Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 10, 1997, 93–106

A HAMILTONIAN SYSTEM WITH AN EVEN TERM

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1. Introduction

In this paper we study, using variational methods, a Hamiltonian system of the form $-u'' + u = h(t)V(u)$, where h and V are differentiable, h is positive, bounded, and bounded away from zero, and V is a "superquadratic" potential. That is, V behaves like q to a power greater than 2, so $|V(q)| = o(|q|^2)$ for $|q|$ small and $V(q) > O(|q|^2)$ for $|q|$ large. To prove that a solution homoclinic to zero exists, one must assume additional hypotheses on h (see [EL] for a counterexample). In [R1], solutions were found when h is assumed to be periodic. In [STT], solutions were found when h is almost periodic (a weaker condition than periodicity). In [MNT], a condition yet weaker than almost periodic is defined, and solutions to the equation are found when h satisfies that condition. Like periodicity and almost periodicity, this condition assumes basically that h is similar to translates of itself, that is, for certain large values of T, the functions $t \mapsto h(t)$ and $t \mapsto h(t+T)$ are close to each other. Other ways to guarantee solutions involve making $|h'|$ small: see papers such as [FW], [WZ], and [FdP] on the nonlinear Schrödinger equation, and [A] for a novel example of an h which "oscillates slowly".

In this paper we attempt to find solutions to the system without assuming that h satisfies any kind of time-recurrence property or restriction on h'. We assume two conditions: first, that h is even ($h(-t) = h(t)$). Therefore it is convenient to treat the system as a system on the half-line $\mathbb{R}^+ = [0, \infty)$. Second,

h only takes on a small range of values, with the variation in h depending on V. Here is a statement of the theorem:

Theorem 1.0. *Let $n \geq 1$ and V satisfy*

(V_1) $V \in C^2(\mathbb{R}^n, \mathbb{R})$,

(V_2) $V'(0) = 0$, $V''(0) = 0$ *and*

(V_3) *there exists* $p > 1$ *such that* $V''(q)q\cdot q \geq pV'(q)\cdot q > 0$ *for all* $q \in \mathbb{R}^n \backslash \{0\}$.

Then there exists $d > 0$ with the property that if h satisfies

(h_1) $h \in C^1(\mathbb{R}^+, \mathbb{R})$,

(h_2) $h'(0) = 0$ *and*

(h_3) $1 \leq h(t) \leq 1 + d$ *for all* $t \in \mathbb{R}$,

then the Hamiltonian system

$$-u'' + u = h(t)V'(u) \tag{$*$}$$

has a non-zero solution v on $\mathbb{R}^+$, satisfying $v'(0) = 0$ and $v(t) \to 0$, $v'(t) \to 0$ as $t \to \infty$.

An example of V satisfying (V_1)–(V_3) is $V(q) = |q|^{p+1}$ with $p > 1$. Condition (V_3) is a little stronger than growth conditions found in previous papers such as [Sé] or [CMN]. The conditions on h are fairly weak; h need not be periodic, or monotone, or tend to a single value as $t \to \infty$ like in [BL]. If h has a lower bound other than 1, then h and V can be rescaled so that (h_3) is satisfied and the problem reduces to the one in the theorem statement.

Plan of Proof. We give a variational formulation of the problem. Let $E = W^{1,2}(\mathbb{R}^+)$ along with the inner product

$$(u, w) = \int_0^\infty \big(u' \cdot w' + uw\big)\, dt$$

for $u, w \in E$ and the associated norm $\|u\| \equiv \|u\|_{W^{1,2}(\mathbb{R}^+)}$. Then the functional $I \in C^2(E, \mathbb{R})$ corresponding to ($*$) is

$$I(u) = \frac{1}{2}\|u\|^2 - \int_0^\infty h(t)V(u(t))\, dt.$$

Any critical point v of I satisfies the differential system ($*$), with $v(t) \to 0$ and $v'(t) \to 0$ as $t \to \infty$. Also, any critical point of I satisfies the boundary condition $v'(0) = 0$. Suppose v is a critical point of I. Define $h_2(t) = h(|t|)$ for $t \in \mathbb{R}$. Then, since $h'(0) = 0$, $h_2 \in C^1(\mathbb{R}, \mathbb{R})$. Define the functional I_2 on $W^{1,2}(\mathbb{R})$ by $I_2(u) = \|u\|^2_{W^{1,2}(\mathbb{R})}/2 - \int_{\mathbb{R}} h_2(t)V(u(t))\, dt$ and $v_2 \in W^{1,2}(\mathbb{R})$ by $v_2(t) = v(|t|)$. Then it is easy to verify that v_2 is a critical point of I_2, and therefore a classical solution of the system $-u'' + u = h_2(t)V'(u)$ on the entire real line. Since h_2 is an even function of t, and $h_2 \in C^1(\mathbb{R})$, $v_2'(0) = 0$, so $v'(0) = 0$.

