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LUSIN PROPERTIES AND INTERPOLATION OF SOBOLEV SPACES

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1. Introduction and preliminaries

Our purpose is first to survey some results on Lusin properties of functions together with some applications to show their significance. Then we will give a new form of the Lusin property for Sobolev functions and apply it to interpolation of Sobolev spaces.

Let D be a Lebesgue measurable set in $\mathbb{R}^n$ and k a nonnegative integer. A real measurable function u defined on D is said to have the *Lusin property of order k* if for any $\varepsilon > 0$ there is a C^k-function g on $\mathbb{R}^n$ such that $|\{x \in D : u(x) \neq g(x)\}| < \varepsilon$, where we use the notation $|A|$ for the Lebesgue measure of a set A in $\mathbb{R}^n$. Unless explicitly stated otherwise a function defined on a measurable subset D of $\mathbb{R}^n$ will be assumed to be real measurable and finite almost everywhere on D. For a C^k-function g, the polynomial

$$\pi_k(g;x,y) := \sum_{|\alpha|\leq k} \frac{1}{\alpha!} D^\alpha g(x)(y-x)^\alpha$$

is called the *k-Taylor polynomial* of g at x. Polynomials of this form are sometimes referred to as polynomials centered at x. We refer to [23, p. 2] for the standard multi-index notation which appears in the preceding formula.

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A classical theorem of Lusin states that measurable functions which are finite almost everywhere have the Lusin property of order zero. It then follows by the Lebesgue density theorem that finite almost everywhere measurable functions are approximately continuous almost everywhere. Conversely, one verifies by using Vitali's covering theorem that functions which are approximately continuous almost everywhere have the Lusin property of order zero. Since approximate continuity is a kind of weak regularity for functions, a question immediately presents itself: whether with increased degree of some form of weak regularity, a function will have the Lusin properties of higher order. It was Federer who discovered that a function which is totally differentiable almost everywhere has the Lusin property of order 1. Whitney's result [22] for approximately totally differentiable functions is, to our knowledge, the first definite result in this direction. We shall describe in §2 some results concerning characterizations of functions which have the Lusin property of order k for $k \geq 1$.

For a nonnegative integer k and a real number $p \geq 1$, a function u defined on a measurable subset D of $\mathbb{R}^n$ is said to have the *strong* (k,p)*-Lusin property* if for any $\varepsilon > 0$ there is a C^k-function g defined on $\mathbb{R}^n$ such that $|D_g^u| < \varepsilon$ and $\|g\|_{k,p}(D_g^u) < \varepsilon$, where $D_g^u = \{x \in D : u(x) \neq g(x)\}$ and

$$\|g\|_{k,p}(D_g^u) := \sum_{|\alpha| \leq k} \|D^\alpha g\|_{L^p(D_g^u)}.$$

We have shown in [12] that if D is a Lipschitz domain, then functions of the Sobolev space $W_p^k(D)$ have the strong (k,p)-Lusin property. When $p > 1$, this result has been extended by Michael and Ziemer [16] as follows: Let k, m be positive integers with $m \leq k, (k-m)p < n$ and let D be an arbitrary nonempty open subset of $\mathbb{R}^n$. Then for $u \in W_p^k(D)$ and $\varepsilon > 0$, there exists a C^m-function g on D such that

$$R_{k-m,p}(D_g^u) < \varepsilon \quad \text{and} \quad \|g\|_{k,p}(D) < \varepsilon,$$

where $R_{k-m,p}$ is a Riesz capacity and $\|g\|_{k,p}(D)$ is the Sobolev norm of g in $W_p^k(D)$.

We remark here that the strong $(1,1)$-Lusin property for $u \in W_p^k(D)$ is a consequence of a more general result of Michael [15]: Let u be a function of bounded variation with compact support on $\mathbb{R}^n$. Then for each $\varepsilon > 0$, there is a Lipschitz function g on $\mathbb{R}^n$ such that $|D_g^u| < \varepsilon$ and $|\mathrm{Var}(u) - \mathrm{Var}(g)| < \varepsilon$, where $\mathrm{Var}(f)$ denotes the total variation of a function f. As shown in [2] the Lusin property established in [16] is closely related to some pointwise inequalities for Sobolev functions.

Closely related to the strong $(1,p)$-Lusin property is a Lusin type property which we shall take up in §3. There we shall also prove another strong Lusin type property for functions in $W^k_\infty(D)$ with application to interpolation of Sobolev spaces.

