Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 9, 1997, 297–312

SOME QUALITATIVE PROPERTIES OF THE SOLUTIONS OF AN ELLIPTIC EQUATION VIA MORSE THEORY

Giuseppina Vannella

1. Introduction

We consider the following problem:

$$(P)\qquad \begin{cases} u \in C^2(\overline{\Omega}), \\ -\Delta u + \dfrac{1}{\varepsilon^2} F'(u(x)) = 0 \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{in } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain with sufficiently regular boundary ($n \geq 3$), $\varepsilon > 0$ is a real number and $F \in C^2(\mathbb{R})$ is a real function which satisfies the following assumptions:

(i) F is even,
(ii) 0 is a local maximum for F, with $F(0) = 1$ and $F''(0) < 0$,
(iii) $F(\mathbb{R}) \subset \mathbb{R}^+$ and F vanishes at (and only at) 1 and -1,
(iv) $\exists a > 0\ \forall t \geq 1$, $F''(t) \geq a$,
(v) $\exists p \in \,]2, 2^*[\ \exists b, c \geq 0\ \forall t \in \mathbb{R}$, $|F''(t)| \leq b|t|^{p-2} + c$ (here $2^* = 2n/(n-2)$).

Let us remark that from (iii) it follows that $F'(1) = F'(-1) = 0$. Moreover, since F'' is even and continuous, we see from (ii) and (iii) that

$$\exists \beta \in \,]0,1[\ \forall t \in \,]-\beta, \beta[, \quad F''(t) < 0 \quad \text{and} \quad F''(-\beta) = F''(\beta) = 0.$$

1991 *Mathematics Subject Classification.* Primary 35J65; Secondary 58E05.

An example of a function F satisfying (i)–(v) is

$$F(t) = \frac{2}{\alpha}|t|^{2+\alpha} - \left(1 + \frac{2}{\alpha}\right)t^2 + 1$$

where $0 < \alpha < 2^* - 2 = 4/(n-2)$.

Problems like (P), with small ε, are used to model some phase transition problems.

We shall prove (see Property 2.5) that the solutions of (P) correspond to critical points of the functional defined on the Sobolev space $H^1(\Omega)$ by

$$E_\varepsilon(u) = \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon}\int_\Omega F(u(x))\,dx.$$

Some related results are due to Modica and Mortola ([11], [12]) who studied the asymptotic behaviour of minimum points of functionals of type E_ε as $\varepsilon \to 0$.

Let us also mention the work of Passaseo ([14]) who first considered the existence of critical points of such functionals which are not minimum points.

In this paper we shall prove the existence of critical points of E_ε using Morse theory. Since they may a priori be degenerate, standard Morse theory cannot be applied directly and we shall use its generalization recently developed by Benci and Giannoni in [2]. Apart from giving multiplicity results, the use of Morse theory allows us to find estimates on the size of the "phase transition zone". More precisely, if u is a critical point of E_ε and $\alpha \in]0,\beta[$, the "phase transition zone" of u is defined by

$$\Gamma_\alpha(u) = \{x \in \Omega \mid -\alpha < u(x) < \alpha\},$$

i.e. the part of Ω on which u assumes values near zero.

Roughly speaking, we prove that for any fixed $\alpha < \beta$ and $j \in \mathbb{N}$, as $\varepsilon \to 0$, all critical points u of E_ε whose restricted Morse index is less than j (see Definition 2.1) tend to concentrate their values outside $]-\alpha,\alpha[$. More precisely, we have the following result:

Theorem 1.1. *For fixed $\alpha \in]0,\beta[$, let ε, l be positive numbers, u a critical point of E_ε having restricted Morse index $m(u)$, and $N(u,l)$ the greatest number of disjoint open n-dimensional hypercubes of side l which can be contained in $\Gamma_\alpha(u)$. There exists a nonincreasing infinitesimal sequence $(k_j)_{j\geq 1}$ such that, if $\varepsilon < lk_j$, then $N(u,l) \leq m(u)/j$. Moreover, for fixed $j \in \mathbb{N}$ and $l > 0$, there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ and any critical point u of E_ε satisfying $m(u) \leq j$, we have $N(u,l) = 0$.*

Theorem 1.1 is a restatement of Theorem 4.3 and Corollary 4.6. Its proof is given at the end of the last section.

The article is organized as follows. In Section 2 we derive some properties of the functional E_ε. In Section 3 we give a short presentation of the generalized Morse theory and we show how our functional relates to it. In Section 4 we state our main results. Finally, we give the proof of these results in Section 5.

2. Preliminaries

Since F satisfies assumption (v), standard computations show that E_ε is C^2 on $H^1(\Omega)$. Moreover, for each $u, v, w \in H^1(\Omega)$,

$$dE_\varepsilon(u)(v) = \varepsilon \int_\Omega (\nabla u \mid \nabla v) + \frac{1}{\varepsilon} \int_\Omega F'(u(x))v(x)\,dx$$

and

$$d^2E_\varepsilon(u)(v, w) = \varepsilon \int_\Omega (\nabla v \mid \nabla w) + \frac{1}{\varepsilon} \int_\Omega F''(u(x))v(x)w(x)\,dx. \tag{1}$$

Definition 2.1. We recall that $u \in H^1(\Omega)$ is a *critical point* for E_ε if $dE_\varepsilon(u) = 0$. We denote by K_{E_ε} the set of these points.

Moreover, we say that E_ε *satisfies the Palais–Smale condition* (or briefly, that E_ε satisfies P.S.) if any sequence $(u_n)_{n\in\mathbb{N}}$ such that

$$E_\varepsilon(u_n) \underset{n}{\rightarrow} c \in \mathbb{R} \quad \text{and} \quad dE_\varepsilon(u_n) \underset{n}{\rightarrow} 0$$

has a subsequence which converges in $H^1(\Omega)$.