We will prove via an indirect argument that a critical point of I exists. First we define a submanifold $\mathcal{S}$ of $E = W^{1,2}(\mathbb{R}^+)$ with the property that $\inf_{u\in\mathcal{S}} I(u) = c$, where c is the mountain-pass value associated with I. Then we take a sequence $(u_m) \subset E$ with $I(u_m) \to c$ and $I'(u_m) \to 0$ as $m \to \infty$. It is not apparent whether I satisfies the Palais–Smale condition, so it is not clear whether (u_m) converges. But we can show that (u_m) is a bounded sequence, so it has a weak limit. This weak limit point must be a critical point of I. If the limit point is not zero, then Theorem 1.0 is proven.

If (u_m) converges weakly to zero, then matters are more complicated. In this case, we can construct a sequence (y_m) with $I(y_m) \le c/2 + o(m)$, where $o(m) \to 0$ as $m \to \infty$, and y_m "close" to $\mathcal{S}$. For large enough m, we can use y_m to construct $z \in \mathcal{S}$ with $I(z) < c$. This is impossible, so (u_m) has a nonzero weak limit, and there exists v satisfying Theorem 1.0. □

This paper is organized as follows: in Section 2 we explore the mountain-pass structure of the functional I, define the manifold $\mathcal{S}$, and obtain some quantitative estimates. Section 3 contains the main proof of Theorem 1.0, the "splitting" argument to obtain the sequence $(y_m) \subset \mathcal{S}$ in the indirect argument above. Section 4 contains a computation of d for a specific function V.

2. Mountain–pass structure of I

Before defining $\mathcal{S}$, let us explore the related mountain-pass structure of I. Define the set of paths

$$\Gamma = \{\, \gamma \in C([0,1], E) \mid \gamma(0) = 0,\ I(\gamma(1)) < 0 \,\}. \tag{2.0}$$

Integrating (V_3) yields

$$V'(q)q \ge (p+1)V(q) \tag{2.1}$$

for all $q \in \mathbb{R}^n$. For $\lambda > 1$, the above implies

$$V(\lambda q) \ge \lambda^{p+1} V(q) \tag{2.2}$$

for all $q \in \mathbb{R}^n$. Thus it is easy to show that for any $u \in E \setminus \{0\}$, $I(\lambda u) \to -\infty$ as $\lambda \to \infty$, and Γ is well defined. Define the minimax value

$$c = \inf_{\gamma\in\Gamma} \max_{\theta\in[0,1]} I(\gamma(\theta)). \tag{2.3}$$

Let us obtain a positive lower bound for c. Let $\beta > 0$ satisfy

$$|q| \le \beta \Rightarrow V'(q)\cdot q \le |q|^2/8. \tag{2.4}$$

This is possible by (V_1)–(V_2). From now on assume, without loss of generality, that

$$d \le 1. \tag{2.5}$$

Then $h(t) \leq 2$ for all $t \geq 0$. If $\|u\| \leq \beta$, then $\|u\|_{L^\infty(\mathbb{R}^+)} \leq \beta$ (see Appendix), and (by (2.1))

$$\begin{aligned} (2.6)\quad I(u) &= \frac{1}{2}\|u\|^2 - \int_0^\infty h(t)V(u)\,dt \geq \frac{1}{2}\|u\|^2 - \frac{2}{p+1}\int_0^\infty V'(u)\cdot u\,dt \\ &\geq \frac{1}{2}\|u\|^2 - (1)\int_0^\infty \frac{1}{8}|u|^2\,dt \geq \frac{1}{2}\|u\|^2 - \frac{1}{8}\|u\|^2 = \frac{3}{8}\|u\|^2 \geq 0. \end{aligned}$$

Therefore any mountain-pass curve must cross the sphere $\{\|u\| = \beta\}$, that is, if $\gamma \in \Gamma$, there exists $\theta^* \in [0,1]$ with $\|\gamma(\theta^*)\| = \beta$. So the above implies

$$\max_{\theta\in[0,1]} I(\gamma(\theta)) \geq I(\gamma(\theta^*)) \geq 3\|\gamma(\theta^*)\|^2/8 = 3\beta^2/8. \tag{2.7}$$

Since γ is an arbitrary element of Γ,

$$c \geq 3\beta^2/8. \tag{2.8}$$

Note that this estimate does not depend on d, as long as $d \leq 1$.

There is another way to describe c (we will need both characterizations). Define

$$\mathcal{S} = \{u \in E \mid u \neq 0,\ I'(u)u = 0\}. \tag{2.9}$$

In [R2] it is proven, under weaker growth hypotheses on V than (V_3), that

$$\inf_{u\in\mathcal{S}} I(u) = c. \tag{2.10}$$

In fact, for any $u \in \mathcal{S}$, the function $s \mapsto I(su)$ is strictly increasing on $0 < s < 1$, attains a maximum of $I(u)$ at $s = 1$, and decreases to $-\infty$ on $1 < s < \infty$. The following lemma gives estimates how quickly $I(su)$ changes when s is near 1.

