\alpha^{(0)}$, i.e. $b(0) = \alpha^{(0)}$. A positive cycle is a part $\{b(s), 0 \le s \le \tau\}$ such that

(i) $b(s) \ne \alpha^{(-1)}$ for all $0 \le s \le \tau$,
(ii) $b(s) = \alpha^{(1)}$ for at least one $s, 0 \le s \le \tau$; denote by s_0 the minimal s with this property,
(iii) $b(s) \ne \alpha^{(0)}$ for $s_0 \le s < \tau$,
(iiii) $b(\tau) = \alpha^{(0)}$.

In an analogous way, one can define negative cycles. We shall study the distribution of the length τ of positive cycles using the ideas of [Si3], [Si4]. Any cycle has the following structure. A trajectory goes out of a $\alpha^{(0)}$ and at some random moment ξ_1 reaches $\alpha^{(1)}$ not coming to α^{-1} in between. After that, it has several positive cycles which start from $\alpha^{(1)}$. We denote the number of these cycles by ν_1 and their lengths by $\tau_1, \dots, \tau_{\nu_1}$. After the last cycle there follows a final piece of the trajectory when it goes out of $\alpha^{(1)}$ and comes to a $\alpha^{(0)}$ earlier than to $\alpha^{(2)}$. Let the length of this piece be η_1. We denote by $p^{(+)}_{\alpha^{(0)}}(t)$ the probability density that the length of the positive cycle is t. Using strong Markov property of the process b we can write

$$\begin{aligned}(11) \qquad p^{(+)}_{\alpha^{(0)}}(t) = \sum_{\nu_1=0}^{\infty} \int_0^t du q_{\alpha^{(0)},\xi_1}(u) \int p_{\alpha^{(1)}}(u_1) \dots p_{\alpha^{(1)}}(u_{\nu_1}) \\ \cdot q_{\alpha^{(1)},\xi_2}(t-(u+u_1+u_{\nu_1}))du_1 \dots du_{\nu_1}.\end{aligned}$$

Here q_{α,ξ_1}, $q_{T\alpha,\xi_2}$ are the densities of the distributions of ξ_1, ξ_2 respectively. It is tacitly assumed that these densities are zero if the values of the arguments are negative.

We introduce Laplace transforms

$$\varphi^{(+)}_{\alpha}(\lambda) = \int_0^\infty e^{-\lambda t} p^{(+)}_{\alpha}(t)\, dt, \psi_{\alpha,\xi_1}(\lambda) = \int e^{-\lambda t} q_{\alpha,\xi_1}(t)\, dt,$$

$$\psi_{T\alpha,\xi_2}(\lambda) = \int_0^\infty e^{-\lambda t} q_{T\alpha,\xi_2}(t)\, dt.$$

Multiplying both sides of (10) by $e^{-\lambda t}$ and integrating over t from 0 to ∞ we arrive at the expression

$$\varphi_\alpha^{(+)}(\lambda) = \psi_{\alpha,\xi_1}(\lambda)\frac{1}{1-\varphi_{T\alpha}^{(+)}(\lambda)} \cdot \psi_{T\alpha,\xi_2}(\lambda). \tag{12}$$

It is easy to see that the distributions of ξ_1 and ξ_2 decay exponentially. Therefore ψ_{α,ξ_1}, $\psi_{T\alpha,\xi_2}$ are analytic functions of λ in some neighbourhood of $\lambda = 0$. The methods and the arguments in [Si3] give a possibility to show that $\varphi_\alpha^{(+)}(\lambda) = \varphi_\alpha^{(+)}(0) - C(\alpha)\lambda^{1/2}(1+0(1))$ as $\lambda \to 0$. Here $C(\alpha) > 0$ depends only on $\alpha \in \mathrm{Tor}^d$. It follows from Tauberian theorems for Laplace transforms that $p_\alpha^{(+)}(t) \sim C_1(\alpha)/t^{3/2}$ as $t \to \infty$. In other words, the distribution $p_\alpha^{(+)}(t)$ belongs to the domain of attraction of the one-sided stable law with exponent $\alpha = 1/2$ (see [GK], [F]). In the same way, one can study the asymptotics of the distribution of the lengths of negative cycles.

We return back to our diffusion process $b(s)$, $s > 0$. Each realization determines a random walk on the lattice $\{m/\omega_d\}$ which we shall denote by $B(n)$, $0 \le n < \infty$. We have also random moments of time $\mathcal{T}(n)$ which are determined uniquely from the expression

$$B(n) = k \Leftrightarrow b(\mathcal{T}(n)) = k.$$

It is clear that $0 = \mathcal{T}(0) < \mathcal{T}(1) < \ldots < \mathcal{T}(n) < \ldots, \mathcal{T}(n) \to \infty$ as $n \to \infty$.

Let y lie between $\alpha^{(k)}$ and $\alpha^{(k+1)}$ on the orbit $\{S^x\alpha^{(0)}, -\infty < x < \infty\}$. If $b(0) = y$ then one can find a $\mathcal{T}(n)$ such that $b(\mathcal{T}(n)) = \alpha^{(k)}$ or $\alpha^{(k+1)}$ and $b(s)$ remains within the interval $(\alpha^{k-1)}, \alpha^{(k+1)})$ in the first case and inside the interval $(\alpha^{(k)}, \alpha^{(k+2)}$ in the second case when s changes between $\mathcal{T}(n)$ and t. We can write

$$p(y,0;x,t) = \int dP_k(s)q_k(t-s)$$

in the first case and

$$p(y,0;x,t) = \int dP_{k+1}(s)q'_k(t-s)$$

in the second one where q_k, q'_k are the corresponding conditional probability densities to be at $t = 0$ at y and to remain within the above-mentioned intervals. All these probabilities depend also on $\alpha^{(k)}$ or $\alpha^{(k+1)}$ respectively. It is easy to see that they both decay exponentially as functions of $t-s$. The functions $P_k(s)$ are the distribution functions of $\mathcal{T}(n)$. The same arguments as in [Si3] show that $\mathcal{T}(n)$ can be represented as a sum of k independent random variables where each variable belongs to the domain of attraction of the one-sided stable law with exponent $\alpha = 1/2$. Therefore, the density of the distribution of $\mathcal{T}(n)/k^2$ converges to $(1/\sqrt{2\pi z^3\sigma})\exp\{-\sigma/2z\}$ where $\sigma > 0$ is a constant. If we write $y = k + y_1$, $\mathcal{T}(n) = t - \tau$ where τ is the transition time from k or $k+1$

to y then the convergence mentioned above easily implies that the density of distribution of $(b(0) - b(t))/\sqrt{t} = (y - x)/\sqrt{t}$ converges to the Gaussian density. Other estimates of Theorem 2 can be also easily obtained along these lines.

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