If $u \in K_{E_\varepsilon}$, the *restricted Morse index* of u is the maximal dimension of a subspace of $H^1(\Omega)$ on which $d^2E_\varepsilon(u)$ is negative definite; it is denoted by $m(u)$.

The *nullity* of u is the dimension of the kernel of $d^2E_\varepsilon(u)$ (i.e. the subspace consisting of all v such that $d^2E_\varepsilon(u)(v, w) = 0$ for all $w \in H^1(\Omega)$).

The *large Morse index* is the sum of the restricted Morse index and the nullity; it is denoted by $m^*(u)$.

A critical point u is called *nondegenerate* if its nullity is 0, while in the other case it is called *degenerate*.

Example 2.2. Let u_0 be the function constantly equal to 0. Since $\nabla u_0 = 0$ and $F'(0) = 0$ we have

$$dE_\varepsilon(u_0)(v) = \varepsilon \int_\Omega (\nabla u_0 \mid \nabla v) + \frac{1}{\varepsilon} \int_\Omega F'(0)v(x)\,dx = 0 \quad \text{for all } v \in H^1(\Omega)$$

and so u_0 is a critical point of E_ε. Two other examples of critical points are given by u_1 and u_{-1}, the functions constantly equal to 1 and to -1 respectively, since $dE_\varepsilon(u_1) = dE_\varepsilon(u_{-1}) = 0$. Concerning the Morse index of these critical points, as by (iv) $F''(\pm 1) > 0$, we see that

$$m^*(u_1) = m^*(u_{-1}) = 0.$$

Thus u_1 and u_{-1} are nondegenerate critical points.

In Lemma 4.1 we will compute the large and restricted Morse index of u_0.

REMARK 2.3. Since E_ε is an even functional, dE_ε is odd and d^2E_ε is even. Thus if $u \in H^1(\Omega)$ is a critical point of E_ε different from u_0, then also $-u$ is a critical point and moreover $d^2E_\varepsilon(u) = d^2E_\varepsilon(-u)$. So u and $-u$ have the same restricted and large Morse index.

Next we shall derive some properties of the critical points of E_ε.

The first one follows from the maximum principle. The proof we give is closely related to the one given in [5].

PROPERTY 2.4. *If u is a critical point of E_ε then $|u| \le 1$ a.e. in Ω.*

PROOF. Let $G \in C^1(\mathbb{R}, \mathbb{R})$ be a function such that

$$\forall t \le 0,\ G(t) = 0 \quad \text{and} \quad \forall t > 0,\ 0 < G'(t) \le M,$$

where $M > 0$ is a real number.

Let $u \in H^1(\Omega)$ be a critical point of E_ε. Then $v(x) = G(u(x) - 1) \in H^1(\Omega)$ and setting

$$\Omega_1 = \{x \in \Omega \mid u(x) > 1\}$$

we see that

$$\forall x \in \Omega \setminus \Omega_1, \quad G(u(x) - 1) = 0 \quad \text{and} \quad G'(u(x) - 1) = 0.$$

Moreover, by assumption (iv) and since $F'(1) = 0$,

$$\forall x \in \Omega_1, \quad F'(u(x)) \ge a(u(x) - 1),$$

so

$$\begin{aligned} 0 = dE_\varepsilon(u)(v) &= \varepsilon \int_\Omega G'(u-1)|\nabla u|^2 + \frac{1}{\varepsilon} \int_\Omega F'(u(x))G(u(x) - 1)\,dx \\ &= \varepsilon \int_{\Omega_1} G'(u-1)|\nabla u|^2 + \frac{1}{\varepsilon} \int_{\Omega_1} F'(u(x))G(u(x) - 1)\,dx \\ &\ge \varepsilon \int_{\Omega_1} G'(u-1)|\nabla u|^2 + \frac{1}{\varepsilon} \int_{\Omega_1} a(u(x) - 1)G(u(x) - 1)\,dx \\ &= \varepsilon \int_{\Omega} G'(u-1)|\nabla u|^2 + \frac{1}{\varepsilon} \int_{\Omega} a(u(x) - 1)G(u(x) - 1)\,dx. \end{aligned}$$

Therefore

$$\frac{\varepsilon^2}{a} \int_\Omega G'(u-1)|\nabla u|^2 \le -\int_\Omega (u(x) - 1)G(u(x) - 1)\,dx.$$

In this inequality the left term is positive and the right one negative, so both terms have to vanish and in particular

$$\int_\Omega (u(x) - 1)G(u(x) - 1)\,dx = 0.$$

This shows that $u \le 1$ a.e. in Ω, since $tG(t) \ge 0$ for all $t \in \mathbb{R}$ and $tG(t) = 0 \Leftrightarrow t \le 0$.

On the other hand, by Remark 2.3, $-u$ is also a critical point, so we have $-u \le 1$ a.e. in Ω and we finally get $|u| \le 1$ a.e. in Ω. $\square$

The second property insures the regularity of the critical points of E_ε.

Property 2.5. *If $u \in H^1(\Omega)$ is a critical point of E_ε, then*

$$\begin{cases} u \in C^2(\overline{\Omega}), \\ -\Delta u + \dfrac{1}{\varepsilon^2} F'(u(x)) = 0 \quad \textit{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \quad \textit{in } \partial\Omega. \end{cases}$$

Proof. First note that by standard regularity theory we find, using the previous property, that $u \in H^2(\Omega)$. Then taking $v \in C_0^\infty(\Omega) \subset H^1(\Omega)$ we have

$$\begin{aligned} 0 = dE_\varepsilon(u)(v) &= \varepsilon \int_\Omega (\nabla u \mid \nabla v) + \frac{1}{\varepsilon}\int_\Omega F'(u(x))v(x)\,dx \\ &= \int_\Omega \Big(-\varepsilon\Delta u + \frac{1}{\varepsilon}F'(u(x))\Big)v(x)\,dx \end{aligned}$$

and so

$$-\varepsilon\Delta u + \frac{1}{\varepsilon}F'(u(x)) = 0 \quad \text{a.e. in } \Omega.$$

Next we know that

- F' is a C^1 function,
- $|u| \le 1$ a.e. in Ω,
- $\Delta u = \frac{1}{\varepsilon^2}F'(u(x))$ a.e. in Ω,
- $\partial\Omega$ is sufficiently regular.