LEMMA 2.11. *Let $u \in E$ and define $g(s) = I(su)$ for $s \geq 0$. Assume $p \leq 2$. Then*

$$s \geq 1 \Rightarrow g'(s) \leq g'(1)s^p - (p-1)(s-1)\|u\|^2/4 \tag{i}$$

and

$$1/2 \leq s \leq 1 \Rightarrow g'(s) \geq g'(1)s^p + (p-1)(1-s)\|u\|^2/4. \tag{ii}$$

PROOF. Let $u \in E$ and define $g(s) = I(su)$. Then

$$\left\{ \begin{aligned} g(s) &= \frac{1}{2}s^2\|u\|^2 - \int_0^\infty h(t)V(su)\,dt, \\ g'(s) &= s\|u\|^2 - \int_0^\infty h(t)V'(su)\cdot u\,dt, \\ g''(s) &= \|u\|^2 - \int_0^\infty h(t)V''(su)u\cdot u\,dt. \end{aligned} \right. \tag{2.12}$$

By (V_3), we have

$$g''(s) = \|u\|^2 - \frac{1}{s^2}\int_0^\infty h(t)V''(su)(su)\cdot(su)\,dt \tag{2.13}$$
$$\le \|u\|^2 - \frac{p}{s^2}\int_0^\infty h(t)V'(su)\cdot(su)\,dt$$
$$= \|u\|^2 - \frac{p}{s}\int_0^\infty h(t)V'(su)\cdot u\,dt$$
$$= \|u\|^2 - p(s\|u\|^2 - g'(s))/s = pg'(s)/s - (p-1)\|u\|^2.$$

Therefore,

$$\frac{d}{ds}[s^{-p}g'(s)] = s^{-p}g''(s) - ps^{-p-1}g'(s) \tag{2.14}$$
$$= s^{-p}(g''(s) - pg'(s)/s) \le -(p-1)s^{-p}\|u\|^2.$$

If $s \ge 1$, then integrating the above from 1 to s yields

$$\begin{gathered} s^{-p}g'(s) - g'(1) \le -(p-1)\|u\|^2\int_1^t s^{-p}\,ds = -(1-s^{-p+1})\|u\|^2, \\ g'(s) \le s^p g'(1) - (s^p - s)\|u\|^2. \end{gathered} \tag{2.15}$$

If $s \le 1$, then integrating (2.14) from s to 1 yields

$$\begin{gathered} g'(1) - s^{-p}g'(s) \le -(p-1)\|u\|^2\int_s^1 t^{-p}\,dt = (1-s^{-p+1})\|u\|^2, \\ g'(s) \ge s^p g'(1) + (s - s^p)\|u\|^2. \end{gathered} \tag{2.16}$$

If $s \ge 1$, then by the mean value theorem, there exists $\lambda \ge s \ge 1$ with

$$s^p - s \ge s^{p-1} - 1 \ge (p-1)\lambda^{p-2}(s-1) \ge (p-1)(t-1). \tag{2.17}$$

If $s \in [1/2, 1]$, then $1/s \ge 1$, so by the above,

$$s - s^p = s^{p+1}(1/s^p - 1/s) \ge (p-1)s^{p+1}(1/s - 1) \tag{2.18}$$
$$= (p-1)s^p(1-s) = (1/2)^p(p-1)(1-s)$$
$$\ge (p-1)(1-s)/4.$$

Lemma 2.11 follows from (2.15)–(2.18). □

We have a lower bound for c that is independent of d. We also need an upper bound for c that is independent of d. Define the functional

$$I^+(u) = \frac{1}{2}\|u\|^2 - \int_0^\infty V(u(t))\,dt. \tag{2.19}$$

Then $I^+(u) \ge I(u)$ for all $u \in E$. Define the mountain-pass value c^+, similar to c, by defining the set of paths

$$\Gamma^+ = \{\, g \in C([0,1], E) \mid g(0) = 0,\ I^+(g(1))) < 0 \,\}, \tag{2.20}$$

and setting

$$c^+ = \inf_{g\in\Gamma^+} \max_{\theta\in[0,1]} I^+(g(\theta)). \tag{2.21}$$

c^+ depends only on V, not on d. Using the mountain-pass characterization of c (2.3), it is easy to see that $c^+ \geq c$ because $I^+(u) \geq I(u)$ for all $u \in E$. We will estimate c^+ in terms of β and V in Section 4.

It is well known that (V_3) or a weaker condition implies that Palais–Smale sequences of I are bounded, even that $\mathcal{S} \cap \{u \mid I(u) \leq D\}$ is bounded for any $D \in \mathbb{R}$. We want an estimate on $\|u\|$ for when $I(u)$ is small and u is "almost" in $\mathcal{S}$:

Lemma 2.22. *If $p \leq 2$, $|I'(u)u| \leq c^+$ and $I(u) \leq 2c^+$, then*

$$\|u\| \leq \sqrt{\frac{14c^+}{p-1}} \equiv B. \tag{2.23}$$

Proof. By (2.1) we have

$$\begin{aligned} -c^+ \leq I'(u)u = \|u\|^2 - \int_{\mathbb{R}} hV'(u)\cdot u \leq \|u\|^2 - (p+1)\int_{\mathbb{R}} hV(u) = \\ = (p+1)I(u) - \left(\frac{p-1}{2}\right)\|u\|^2 \leq 6c^+ - \left(\frac{p-1}{2}\right)\|u\|^2, \end{aligned}$$

so

$$\|u\|^2 \leq \left(\frac{2}{p-1}\right)7c^+ = \frac{14c^+}{p-1}. \qquad \square$$