In the remaining part of this section we consider some preliminary results. For a function u defined on an open set D the maximal function of u, Mu, is defined by

$$Mu(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)\cap D} |u(y)|\, dy, \quad x \in \mathbb{R}^n,$$

where $B(x,r)$ is the ball with center x and radius r. For properties of maximal functions we refer to [19] and [23]. We shall also need the modified maximal function of u, $M_0(u)$, which is defined by

$$M_0u(x) := \sup_{0<r\le\delta(x)} \frac{1}{|B(x,r)|} \int_{B(x,r)} |u(y)|\, dy, \quad x \in D,$$

where $\delta(x)$ is $\min\{1,(1/2)\mathrm{dist}(x,\partial D)\}$. If u is locally integrable on D, then for any compact subset F of D, $M_0u(x) \le M(uI_S)(x)$ for $x \in F$, with $S := \bigcup_{x\in F} B(x,\delta(x))$ and I_S being the indicator function of S. Since the closure of S is a compact subset of D, $uI_S \in L^1(D)$, hence M_0u is finite almost everywhere on D. The Sobolev space $W^k_p(D)$ will always be understood with D an open subset of $\mathbb{R}^n$ and $W^k_p(\mathbb{R}^n)$ will be simply denoted by W^k_p. We shall denote by $W^k_{\mathrm{loc}}(D)$ the space of all those locally integrable functions on D whose generalized partial derivatives up to order k are also locally integrable. For $u \in W^k_{\mathrm{loc}}(D)$, the generalized partial derivatives $D^\alpha u, |\alpha| \le k$, will sometimes be written as u_α. If $u \in W^k_{\mathrm{loc}}(D)$, then for almost all $x \in D, u_\alpha(x)$ is defined for all α with $|\alpha| \le k$. For such x we let

$$\pi_k(u;x,y) := \sum_{|\alpha|\le k} \frac{1}{\alpha!} u_\alpha(x)(y-x)^\alpha$$

and call it the k-*Taylor polynomial* of u at x. Hence the k-Taylor polynomial $\pi_k(u;x,y)$ of u is defined for almost all x in D. We shall need the following lemma concerning k-Taylor polynomials:

Lemma 1. *If $u \in W^k_{\mathrm{loc}}(D)$, then for almost all $x \in D$ and $r \le \delta(x)$ we have*

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} \frac{|u(y)-\pi_k(u;x,y)|}{|y-x|^k}\, dy \le 2 \sum_{|\alpha|=k} M_0(u_\alpha)(x); \tag{1}$$

and

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \frac{|u(y)-\pi_k(u;x,y)|}{|y-x|^k}\, dy = 0. \tag{2}$$

Proof. If $u \in W^k_{\mathrm{loc}}(D)$, then, as shown in [12, p. 648], for $B(x,r) \subset D$ we have

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} \frac{|u(y)-\pi_k(u;x,y)|}{|y-x|^k}\,dy \tag{3}$$
$$\leq k\int_0^1 (1-t)^{k-1}\Big\{\sum_{|\alpha|=k}\frac{1}{|B(x,rt)|}\int_{B(x,rt)}|u_\alpha(y)-u_\alpha(x)|\,dy\Big\}\,dt.$$

Since

$$\frac{1}{|B(x,rt)|}\int_{B(x,rt)}|u_\alpha(y)-u_\alpha(x)|\,dy \leq 2M_0(u_\alpha)(x) \tag{4}$$

for almost all $x \in D$, (1) follows from (3). Let x be a Lebesgue point of all u_α with $|\alpha| = k$ for which both (3) and (4) hold and $\sum_{|\alpha|=k} M_0(u_\alpha)(x) < \infty$. For this x, since (4) holds, if we let $r \to 0$ on both sides of (3), we may use the Lebesgue bounded convergence theorem for the right hand side of (3) and the integration in t to obtain (2). This shows that (2) holds for almost all x.

For a closed set F we denote by $t^k(F)$ the class of all those functions u with the property that there is a family $\{p(x,y)\}_{x\in F}$ of polynomials in y of degree $\leq k$ and a constant $M \geq 0$ such that

1) $u(x) = p(x,x)$, $x \in F$;
2) $|D^\alpha p(x,x)| \leq M$, $|D^\alpha p(y,y) - D^\alpha p(x,y)| \leq M|y-x|^{k-|\alpha|}$, $x,y \in F$;
3) $\lim_{y\to x} |D^\alpha p(y,y) - D^\alpha p(x,y)| \cdot |y-x|^{-k+|\alpha|} = 0$ uniformly on every compact subset of F.