So using regularity results as in [4, 1.5B], we deduce that $u \in C^2(\overline{\Omega})$. Finally, for all $v \in C^1(\overline{\Omega}) \cap H^1(\Omega) \supset C^2(\overline{\Omega})$,

$$\begin{aligned} \int_{\partial\Omega} \frac{\partial u}{\partial n} v\,d\sigma &= \int_\Omega (\nabla u \mid \nabla v) + \int_\Omega \Delta u v \\ &= \int_\Omega (\nabla u \mid \nabla v) + \frac{1}{\varepsilon^2}\int_\Omega F'(u(x))v(x)\,dx \\ &\quad - \frac{1}{\varepsilon^2}\int_\Omega F'(u(x))v(x)\,dx + \int_\Omega \Delta u v \\ &= \frac{1}{\varepsilon} dE_\varepsilon(u)(v) - \int_\Omega \Big(-\Delta u + \frac{1}{\varepsilon^2}F'(u(x))\Big)v(x)\,dx = 0 \end{aligned}$$

and so $\partial u/\partial n = 0$ in $\partial\Omega$. $\square$

Property 2.6. *E_ε satisfies P.S.*

Proof. Let $(u_n) \subset H^1(\Omega)$ be a sequence such that $E_\varepsilon(u_n) \to c$ and $dE_\varepsilon(u_n) \to 0$. We need to find a subsequence of (u_n) which converges in $H^1(\Omega)$.

Clearly E_ε is coercive. Thus (u_n) is bounded and there exists a subsequence (u_{k_n}) of (u_n) which weakly converges to an element $\overline{u} \in H^1(\Omega)$.

Now considering the function $g \in C^1(\mathbb{R},\mathbb{R})$ given by

$$g(t) = \frac{1}{\varepsilon}F'(t) - \varepsilon t,$$

we know by assumption (v) that

$$\exists b_1, c_1 \geq 0, \quad |g(t)| \leq b_1|t|^{p-1} + c_1 \tag{2}$$

where $p < 2^* = 2n/(n-2)$.

Denoting by $H^{-1}(\Omega)$ the dual space of $H^1(\Omega)$, by $\Phi : H^1(\Omega) \to H^{-1}(\Omega)$ the Nemytskiĭ operator relative to g, i.e.

$$\Phi(u)(v) = \int_\Omega g(u(x))v(x)\,dx,$$

and by $\mathcal{L} : H^1(\Omega) \to H^{-1}(\Omega)$ the Riesz isomorphism, i.e.

$$\mathcal{L}(u)(v) = (u,v)_{H^1(\Omega)} = \int_\Omega (\nabla u \mid \nabla v) + \int_\Omega u(x)v(x)\,dx,$$

we have $dE_\varepsilon = \varepsilon\mathcal{L} + \Phi$ and so

$$\lim_{n\to\infty}[\varepsilon\mathcal{L}(u_n) + \Phi(u_n)] = \lim_{n\to\infty} dE_\varepsilon(u_n) = 0.$$

Moreover, as (2) holds, by Carathéodory's theorem Φ is completely continuous, i.e. if (v_n) weakly converges to v_0, then $\Phi(v_n)$ strongly converges to $\Phi(v_0)$, thus $\lim_{n\to\infty}\Phi(u_{k_n}) = \Phi(\overline{u})$. Therefore $\mathcal{L}(u_{k_n})$ converges in $H^{-1}(\Omega)$ and, as $\mathcal{L}$ is an isomorphism, (u_{k_n}) strongly converges in $H^1(\Omega)$. □

PROPERTY 2.7. *K_{E_ε} is a compact set.*

PROOF. Let us first show that K_{E_ε} is a bounded subset of $H^1(\Omega)$. If u is a critical point of E_ε we have

$$\varepsilon\int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon}\int_\Omega F'(u(x))u(x)\,dx = dE_\varepsilon(u)(u) = 0.$$

The second term is uniformly bounded with respect to u by Property 2.4 and consequently so is the first one. Still using Property 2.4, we see that also $\int_\Omega |u(x)|^2\,dx$ is uniformly bounded and so

$$\exists M > 0\ \forall u \in K_{E_\varepsilon}, \quad \|u\|_{H^1(\Omega)} \leq M.$$

Now since E_ε satisfies P.S., this shows that K_{E_ε} is compact. □

3. Classical and generalized Morse theory

We recall that classical Morse theory deals with functionals

(a) of class C^1 and twice differentiable in a neighbourhood of their critical points,
(b) whose critical points are not degenerate,
(c) which satisfy P.S.

In the whole section let V be a Hilbert manifold and A an open subset of V. We will denote by $\mathcal{M}(\overline{A})$ the set of functionals on A which satisfy (a)–(c) and can be extended to a function of class C^1 in a neighbourhood of $\overline{A}$.

Now let f belong to $\mathcal{M}(\overline{A})$ and K_f be the set of critical points of f. The *Morse polynomial* of a subset K of K_f is defined by

$$m_\lambda(K, f) = \sum_{x \in K} \lambda^{m(x)}$$

with the convention that $\lambda^\infty = 0$. Thus $m(\lambda)$ is a polynomial $\sum_k a_k \lambda^k$ whose coefficients a_k are integers representing the number of critical points in K having Morse index k.