3. Splitting

This section contains the "splitting" argument that is the core of the proof of Theorem 1.0. By Ekeland's Variational Principle ([MW]), there exists a Palais–Smale sequence $(u_m) \subset E$ with $I(u_m) \to c$ and $I'(u_m) \to 0$ as $m \to \infty$. By arguments of [CR], (u_m) is bounded. Therefore it has a subsequential weak limit $\overline{u}$. Also by [CR], $\overline{u}$ is a critical point of I, and u_m converges to $\overline{u}$ in $W^{1,2}([0,R])$ for each $R > 0$. If $\overline{u} \neq 0$, then Theorem 1.0 is proven. In fact, in this case, $I(\overline{u}) \leq c$ (see [CR]). $I(\overline{u}) \geq c$ because by the observations following (2.10), for large enough T, $\theta \mapsto T\theta\overline{u}$ defines a path in Γ, along which the maximum value of I is c. Thus $I(\overline{u}) = c$.

We will show that if d is chosen small enough, in terms of V, then the case $\overline{u} = 0$ is impossible. The argument is indirect. Suppose $\overline{u} = 0$. Define the cutoff function $\varphi \in C(\mathbb{R}^+, [0,1])$ by $\varphi(t) = t$ for $0 \leq t \leq 1$, $\varphi \equiv 1$ on $[1,\infty)$. Define $w_m = \varphi u_m$. $\|u_m\|_{W^{1,2}([0,1])} \to 0$ as $m \to \infty$, and it is easy to verify that $\|u_m - w_m\| \to 0$ as $m \to \infty$. I'', I', and I are bounded on bounded subsets of E. For example, to prove for I'', let $K > 0$ and suppose $\|u\| \leq K$. Then

$\|u\|_{L^\infty(\mathbb{R}^+)} \le K$ (see Appendix). Let $C > 0$ satisfy $|V''(q)xy| \le C$ for all $|q| \le K$, $|x| \le 1$, $|y| \le 1$. Let $v, w \in E$. Then

$$
\begin{aligned}
|I''(u)(v,w)| &= \left|(v,w) - \int_0^\infty h(t)V''(u)v\cdot w\,dt\right| \\
&\le \|v\|\|w\| + \int_0^\infty 2C|v||w|\,dt \\
&\le \|v\|\|w\| + 2C\|v\|_{L^2(\mathbb{R}^+)}\|w\|_{L^2(\mathbb{R}^+)} \\
&\le (1+2C)\|v\|\|w\|.
\end{aligned}
\tag{3.0}
$$

Since I'', I' and I are bounded on bounded subsets of E, and (u_m) is a bounded sequence, it follows that $I(w_m) \to c$ and $I'(w_m)w_m \to 0$ as $m \to \infty$.

Let $\varepsilon > 0$ satisfy

$$\varepsilon < \beta^2/4 \tag{3.1}$$

where β is from (2.4). ε will fixed more precisely later. Since $w_m \to 0$ in $W^{1,2}([0,1])$ (and thus in $L^\infty([0,1])$), we may choose m large enough so that

$$\|w_m\|_{L^\infty([0,1])} < \beta, \tag{3.2}$$

$$I(w_m) < 7c/6, \tag{3.3}$$

and

$$|I'(w_m)w_m| < \varepsilon. \tag{3.4}$$

For convenience define

$$w = w_m. \tag{3.5}$$

We will choose a "cutting point" $\widehat{t} > 0$, and split w into two functions, $w^{(1)} = w|_{[0,\widehat{t}]}$ (the restriction of w to $[0,\widehat{t}]$), and $w^{(2)} = w|_{[\widehat{t},\infty]}$. Functions $w^{(1)}$ and $w^{(2)}$ can be transformed into z_1 and z_2 respectively in E: $w^{(1)}$ into z_1, by reflecting over $t = \widehat{t}/2$; and $w^{(2)}$ into z_2, by translating by a factor of $\widehat{t}$ to the left. If d is small enough and $\widehat{t}$ is chosen carefully, $I'(z_1)z_1$ and $I'(z_2)z_2$ are both very close to zero, but either $I(z_1)$ or $I(z_2)$ is significantly less than c. Using Lemma 2.11, we then choose $\bar{s}$ very close to 1 so that $\bar{s}z_* \in \mathcal{S}$ but $I(\bar{s}z_*) < c$, where $* = 1$ or 2. This contradicts the fact that $\inf\{I(u) \mid u \in \mathcal{S}\} = c$, proving Theorem 1.0.

Let us choose $\widehat{t}$. We claim that $\|w_m\|_{L^\infty(\mathbb{R}^+)} > \beta$ for large m: since $I(w_m) \to c$ and $I(0) \neq c$, $\|w_m\|$ is bounded away from 0 for large m. If $\|w_m\|_{L^\infty(\mathbb{R}^+)} \le \beta$, then by (2.4),

$$
\begin{aligned}
I'(w_m)(w_m) &= \|w_m\|^2 - \int_0^\infty hV'(w_m)\cdot w_m\,dt \\
&\ge \|w_m\|^2 - \int_0^\infty 2\left(\frac{1}{8}\right)|w_m|^2\,dt \ge \frac{3}{4}\|w_m\|^2.
\end{aligned}
\tag{3.6}
$$