The smallest such M is denoted by $N(u;t^k;F)$. In the above we have used $D^\alpha p(x,y)$ to denote the value of $D^\alpha p(x,z)$ when $z = y$. Here D^α is with respect to z. Finally, we denote $D^\alpha p(x,x)$ by $D^\alpha u(x)$. By combining arguments of [19, Chapter 6] and [5, 3.1.14] one obtains the following lemma which is slightly stronger than Whitney's extension theorem:

Lemma 2 (Whitney's Extension Theorem). *There is a constant $C > 0$ depending only on n and k such that for each $u \in t^k(F)$ there is $g \in C^k(\mathbb{R}^n)$ with $g|_F = u$ and $N(g;t^k;\mathbb{R}^n) \leq CN(u;t^k;F)$.*

In the proof of Theorem (3.5) in [14] we have actually proved the following lemma:

Lemma 3. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and let F be a closed subset of $\mathbb{R}^n$. Suppose that for each $x \in F$ there is a polynomial $p(x,y)$ in y of degree at most k such that for some constant $M > 0$ we have*

$$\sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|u(y)-p(x,y)|\,dy \leq M, \quad x \in F,$$
$$|D^\alpha p(x,x)| \leq M, \quad x \in F,\ |\alpha| \leq k;$$

and

$$\lim_{r\to 0} r^{-k}\frac{1}{|B(x,r)|}\int_{B(x,r)}|u(y)-p(x,y)|\,dy=0$$

uniformly on each compact subset of F. Then $u\in t^k(F)$ and $N(u;t^k;F)\le C(n,k)M$.

2. Approximate Taylor polynomials and Lusin property of order k

If a function u has the Lusin property of order k on D, then it is clear that for almost every x of D there is a C^k-function g such that the set $\{z\in D: u(z)=g(z)\}$ contains x and has density one at x. Thus the following condition holds at almost every point x of D:

$$\operatorname{ap}\lim_{y\to x}\frac{|u(y)-\pi_k(g;x,y)|}{|y-x|}=0, \tag{1}$$

and hence also does the condition

$$\operatorname{ap}\limsup_{y\to x}\frac{|u(y)-\pi_{k-1}(g;x,y)|}{|y-x|^k}<\infty. \tag{2}$$

We recall that $\operatorname{ap}\lim_{y\to x}u(y)=l$ means that the set $\{y\in D:|u(y)-l|\le\varepsilon\}$ has density one at x for any $\varepsilon>0$, and that $\operatorname{ap}\limsup_{y\to x}u(y)$ is the infimum of all those $\lambda\in\mathbb{R}$ such that the set $\{y\in D: u(y)>\lambda\}$ has density zero at x.

Now some definitions are in order. A function u defined on D is said to have *approximate* $(k-1)$*-Taylor polynomial* at x if there is a polynomial $p(x,y)$ in y of degree at most $k-1$ such that

$$\operatorname{ap}\limsup_{y\to x}\frac{|u(y)-p(x,y)|}{|y-x|^k}<\infty; \tag{3}$$

while u will be said to be *approximately differentiable of order* k at x if there is a polynomial $p(x,y)$ in y and of degree at most k such that

$$\operatorname{ap}\lim_{y\to x}\frac{|u(y)-p(x,y)|}{|y-x|^k}=0. \tag{4}$$

If (4) is replaced by

$$\lim_{y\to x}\frac{|u(y)-p(x,y)|}{|y-x|^k}=0, \tag{5}$$

then u is said to be *differentiable of order* k at x. From (1) and (2), if u has the Lusin property of order k on D, then it is approximately differentiable of order k and has approximate $(k-1)$-Taylor polynomial at almost every point of D. If u is approximately differentiable (differentiable) of order 1 at x, it will be simply said to be *approximately differentiable* (*differentiable*) at x.

We have shown in [13] the following theorem relating those properties of functions defined above:

Theorem 1. *For a measurable function u defined on D the following statements are equivalent:*

(I) *u has the Lusin property of order k on D.*
(II) *u has approximate $(k-1)$-Taylor polynomial at almost every point of D.*
(III) *u is approximately differentiable of order k at almost every point of D.*

Remark. Since it follows from (2) of Lemma 1 that u is approximately differentiable of order k almost everywhere on D if $u \in W^k_{\mathrm{loc}}(D)$, we infer from Theorem 1 that u has the Lusin property of order k.

As a consequence of Theorem 1 we also establish in [13] the following theorem:

Theorem 2. *In order for u to be differentiable of order k almost everywhere on D it is necessary and sufficient that at almost every point x of D there is a polynomial $p(x,y)$ in y of degree at most $k-1$ such that*

$$\limsup_{y\to x} \frac{|u(y)-p(x,y)|}{|y-x|^k} < \infty. \tag{6}$$

We now give some remarks concerning Theorems 1 and 2. When $k=1$, the equivalence of statements (I) and (III) in Theorem 1 is due to Whitney [22], while the equivalence of statements (I) and (II) is due to Federer [5, 3.1.16]. In [10], equivalence of (I) and (III) in Theorem 1 is established with the additional assumption in (III) that each u_α is measurable and is approximately differentiable of order $k-|\alpha|$ almost everywhere. Theorem 1 in its generality may be considered as an answer to a question raised by Federer [5, 3.1.17]. Theorem 2 is a generalization of a well-known result of Rademacher [18] and Stepanoff [21] to differentiability of higher order. Theorem 2 is also more general in that we do not assume D to be open.