If $f \in \mathcal{M}(\overline{A})$ is bounded from below, then from classical Morse theory we have the *Morse relation*

$$m_\lambda(K_f) = P_\lambda(A) + (1 + \lambda) Q_\lambda$$

where Q_λ is a formal series in λ with coefficients in $\mathbb{N} \cup \{+\infty\}$, while $P_\lambda(A)$ is the Poincaré polynomial of A:

$$P_\lambda(A) = \sum_{q \in \mathbb{N}} \dim H_q(A, \mathbb{Z}_2) \lambda^q$$

and $H_q(A, \mathbb{Z}_2)$ are the absolute homology groups of A with $\mathbb{Z}_2$ as field of coefficients.

In our case E_ε is a $C^2(H^1(\Omega))$ functional which satisfies P.S., but it could have degenerate critical points and thus could not belong to $\mathcal{M}(H^1(\Omega))$, so it is not possible to apply the classical Morse theory to this functional. Recently, however, Benci and Giannoni [2] have introduced a generalized Morse theory. In the larger class it deals with, the Morse relation still holds if we replace the Morse polynomial with the Morse index (see Definition 3.2). We will see that E_ε belongs to this class.

The family $\mathcal{S}$. We denote by $\mathcal{S}$ the family of formal series in one variable λ with coefficients in $\mathbb{N} \cup \{+\infty\}$.

For each $k \in \mathbb{N}$ and $P \in \mathcal{S}$ we denote by $c_k : \mathcal{S} \to \mathbb{N} \cup \{+\infty\}$ the function which sends every polynomial P into its kth coefficient, so that

$$P = \sum_{k \in \mathbb{N}} c_k(P) \lambda^k.$$

On $\mathcal{S}$ a relation of total order and a notion of limit are defined in the following way:

(o) $P < Q \Leftrightarrow \exists k_0 \in \mathbb{N}\ \forall k < k_0,\ c_k(P) = c_k(Q)$ and $c_{k_0}(P) < c_{k_0}(Q)$;
(l) $Q = \lim_{n \to \infty} P_n \Leftrightarrow \forall k \in \mathbb{N}$, $\lim_{n \to \infty} c_k(P_n) = c_k(Q)$.

Under the topology induced by (l), $\mathcal{S}$ is a compact set.

If A is a subset of $\mathcal{S}$, let $\overline{A}$ denote the closure of A. We have

$$P \in \overline{A} \Leftrightarrow \exists (P_n)_{n\in\mathbb{N}} \subset A, \quad \lim_{n\to\infty} P_n = P.$$

Finally, we put $\inf A = \min \overline{A}$ and $\sup A = \max \overline{A}$. It can be proved that for any set $A \subset \mathcal{S}$, $\inf A$ and $\sup A$ exist and are unique.

The class $\mathcal{F}(\overline{A})$ and the Morse index. Now we will describe a class $\mathcal{F}(\overline{A})$ of C^1 functionals where the generalized Morse index will be defined.

For any $\varepsilon > 0$ and $B \subset V$ we put

$$N_\varepsilon(B) = \{x \in V \mid d(x, B) < \varepsilon\}$$

where d is the distance on the Hilbert space V. For an open subset A of V and $f \in C^1(\overline{A})$ we set

$$\mathcal{M}_f^\varepsilon(\overline{A}) = \{g \in \mathcal{M}(\overline{A}) \mid g(x) = f(x)\ \forall x \notin N_\varepsilon(K_f) \cap \overline{A}\}.$$

The set $\mathcal{F}(\overline{A})$ is defined by

$$\mathcal{F}(\overline{A}) = \{f \in C^1(\overline{A}) \mid \mathcal{M}_f^\varepsilon(\overline{A}) \neq \emptyset\ \forall \varepsilon > 0\}.$$

Definition 3.1. Let $f \in C^1(\overline{A})$. A compact set $K \subset K_f$ is called *isolated* if there exists an open set ω such that $K = K_f \cap \omega$. The set ω is called an *isolating set* for K.

Finally, we introduce the definition of Morse index for an isolated critical set.

Definition 3.2. Let K be an isolated critical set of f, and let ω be an isolating set for K. The *Morse index* of K is the formal series

$$i_\lambda(K, f) = i_\lambda(K, f, \omega) = \sup_{\varepsilon>0}\Big(\inf_{g\in\mathcal{M}_f^\varepsilon(\overline{\omega})} m_\lambda(K_g, g)\Big).$$

It is easy to see that the index of an isolated critical set does not depend on the isolating set ω.

Definition 3.3. Let $x \in K_f \cap \overline{A}$ be an isolated critical point. The *multiplicity* of x is the integer $i_1(\{x\}, f)$. Analogously one can define the multiplicity of an isolated critical set K, denoted by $i_1(K, f)$.

It is easy to see that any nondegenerate critical point is isolated and its multiplicity is one.

We will use the following results about the Morse index.

Proposition 3.4. *Let f be a functional belonging to $\mathcal{F}(\overline{A})$:*

(i) *if x_0 is a nondegenerate critical point of f, then*

$$i_\lambda(\{x_0\}, f) = \lambda^{m(x_0)},$$

(ii) *if* $K_1, K_2 \subset K_f$ *are isolated compact sets and* $K_1 \cap K_2 = \emptyset$*, then*

$$i_\lambda(K_1 \cup K_2, f) = i_\lambda(K_1, f) + i_\lambda(K_2, f),$$

(iii) *if* f *is bounded from below, then*

$$i_\lambda(K_f, f) = P_\lambda(\overline{A}) + (1+\lambda)Q_\lambda \tag{3}$$

where $Q_\lambda \in \mathcal{S}$.

A proof of this proposition is given in [2] (Theorems 5.8–5.9).

Let us show that $E_\varepsilon \in \mathcal{F}(H^1(\Omega))$. First we note that in [2] (see Ex. 5.2) it is established that if $(\,,)_V$ denotes the inner product of a Hilbert space V, $\Psi \in C^1(V)$ is a function whose gradient is completely continuous, and $f(x) = (x,x)_V + \Psi(x)$ satisfies P.S. and is bounded in A, then $f \in \mathcal{F}(\overline{A})$.