This cannot happen for large m, since $\|w_m\|$ is bounded away from 0 for large m and $I'(w_m)w_m \to 0$. Since $\|w_m\|_{L^\infty(\mathbb{R}^+)} > \beta$ for large m, we may define

$$t_0 = \min\{\, t \mid |w(t)| \ge \beta \,\} < t_1 = \max\{\, t \mid |w(t)| \ge \beta \,\}. \tag{3.7}$$

By (3.2), $1 < t_0 < t_1$. By (3.4),

$$|I'(w)w| = \left| \int_0^\infty |w'|^2 + |w|^2 - h(t)V'(w) \cdot w \, dt \right| < \varepsilon. \tag{3.8}$$

We will choose the cutting point $\widehat{t}$ between t_0 and t_1 so that the integral above, evaluated only from 0 to $\widehat{t}$, is zero (and the integral evaluated from $\widehat{t}$ to ∞ is also close to zero). For $t < t_0$, $|w(t)| < \beta$, and since by (2.5) $h(t) \le 2$ for all $t \ge 0$,

$$|h(t)V'(w(t))w(t)| \le 2|w(t)|^2/8 = |w(t)|^2/4 \tag{3.9}$$

by the definition (2.4) of β. Therefore

$$\begin{aligned} &\int_0^{t_0} |w'|^2 + |w|^2 - h(t)V'(w(t)) \cdot w(t) \, dt \\ &\qquad \ge \frac{3}{4} \int_0^{t_0} |w'|^2 + |w|^2 \, dt = \frac{3}{4} \|w\|^2_{W^{1,2}([0,t_0])} \\ &\qquad \ge \frac{3}{4} \|w\|^2_{W^{1,2}([0,t_0])} \ge \frac{3}{16} \|w\|^2_{L^\infty([0,t_0])} = \frac{16}{3}\beta^2, \end{aligned} \tag{3.10}$$

using an embedding in the Appendix, and the fact that $\|w\|_{L^\infty([0,t_0])} = \beta$. By similar reasoning to (3.9)–(3.10), and using the other embedding in the Appendix,

$$\begin{aligned} &\int_{t_1}^{\infty} |w'|^2 + |w|^2 - h(t)V'(w(t)) \cdot w(t) \, dt \\ &\qquad \ge \frac{3}{4} \|w\|^2_{W^{1,2}([t_1,\infty])} \ge \frac{3}{4} \|w\|^2_{L^\infty([t_1,\infty])} = \frac{3}{4}\beta^2. \end{aligned} \tag{3.11}$$

By (3.8), (3.11), and (3.1),

$$\begin{aligned} &\int_0^{t_1} |w'|^2 + |w|^2 - h(t)V(w(t))w(t) \, dt \\ &\qquad = \int_0^\infty |w'|^2 + |w|^2 - h(t)V'(w) \cdot w \, dt \\ &\qquad\quad - \int_{t_1}^\infty |w'|^2 + |w|^2 - h(t)V'(w) \cdot w \, dt \\ &\qquad < \varepsilon - \frac{3}{4}\beta^2 < \frac{1}{4}\beta^2 - \frac{3}{4}\beta^2 < 0. \end{aligned} \tag{3.12}$$

The above integral is negative but the integral from 0 to t_0 of the same integrand is positive by (3.10). Therefore there exists $\widehat{t} \in (t_0, t_1)$ with

$$\int_0^{\widehat{t}} |w'|^2 + |w|^2 - h(t)V'(w(t)) \cdot w(t) \, dt = 0. \tag{3.13i}$$

By the above and (3.8), we have similarly,

$$\left| \int_{\widehat{t}}^{\infty} |w'|^2 + |w|^2 - h(t)V'(w(t)) \cdot w(t)\, dt \right| < \varepsilon. \tag{3.13ii}$$

By (3.3),

$$\int_0^{\widehat{t}} \frac{1}{2}|w'|^2 + \frac{1}{2}|w|^2 - h(t)V(w(t))\, dt < \frac{7}{12}c \tag{3.14i}$$

or

$$\int_{\widehat{t}}^{\infty} \frac{1}{2}\dot{w}^2 + \frac{1}{2}w^2 - h(t)V(w(t))\, dt < \frac{7}{12}c. \tag{3.14ii}$$

If the former case, (3.14)(i), holds, define $z \in E$ by reflecting w over $t = \widehat{t}/2$, that is,

$$z(t) = \begin{cases} w(\widehat{t} - t) & 0 \le t \le \widehat{t}, \\ 0 & t \ge \widehat{t}. \end{cases} \tag{3.15}$$

If the latter case, (3.14ii), holds, define $z \in E$ by $z(t) = w(t + \widehat{t})$. In future arguments, we assume for convenience that the latter case holds. Arguments for the former case are very similar.

By the discussion preceding Lemma 2.11, there exists a unique $\bar{s} > 0$ with the property that $\bar{s}z \in \mathcal{S}$. We will prove that, if one assumes d to be small enough, then $I(\bar{s}z) < c$. This is impossible, and Theorem 1.0 follows. Recall ε from (3.1), and define ε more precisely by

$$\varepsilon = \frac{(p-1)\beta^2}{60}. \tag{3.16}$$

Set

$$d = \frac{\varepsilon}{B^2} = \frac{(p-1)\beta^2}{60} \cdot \frac{(p-1)}{14c^+} = \frac{(p-1)^2\beta^2}{840c^+}. \tag{3.17}$$

Assume from now on that

$$p \le 2. \tag{3.18}$$

Then, as we have been assuming, $d \le 1$, using (2.8) and $c^+ \ge c$. The following estimate, which uses (2.8), will be useful later:

$$\varepsilon = \frac{(p-1)\beta^2}{60} \le \frac{(p-1)}{60} \cdot \frac{8}{3}c < \frac{(p-1)c}{22} \le \frac{c}{22} \le \frac{c^+}{22}. \tag{3.19}$$

We will show that $|I'(z)z| < 3\varepsilon$, while $I(z) < 2c/3$. This will imply that the function $g(s) = I(sz)$ has a maximum for $s \ge 0$ that is less than c, which is impossible. We estimate $I'(z)z$ by comparing the integral for $I'(z)z$ to that for $I'(w)w$ in (3.13ii) and by (3.13ii) and (V_3)

$$
\begin{aligned}
(3.20)\qquad |I'(z)z| &= \left| \int_0^\infty |z'(t)|^2 + |z(t)|^2 - h(t)V'(z(t)) \cdot z(t)\, dt \right| \\
&= \left| \int_0^\infty |w'(t+\widehat{t})|^2 + |w(t+\widehat{t})|^2 - h(t)V'(w(t+\widehat{t})) \cdot w(t+\widehat{t})\, dt \right| \\
&= \left| \int_{\widehat{t}}^\infty |w'(t)|^2 + |w(t)|^2 - h(t-\widehat{t})V'(w(t)) \cdot w(t)\, dt \right| \\
&= \left| \int_{\widehat{t}}^\infty |w'(t)|^2 + |w(t)|^2 - h(t)V'(w(t)) \cdot w(t)\, dt \right| \\
&\quad + \left| \int_{\widehat{t}}^\infty (h(t) - h(t-\widehat{t}))V'(w(t)) \cdot w(t)\, dt \right| \\
&= \varepsilon + d \int_{\widehat{t}}^\infty V'(w(t)) \cdot w(t) \le \varepsilon + d \int_0^\infty V'(w(t)) \cdot w(t) \\
&= \varepsilon + d(\|w\|^2 - I'(w)w) \\
&\le \varepsilon + d(B^2 + \varepsilon) \le 2\varepsilon + dB^2 \le 3\varepsilon.
\end{aligned}
$$

In the last line we use (2.5) ($d \le 1$), and Lemma 2.22 with (3.3), (3.4) and (3.19).

Now we estimate $I(z)$ by comparing the integral for $I(z)$ to that for $I(w)$; we assume case (3.14ii) holds, so z equals w translated $\widehat{t}$ units to the left. Recall that w satisfies (3.2)–(3.4). By (3.3) and (2.1) we have

$$
\begin{aligned}
(3.21)\qquad I(z) &= \int_0^\infty \frac{1}{2}|z'(t)|^2 + \frac{1}{2}|z(t)|^2 - h(t)V(z(t))\, dt \\
&= \int_0^\infty \frac{1}{2}|w'(t+\widehat{t})|^2 + \frac{1}{2}|w(t+\widehat{t})|^2 - h(t)V(w(t+\widehat{t}))\, dt \\
&= \int_{\widehat{t}}^\infty \frac{1}{2}|w'(t)|^2 + \frac{1}{2}|w(t)|^2 - h(t-\widehat{t})V(w(t))\, dt \\
&= \int_{\widehat{t}}^\infty \frac{1}{2}|w'(t)|^2 + \frac{1}{2}|w(t)|^2 - h(t)V(w(t))\, dt \\
&\quad + \int_{\widehat{t}}^\infty (h(t) - h(t-\widehat{t}))V(w(t))\, dt \\
&< \frac{7}{12}c + d \int_{\widehat{t}}^\infty V(w(t))\, dt \le \frac{7}{12}c + d \int_{\widehat{t}}^\infty V'(w(t)) \cdot w(t)\, dt \\
&\le \frac{7}{12}c + d(B^2 + \varepsilon) < \frac{7}{12}c + 2\varepsilon < \frac{2}{3}c.
\end{aligned}
$$

In the last line, we estimate the last integral using the calculation at the end of (3.20), and also use (3.16), $d \le 1$, and (3.19).

We have $z \in E$ with $I(z) < 2c/3$ and $|I'(z)z| < 3\varepsilon$. By choice of the cutting point $\widehat{t}$ between t_0 and t_1 (3.7), and the definition of z as a reflection or translation of w (see (3.15) and the remark following it), $\|z\|_{L^\infty(\mathbb{R}^+)} \ge |z(0)| = \beta$, so $\|z\| \geq \beta$. Defining $g(s) = I(sz)$ as in Lemma 2.11, $g(1) = I(z) < 2c/3$ and $|g'(1)| = |I'(z)z| < 3\varepsilon$. We will show that $g'(5/4) < 0$ and $g'(3/4) > 0$. Therefore there exists $\bar{s} \in (3/4, 5/4)$ with $g'(\bar{s}) = I'(\bar{s}z)z = 0$, so $\bar{s}z \in \mathcal{S}$. Then we prove that for all $s \in [3/4, 5/4]$, $g(s) < c$. This contradicts the fact that $I(\bar{s}z) \geq c$, proving Theorem 1.0. By Lemma 2.11(i), since $p \in (1, 2]$ and $\|z\| > \beta$,

$$g'(5/4) \leq g'(1) - ((p-1)/4)(\|z\|^2/4) \leq 3\varepsilon - (p-1)\beta^2/16 < 0 \tag{3.22}$$

using the definition (3.16) of ε. Similarly,

$$g'(3/4) \geq g'(1) + ((p-1)/4)(\|z\|^2/4) \geq -3\varepsilon + (p-1)\beta^2/16 > 0. \tag{3.23}$$