Slightly modifying the proof we can replace the hypothesis that f is bounded by the assumption that K_f is compact.

Now E_ε can be written as

$$E_\varepsilon(u) = \frac{\varepsilon}{2}(u,u)_{H^1(\Omega)} + \Psi(u)$$

where Ψ' is completely continuous (see proof of Property 2.6). Moreover, E_ε satisfies P.S. and K_{E_ε} is compact by Property 2.6, respectively 2.7. Consequently, E_ε belongs to $\mathcal{F}(H^1(\Omega))$ and by (3), as E_ε is bounded from below, we have

$$i_\lambda(K_{E_\varepsilon}, E_\varepsilon) = P_\lambda(H^1(\Omega)) + (1+\lambda)Q_\lambda \tag{4}$$

where $Q_\lambda \in \mathcal{S}$.

Finally, we note for future reference that with the same hypotheses on f, we have

Proposition 3.5. *If* K *is an isolated critical set of* f *with*

$$i_\lambda(K, f) = \sum_{k \in \mathbb{N}} a_k \lambda^k$$

and if $a_{k_0} \neq 0$ *then*

$$\exists x \in K, \quad m(x) \le k_0 \le m^*(x).$$

This result is also obtained by slightly modifying a result of [2] (see Theorem 5.10).

4. Statements of the results

In this section we state the main results of this paper. The proofs will be given in the next section.

Let $0 = \lambda_1 \le \lambda_2 \le \ldots$ denote the eigenvalues of $-\Delta$ on Ω with Neumann boundary conditions, and for $k \ge 2$ let $\mu_k = \sqrt{-F''(0)/\lambda_k}$.

Lemma 4.1. *u_0 is a nondegenerate critical point of E_ε if and only if $\varepsilon \neq \mu_k$ for all $k \geq 2$. Furthermore, if $\varepsilon \in \left]\mu_{k+1}, \mu_k\right[$, then the restricted Morse index of u_0 is equal to k. This shows in particular that $m(u_0) \geq 1$ for all $\varepsilon > 0$.*

Now we will state an existence result which differs from [14] since it gives information on the Morse index of solutions.

Theorem 4.2. *If $\varepsilon \in \left]\mu_{k+1}, \mu_k\right[$, then there are at least $2k$ critical points of E_ε (counted with multiplicities) different from u_0 whose Morse index is less than k. Moreover, if the number of these critical points is finite, then for each $h < k$ there is a couple of critical points u and $-u$ such that $m(\pm u) \leq h \leq m^*(\pm u)$.*

Let $\alpha \in \left]0, \beta\right[$, $l > 0$, and let Q_l be an open n-dimensional hypercube of side l. Finally, let $0 < a_1(l) < a_2(l) \leq a_3(l) \leq \ldots$ be the eigenvalues of $-\Delta$ on Q_l with Dirichlet boundary conditions, namely:

$$\begin{aligned}
&a_1(l) = n\pi^2/l^2,\\
&a_2(l) = a_3(l) = \ldots = a_{n+1}(l) = (n+3)\pi^2/l^2,\\
&a_{n+2}(l) = a_{n+3}(l) = \ldots = a_{n+1+n(n-1)/2}(l) = (n+6)\pi^2/l^2,\\
&\ldots
\end{aligned}$$

Let $u \in K_{E_\varepsilon}$. We recall that the *phase transition zone* of u is the set

$$\Gamma_\alpha(u) = \{x \in \Omega \mid -\alpha < u(x) < \alpha\}.$$

We denote by $N(u, l)$ the greatest number of disjoint open hypercubes of side l which can be contained in $\Gamma_\alpha(u)$.

Setting

$$\delta(\alpha) = \min_{t \in [-\alpha, \alpha]} |F''(t)|,$$

we have

$$\forall t \in \left]-\alpha, \alpha\right[, \quad F''(t) \leq -\delta(\alpha). \tag{5}$$

Our next theorem shows how the Morse index of a solution u of (P) can be used to estimate from above the size of its phase transition zone. From now on we assume that $\alpha \in \left]0, \beta\right[$ is arbitrary but fixed.

Theorem 4.3. *Let $l > 0$ be fixed and u a critical point of E_ε having Morse index $m(u)$. Then for all $j \in \mathbb{N}$ such that*

$$\varepsilon < \sqrt{\delta(\alpha)/a_j(l)}$$

we have

$$N(u, l) \leq m(u)/j.$$

In particular, if $\varepsilon < \sqrt{\delta(\alpha)/a_1(l)}$, then $N(u, l) \leq m(u)$.

COROLLARY 4.4. *If $\varepsilon < \mu_2$, then there exists a couple of critical points $u_{1,\varepsilon}$ and $-u_{1,\varepsilon}$ such that*

$$m(\pm u_{1,\varepsilon}) \leq 1 \leq m^*(\pm u_{1,\varepsilon}).$$

Moreover, there are no hypercubes of side $l > \varepsilon\pi\sqrt{(n+3)/\delta(\alpha)}$ contained in $\Gamma_\alpha(u_{1,\varepsilon})$.

COROLLARY 4.5. *There exist two positive constants $\widetilde{a}$ and $\widetilde{c}$ depending only on F but independent of ε and u such that*

$$\forall \varepsilon > 0 \ \forall u \in K_{E_\varepsilon}, \ N(u, \widetilde{c}\varepsilon) \leq \mathcal{A}(\widetilde{a}/\varepsilon^2),$$

where $\mathcal{A}(\widetilde{a}/\varepsilon^2)$ denotes the number of eigenvalues of $-\Delta$ on Ω (with Neumann boundary conditions) which are less than $\widetilde{a}/\varepsilon^2$.