$|g'(1)| < 3\varepsilon$, so for $s \in [1, 5/4]$, Lemma 2.11(i) gives

$$g'(s) \leq g'(1)s^p - (p-1)(s-1)\|z\|^2/2 \leq g'(1)s^p < 3\varepsilon s^p < 3\varepsilon(5/4)^2 < 5\varepsilon, \tag{3.24}$$

and

$$g(s) = g(1) + \int_1^s g'(r)\,dr < 2c/3 + 5\varepsilon(s-1) < 2c/3 + 2\varepsilon < 2c/3 + c/11 < c \tag{3.25}$$

(see (3.19)). For $s \in [3/4, 1]$, Lemma 2.11(ii) gives,

$$\begin{aligned} g'(s) &\geq g'(1)s^p + (p-1)(s-1)\|z\|^2/4 \\ &\geq g'(1)s^p > -3\varepsilon s^p < -3\varepsilon(1)^2 = -3\varepsilon, \end{aligned} \tag{3.26}$$

so, by (3.19),

$$g(s) = g(1) - \int_s^1 g'(r)\,dr < 2c/3 + 3\varepsilon(1-s) < 2c/3 + \varepsilon < 2c/3 + c/22 < c. \tag{3.27}$$

Therefore $g(s) = I(sz) < c$ for all $s \in [3/4, 5/4]$. This is impossible because $\bar{s}z \in \mathcal{S}$ for some $\bar{s} \in [3/4, 5/4]$. The assumption made at the beginning of this section is false. Theorem 1.0 is proven.

4. Determining d — an example

Here we find how to write d, satisfying Theorem 1.0, compactly in terms of β, p, and V. Then we find d for a specific function V.

To compute d using (3.17) we must estimate c^+ as defined in (2.21). Let us find a way to estimate c^+ for *any* V satisfying (V_1)–(V_3) and write it compactly. Recall I^+, Γ^+, and c^+ from (2.19)–(2.21). To define c^+, it suffices to find one element γ of Γ^+ and choose c^+ large enough to guarantee that $c^+ \geq \max_{\theta>0} I^+(g(\theta))$. Define β as in (2.4). Let $\vec{e}_1$ denote the unit vector $[1\ 0 \ldots 0]^T \in \mathbb{R}^n$, and define $w : \mathbb{R}^+ \to \mathbb{R}$ by

$$w(t) = \beta e^{-t}\vec{e}_1. \tag{4.0}$$

A direct calculation yields $\|w\| = \beta$. Since $\|w\|_{L^\infty(\mathbb{R}^+)} = \beta$, ${I^+}'(sw)(w) > 0$ for all $s \in (0,1]$, by (3.6). Thus $I(sw) < I(w)$ for all $s \in (0,1)$. By (2.2),

$$I^+(sw) = \frac{1}{2}s^2\|w\|^2 - \int_0^\infty V(sw)\,dt \le \frac{1}{2}s^2\beta^2 - s^{p+1}\int_0^\infty V(w)\,dt \tag{4.1}$$

for all $s > 1$. $V(r\vec{e}_1)$ is increasing for positive r, so

$$\begin{aligned}\int_0^\infty V(w)\,dt &\ge \int_0^{\ln 2} V(w)\,dt = \int_0^{\ln 2} V(\beta e^{-s}\vec{e}_1)\,ds \\ &> \int_0^{\ln 2} V\Big(\frac{\beta\vec{e}_1}{2}\Big)\,dt = (\ln 2)V\Big(\frac{\beta\vec{e}_1}{2}\Big) > \frac{1}{2}V\Big(\frac{\beta\vec{e}_1}{2}\Big).\end{aligned} \tag{4.2}$$

Therefore

$$I^+(sw) \le \alpha(s) \equiv \frac{1}{2}s^2\Big[\beta^2 - \Big(\frac{1}{2}F\Big(\frac{\beta}{2}\Big)\Big)s^{p-1}\Big] \tag{4.3}$$

for $s > 1$. By elementary calculus, $\alpha(s)$ achieves a maximum over $\{s > 0\}$ of

$$\frac{\beta^2}{2}\Big(\frac{p-1}{p+1}\Big)\Big(\frac{4\beta^2}{(p+1)V(\beta/2)}\Big)^{2/(p-1)} \le \frac{\beta^2}{6}\Big(\frac{2\beta^2}{V(\beta/2)}\Big)^{2/(p-1)}. \tag{4.4}$$