COROLLARY 4.6. *For every $j \in \mathbb{N}$ and $l > 0$ there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ and any critical point u of E_ε satisfying $m(u) \leq j$, we have $N(u, l) = 0$.*

5. Proofs of the results

PROOF OF LEMMA 4.1. Let $(e_n)_{n\in\mathbb{N}}$ be the hilbertian basis of $L^2(\Omega)$ made by the eigenfunctions relative to $(\lambda_n)_{n\in\mathbb{N}}$, i.e.

$$\begin{cases} e_n \in H^1(\Omega) \cap C^\infty(\Omega), \\ -\Delta e_n = \lambda_n e_n \quad \text{in } \Omega, \\ \partial e_n/\partial n = 0 \quad \text{in } \partial\Omega, \end{cases}$$

where $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ and $\lim_{n\to\infty} \lambda_n = +\infty$. If u_0 is degenerate, then

$$\exists v \in H^1(\Omega), v \neq 0 \ \forall w \in H^1(\Omega), \quad d^2E_\varepsilon(u_0)(v, w) = 0.$$

This means that

$$\forall w \in H^1(\Omega), \quad \varepsilon \int_\Omega (\nabla v \mid \nabla w) + \frac{F''(0)}{\varepsilon} \int_\Omega v(x)w(x)\,dx = 0,$$

and thus clearly v solves the problem

$$-\varepsilon\Delta v + \frac{F''(0)}{\varepsilon} v = 0 \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial n} = 0 \quad \text{in } \partial\Omega.$$

In other words, $-F''(0)/\varepsilon^2$ is an eigenvalue of $-\Delta$ on Ω with Neumann conditions, thus u_0 is nondegenerate if these eigenvalues are avoided. Since

$$-F''(0)/\varepsilon^2 = \lambda_k \Leftrightarrow F''(0) = 0 \text{ or } \varepsilon = \sqrt{-F''(0)/\lambda_k}$$

and $F''(0) \neq 0$, setting $\mu_k = \sqrt{-F''(0)/\lambda_k}$ (with $k \geq 2$), u_0 is nondegenerate if and only if $\varepsilon \neq \mu_k$.

Now let e_j be one of the previous eigenfunctions. Then

$$d^2E_\varepsilon(u_0)(e_j,e_j) = \varepsilon\int_\Omega |\nabla e_j|^2 + \frac{F''(0)}{\varepsilon}\int_\Omega e_j^2(x)\,dx$$
$$= \varepsilon\lambda_j + \frac{F''(0)}{\varepsilon} = \frac{1}{\varepsilon}(\varepsilon^2\lambda_j + F''(0)).$$

If $j=1$, then $\lambda_1 = 0$ and $d^2E_\varepsilon(u_0)(e_1,e_1) = F''(0)/\varepsilon < 0$, thus surely $m(u_0)\geq 1$.

If $j\geq 2$, then $\lambda_j > 0$ and

$$\varepsilon^2\lambda_j + F''(0) < 0 \Leftrightarrow \varepsilon < \sqrt{-F''(0)/\lambda_j} = \mu_j$$

so $\varepsilon < \mu_j \leq \mu_{j-1} \leq \ldots \leq \mu_2$ implies that $d^2E_\varepsilon(u_0)$ is negative definite on $\operatorname{span}\{e_1,\ldots,e_j\}$, and thus $m(u_0)\geq j$.

Moreover, if $\varepsilon > \mu_{j+1}$, then

$$\forall k > j, \quad d^2E_\varepsilon(u_0)(e_k,e_k) > 0,$$

i.e. $d^2E_\varepsilon(u_0)$ is positive definite on $\operatorname{span}\{e_{j+1},e_{j+2},\ldots\}$, and thus $m(u_0)=j$.$\square$

Proof of Theorem 4.2. We recall that

$$i_\lambda(K_{E_\varepsilon},E_\varepsilon) = P_\lambda(H^1(\Omega)) + (1+\lambda)Q_\lambda \tag{6}$$

where $Q_\lambda\in\mathcal{S}$.

From Lemma 4.1, we know that if $\varepsilon\neq\mu_k$ $(k\geq 2)$, then u_0 is nondegenerate and thus if we set $K_0 = K_{E_\varepsilon}\setminus\{u_0\}$, then K_0 and $\{u_0\}$ are isolated critical sets. So by (i) and (ii) of Proposition 3.4,

$$i_\lambda(K_{E_\varepsilon},E_\varepsilon) = \lambda^{m(u_0)} + i_\lambda(K_0,E_\varepsilon). \tag{7}$$

Moreover, as seen in Remark 2.3, the set K_0 is symmetric with respect to u_0, in the sense that if $u\in K_0$ then $-u\in K_0$ and they have the same restricted and large Morse index.

In particular, $-u$ has the same degeneracy as u. This means that the coefficients of $i_\lambda(K_0,E_\varepsilon)$ are even integers (unless they are $+\infty$). Thus for all $k\geq 2$, if $\varepsilon\in\,]\mu_{k+1},\mu_k[$, then (7) becomes

$$i_\lambda(K_{E_\varepsilon},E_\varepsilon) = \lambda^k + \sum_{h\in\mathbb{N}} a_h\lambda^h \tag{8}$$

with $a_h\in 2\mathbb{N}\cup\{+\infty\}$.

Since $H^1(\Omega)$ is contractible we have

$$P_\lambda(H^1(\Omega)) = 1. \tag{9}$$

From (8), (9), on setting $Q_\lambda = \sum_{h\in\mathbb{N}} b_h\lambda^h$, (6) becomes

$$\lambda^k + \sum_{h\in\mathbb{N}} a_h\lambda^h = 1 + (1+\lambda)\sum_{h\in\mathbb{N}} b_h\lambda^h. \tag{10}$$

Now let $\widetilde{k}$ be defined by

$$\widetilde{k} = \min(\{h : b_h = +\infty\} \cup \{k\}).$$

We shall prove by induction that $a_h \geq 2$ for all $h < \widetilde{k}$, or more precisely that for each $h = 1, \ldots, \widetilde{k} - 1$ there is $n_h \in \mathbb{N}$ such that

$$b_h = 1 + 2n_h \quad \text{and} \quad a_h = 2 + 2n_{h-1} + 2n_h.$$

For $h = 0$, by (10), $a_0 = 1 + b_0$ and, as a_0 is an even number, b_0 must be odd. Thus there is $n_0 \in \mathbb{N}$ such that $b_0 = 1 + 2n_0$ and $a_0 = 2 + 2n_0 \geq 2$.