The last expression is an upper bound for c^+. Using (3.17), d can be estimated by

$$\begin{aligned}\frac{(p-1)^2\beta^2}{840c^+} &\ge \frac{(p-1)^2\beta^2}{840}\cdot\frac{6}{\beta^2}\cdot\Big(\frac{V(\beta\vec{e}_1/2)}{2\beta^2}\Big)^{2/(p-1)} \\ &= \frac{(p-1)^2}{140}\Big(\frac{V(\beta\vec{e}_1/2)}{2\beta^2}\Big)^{2/(p-1)} \ge d.\end{aligned} \tag{4.5}$$

Let us compute d for the specific case $n = 1$, $1 < p \le 2$, $V(q) = |q|^{p+1}/(p+1)$. We can pick $\beta = (1/8)^{1/(p-1)}$, because

$$V'(q)\cdot q = |q|^{p+1} = |q|^{p-1}|q|^2 \le \beta|q|^2 \tag{4.6}$$

for $|q| \le \beta$. Now,

$$V\Big(\frac{\beta}{2}\Big) = \frac{1}{p+1}\Big(\frac{1}{8}\Big)^{(p+1)/(p-1)} \ge \frac{1}{3}\Big(\frac{1}{8}\Big)^{3/(p-1)}, \tag{4.7}$$

so, using (4.5), d can be estimated by

$$\begin{aligned}\frac{(p-1)^2}{140}\Big(\frac{V(\beta/2)}{2\beta^2}\Big)^{2/(p-1)} &\ge \frac{(p-1)^2}{140}\Big(\frac{1}{6\cdot 8^{3/(p-1)}\cdot 8^{2/(p-1)}}\Big)^{2/(p-1)} \\ &> \frac{(p-1)^2}{140}\Big(\frac{1}{8}\Big)^{(p+4)/(p-1)\cdot(2/(p-1)} \\ &\ge \frac{(p-1)^2}{140}\Big(\frac{1}{8}\Big)^{12/(p-1)^2} \ge d.\end{aligned}$$

Appendix

This brief appendix contains two well-known Sobolev inequalities, along with embedding constants.

Lemma 1. *If* $u \in W^{1,2}([0,\infty);\mathbb{R}^n)$ *then*

(i) $$\|u\|_{L^\infty([0,\infty))} \le \|u\|_{W^{1,2}([0,\infty);\mathbb{R}^n)}.$$

If $a \ge 1$ *and* $u \in W^{1,2}([0,a])$, *then*

(ii) $$\|u\|_{L^\infty([0,a])} \le 2\|u\|_{W^{1,2}([0,\infty);\mathbb{R}^n)}.$$

Proof of (i). Let $u \in W^{1,2}([0,\infty);\mathbb{R}^n)$ and $x_1 \in [0,\infty)$. Let $\varepsilon > 0$. Choose $x_0 \in [0,\infty)$ with $|u(x_0)| < \varepsilon$. Then

$$\begin{aligned} |u(x_1)|^2 &= |u(x_0)|^2 + (|u(x_1)|^2 - |u(x_0)|^2) \\ &< \varepsilon^2 + \left| \int_{x_0}^{x_1} \frac{d}{dx}|u|^2\,dx \right| = \varepsilon^2 + \left| \int_{x_0}^{x_1} 2u \cdot u'\,dx \right| \\ &\le \varepsilon^2 + \left| \int_{x_0}^{x_1} |u'|^2 + |u|^2\,dx \right| \le \varepsilon^2 + \|u\|^2_{W^{1,2}([0,\infty);\mathbb{R}^n)} \end{aligned}$$

via the Cauchy–Schwarz inequality. So $|u(x_1)| \le \|u\|_{W^{1,2}([0,\infty);\mathbb{R}^n)}$ when ε go to zero. Since x_1 is arbitrary, (i) is proven. □

Proof of (ii). Let $a \ge 1$ and $u \in W^{1,2}([0,a])$. Assume $\|u\|_{L^\infty([0,a])} \ge 1$. We will show that $\|u\|_{W^{1,2}([0,\infty);\mathbb{R}^n)} \ge 1/2$.

If $|u(x)| > 1/2$ for all $x \in [0,a]$, then $\|u\|^2_{W^{1,2}([0,\infty))} \ge \int_0^a u^2 > a/4 \ge 1/4$. So suppose $|u(x_0)| \le 1/2$ for some $x_0 \in [0,a]$. Let $x_1 \in [0,a]$ with $|u(x_1)| \ge 1$. Arguing as in part (i) above,

$$\begin{aligned} 1 \le |u(x_1)|^2 &= |u(x_0)|^2 + (|u(x_1)|^2 - |u(x_0)|^2) \\ &< \frac14 + \left| \int_{x_0}^{x_1} \frac{d}{dx}u^2\,dx \right| = \frac14 + \left| \int_{x_0}^{x_1} 2uu'\,dx \right| \\ &\le \frac14 + \left| \int_{x_0}^{x_1} (u')^2 + u^2\,dx \right| \le \frac14 + \|u\|^2_{W^{1,2}([0,1])}. \end{aligned}$$

Therefore $\|u\|^2_{W^{1,2}([0,1])} \ge 3/4 > 1/4$. Part (ii) is proven. □

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Manuscript received January 14, 1997

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