For $h = 1$, by (10), $a_1 = b_0 + b_1 = 1 + 2n_0 + b_1$. Analogously there is $n_1 \in \mathbb{N}$ such that $b_1 = 1 + 2n_1$ and $a_1 = 2 + 2n_0 + 2n_1 \geq 2$.

For $h < \widetilde{k} - 1$ assume that $a_h = 2 + 2n_{h-1} + 2n_h$ and $b_h = 1 + 2n_h$, where $n_h, n_{h-1} \in \mathbb{N}$. By (10),

$$a_{h+1} = b_h + b_{h+1} = 1 + 2n_h + b_{h+1}$$

and thus a_{h+1} is even. So there is $n_{h+1} \in \mathbb{N}$ such that

$$b_{h+1} = 1 + 2n_{h+1} \quad \text{and} \quad a_{h+1} = 2 + 2n_h + 2n_{h+1} \geq 2.$$

Therefore

$$i_1(K_0, E_\varepsilon) = \sum_{h \in \mathbb{N}} a_h \geq \sum_{h \leq \widetilde{k}-1} a_h \geq 2\widetilde{k}.$$

Let us distinguish two cases. If $\widetilde{k} = k$, then there are at least $2k$ critical points for E_ε (counted with multiplicities). Moreover, as $a_h \neq 0$ for all $h < k$, by Proposition 3.5, there exists $u_h \in K_0$ such that

$$m(u_h) \leq h \leq m^*(u_h).$$

By Remark 2.3, $-u_h$ is critical and

$$m(-u_h) \leq h \leq m^*(-u_h),$$

thus the conclusion of Theorem 4.2 holds.

On the contrary, if we have $\widetilde{k} < k$, then $a_0 \geq 2, \ldots, a_{\widetilde{k}-1} \geq 2$ and since $b_{\widetilde{k}} = +\infty$, still by (10), we see that

$$a_{\widetilde{k}} = b_{\widetilde{k}-1} + b_{\widetilde{k}} = +\infty, \quad a_{\widetilde{k}+1} = b_{\widetilde{k}} + b_{\widetilde{k}+1} = +\infty.$$

In this case there are infinitely many critical points having Morse index less than k (if counted with multiplicities). $\square$

PROOF OF THEOREM 4.3. Let Q_l be a hypercube of side $l > 0$ contained in $\Gamma_\alpha(u)$.

For all $i \in \mathbb{N}$ we denote by v_i the eigenfunction of $-\Delta$ corresponding to the eigenvalue $a_i(l)$.

Let j be such that $\varepsilon < \sqrt{\delta(\alpha)/a_j(l)}$. We shall prove that $d^2E_\varepsilon(u)$ is negative definite on v_i for all $i \le j$. Indeed, from (5),

$$\forall x \in \Gamma_\alpha(u), \quad F''(u(x)) \le -\delta(\alpha).$$

Thus

$$\begin{aligned} d^2E_\varepsilon(u)(v_i, v_i) &= \varepsilon \int_\Omega |\nabla v_i|^2 + \frac{1}{\varepsilon} \int_\Omega F''(u(x)) v_i^2(x)\, dx \\ &= \varepsilon \int_{Q_l} |\nabla v_i|^2 + \frac{1}{\varepsilon} \int_{Q_l} F''(u(x)) v_i^2(x)\, dx \\ &\le \varepsilon \int_{Q_l} |\nabla v_i|^2 - \frac{\delta(\alpha)}{\varepsilon} \int_{Q_l} v_i^2(x)\, dx \\ &= \varepsilon \left(a_i(l) - \frac{\delta(\alpha)}{\varepsilon^2} \right) \int_{Q_l} v_i^2(x)\, dx < 0. \end{aligned}$$

If we set $k = N(u,l)$, there are k disjoint hypercubes $Q_l^1, \ldots, Q_l^k$ contained in $\Gamma_\alpha(u)$.

For each $h = 1, \ldots, k$ let $v_1^h, \ldots, v_j^h$ be the eigenfunctions of $-\Delta$ on Q_l^h relative to $a_1(l), \ldots, a_j(l)$ respectively. The functions $v_1^1, v_2^1, \ldots, v_j^1, v_1^2, \ldots, v_j^2, \ldots, v_j^k$ are linearly independent. Indeed, for each $h = 1, \ldots, k$, $v_1^h, \ldots, v_j^h$ are linearly independent both in $L^2(Q_l^h)$ and in $H_0^1(Q_l^h)$, so also in $H^1(\Omega)$. On the other hand, if $h_1 \ne h_2 \in \{1, \ldots, k\}$, then $v_{i_1}^{h_1}$ and $v_{i_2}^{h_2}$ are linearly independent as they have disjoint supports. So $d^2E_\varepsilon(u)$ is negative definite on $\operatorname{span}\{v_1^1, \ldots, v_j^k\}$, whose dimension is $jk = jN(u,l)$. By the definition of (restricted) Morse index,

$$m(u) \ge jN(u,l)$$

and therefore $N(u,l) \le m(u)/j$. □

Proof of Corollary 4.4. Looking into the proof of Theorem 4.2, we see that two cases may occur:

1) if a_0 is finite then $a_1 \ge 2$ (possibly $+\infty$),
2) if $a_0 = +\infty$ then again $a_1 = +\infty$.

In any case $a_1 \ne 0$, and from Proposition 3.5 we deduce that there exists $u_{1,\varepsilon}$ such that $m(u_{1,\varepsilon}) \le 1 \le m^*(u_{1,\varepsilon})$. Obviously by Remark 2.3 we have the same property for $-u_{1,\varepsilon}$. This proves the first assertion. For the second one, if we assume that $l > \varepsilon\pi\sqrt{(n+3)/\delta(\alpha)}$, recalling that $a_2(l) = (n+3)\pi^2/l^2$, we see that $\varepsilon < \sqrt{\delta(\alpha)/a_2(l)}$. Thus by Theorem 4.3 it follows that

$$N(\pm u_{1,\varepsilon}, l) \le m(\pm u_{1,\varepsilon})/2 \le 1/2.$$

That is, $N(\pm u_{1,\varepsilon}, l) = 0$. □

PROOF OF COROLLARY 4.5. From assumption (iv) on F and continuity of F'',

$$\exists \widetilde{a} > 0 \ \forall t \in \mathbb{R}, \quad F''(t) \geq -\widetilde{a}.$$

Let $u \in K_{E_\varepsilon}$ and v be an eigenfunction of $-\Delta$ on Ω (with Neumann boundary condition) relative to an eigenvalue λ. If $d^2E_\varepsilon(u)$ is negative definite on v, then

$$0 > d^2E_\varepsilon(u)(v,v) \geq \varepsilon\left(\lambda - \frac{\widetilde{a}}{\varepsilon^2}\right)\int_\Omega v^2(x)\,dx.$$

Thus $\lambda < \widetilde{a}/\varepsilon^2$, i.e. v belongs to the eigenspace relative to the eigenvalues of $-\Delta$ less than $\widetilde{a}/\varepsilon^2$. Therefore

$$\forall u \in K_{E_\varepsilon}, \quad m(u) \leq \mathcal{A}(\widetilde{a}/\varepsilon^2). \tag{11}$$

Moreover, since $a_1(l) = n\pi^2/l^2$, we have

$$\varepsilon < \sqrt{\delta(\alpha)/a_1(l)} \Leftrightarrow l > \pi\varepsilon\sqrt{n/\delta(\alpha)}.$$

In particular, if $l = \widetilde{c}\varepsilon$, where $\widetilde{c} > \pi\sqrt{n/\delta(\alpha)}$, then by Theorem 4.3 we have

$$\forall u \in K_{E_\varepsilon}, \quad N(u, \widetilde{c}\varepsilon) \leq m(u) \leq \mathcal{A}(\widetilde{a}/\varepsilon^2). \qquad \square$$

PROOF OF COROLLARY 4.6. Let $\varepsilon_0 = \sqrt{\delta(\alpha)/a_{j+1}(l)}$. If $\varepsilon < \varepsilon_0$ and u is a critical point such that $m(u) \leq j$, then from Theorem 4.3,

$$N(u,l) \leq \frac{m(u)}{j+1} \leq \frac{j}{j+1}.$$

Therefore $N(u,l) = 0$. $\square$

PROOF OF THEOREM 1.1. As already shown, the eigenvalues $a_j(l)$ of $-\Delta$ on Q_l with Dirichlet boundary conditions are of the type $a_j(l) = h_j\pi^2/l^2$, where $(h_j)_{j\geq 1}$ is a nondecreasing divergent sequence of integers. So the first part of the theorem directly follows from Theorem 4.3 by putting $k_j = (1/\pi)\sqrt{\delta(\alpha)/h_j}$, while the second part is exactly the statement of Corollary 4.6. $\square$

Acknowledgments. I would like to express my gratitude to Professor Vieri Benci. Throughout the realization of this work I strongly benefited from his pertinent advice and encouragement.

I am also indebted to Dr. Louis Jeanjean with whom I had several fruitful discussions on the subject.

REFERENCES

[1] A. AMBROSETTI AND P. H. RABINOWITZ, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[2] V. BENCI AND F. GIANNONI, *Morse theory for C^1-functionals and Conley blocks*, Topol. Methods Nonlinear Anal. **4** (1994), 365–398.

[3] V. Benci and F. Giannoni, *Morse theory for functionals of class C^1*, C. R. Acad. Sci. Paris Sér. I **315** (1992), 883–888.

[4] M. Berger, *Nonlinearity and Functional Analysis*, Academic Press, London, 1977.

[5] H. Brezis, *Analisi Funzionale*, Liguori, 1986.

[6] A. Dold, *Lectures on Algebraic Topology*, Springer-Verlag, Berlin, 1972.

[7] M. J. Greenberg and J. R. Harper, *Algebraic Topology—a First Course*, Addison-Wesley, Redwood City, 1981.

[8] V. Klee, *Some topological properties of convex sets*, Trans. Amer. Math. Soc. **78** (1955), 30–45.

[9] A. Marino and G. Prodi, *Metodi perturbativi nella teoria di Morse*, Boll. Un. Mat. Ital. (4) **11** (1975), Suppl. Fasc. 3, 1–32.

[10] J. Milnor, *Morse Theory*, Ann. of Math. Stud., vol. 51, Princeton Univ. Press, 1963.

[11] L. Modica, *The gradient theory of phase transitions and the minimal interface criterion*, Arch. Rational Mech. Anal. **98** (1986), 123–142.

[12] L. Modica and S. Mortola, *The Γ-convergence of some functionals*, preprint no. 77-7, Istit. Matem. L. Tonelli, Univ. Pisa, 1977.

[13] R. Palais, *Morse theory on Hilbert manifolds*, Topology **2** (1963), 299–340.

[14] D. Passaseo, *Multiplicity of critical points for some functionals related to the minimal surfaces problem*, Calc. Var. Partial Differential Equations (to appear).

[15] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conf. Ser. in Math., vol. 65, Amer. Math. Soc., 1984.

Manuscript received September 29, 1994

Giuseppina Vannella
Dipartimento di Matematica
Politecnico di Bari
Via Orabona 4
70125 Bari, ITALIA

E-mail address: vannella@sun.dm.uniba.it