Now we mention two applications of Theorem 1 with $k=1$. From its proof, the Corollary on p. 261 of [19] can be restated as the following lemma:

Lemma 4. *Suppose that u is defined in a neighborhood of D and suppose that u has the Lusin property of order 1 on D. If*

$$\limsup_{|y|\to 0} \frac{|u(x+y)+u(x-y)-2u(x)|}{|y|} < \infty \tag{7}$$

almost everywhere on D, then u is differentiable almost everywhere on D.

Since a function which is approximately differentiable almost everywhere on D has the Lusin property of order 1 by Theorem 1, we obtain immediately from Lemma 4 the theorem below:

Theorem 3. *If a measurable function u defined in a neighborhood of D is appoximately differentiable almost everywhere on D and if* (7) *holds at almost every $x \in D$, then u is differentiable almost everywhere on D.*

For another application we will define multiplicity functions for approximately differentiable mappings. We follow the approach in [11]. Let T be a measurable mapping from $D \subset \mathbb{R}^n$ into $\mathbb{R}^m$, $n \leq m$. T is said to be *approximately differentiable* at a point $x \in D$ if its coordinate functions are all approximately differentiable at x. It follows from Theorem 1 that T is approximately differentiable almost everywhere on D if and only if there is a sequence $K = \{K_j\}_{j=1}^\infty$ of compact subsets of D and a sequence $g = \{g_j\}_{j=1}^\infty$ of C^1 mappings from $\mathbb{R}^n$ into $\mathbb{R}^m$ with the following properties:

(i) $K_1 \subset K_2 \subset K_3 \subset \ldots$;
(ii) $|D \setminus \bigcup_{j=1}^\infty K_j| = 0$;
(iii) $T(x) = g_j(x)$ for $x \in K_j$, $j = 1, 2, 3, \ldots$

Such a pair (K, g) will be called a C^1-*representation* of T. If T is approximately differentiable almost everywhere on D, we will simply say that T is an *approximately differentiable mapping.* In this case, if $f^1, \ldots, f^m$ are the coordinate functions of T, then the approximate partial derivatives, still denoted by $\partial f^i/\partial x_j$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, exist almost everywhere on D and hence the function

$$J(T; x) := \left\{ \sum_{1 \leq i_1 < \ldots < i_n \leq m} \left[\frac{\partial(f^{i_1}, \ldots, f^{i_n})}{\partial(x_1, \ldots, x_n)} \right]^2 \right\}^{1/2}$$

is defined almost everywhere on D, where $\partial(f^{i_1}, \ldots, f^{i_n})/\partial(x_1, \ldots, x_n)$ is the usual Wronskian determinant with partial derivatives replaced by approximate partial derivatives; the function $J(T; \cdot)$ will be simply called the *Jacobian* of T. When D is open and T is a C^1 mapping, $J(T; \cdot)$ is the usual Jacobian of T.

We now define multiplicity functions for approximately differentiable mappings and prove the corresponding area formula. Let T be an approximately differentiable mapping from D into $\mathbb{R}^m$, $m \geq n$, and let (K, g) be a C^1-representation of T. Consider a measurable subset E of D, and for a positive integer j and $y \in \mathbb{R}^m$ let

$$m^j(T, E, y; K) := \#\{x \in K_j \cap E : T(x) = y\},$$

where $\#A$ is the cardinality of the set A. Since $m^j(T, E, y; K) = \#g_j^{-1}(y) \cap K_j \cap E$, $m^j(T, E, y; K)$ is an H^n-measurable function in y with H^n being the n-dimensional Hausdorff measure in $\mathbb{R}^m$. Because $m^j(T, E, y; K)$ is nondecreasing in j, the limit $\lim_{j \to \infty} m^j(T, E, y; K)$ exists and will be denoted by $m(T, E, y; K)$. Now, $m(T, E, \cdot; K)$ is a H^n-measurable function on $\mathbb{R}^m$ and is called a *multiplicity function of T relative to E*. When D is open and T is a C^1 mapping, then for any sequence $K = \{K_j\}_{j=1}^\infty$ of compact subsets of D such that $D = \bigcup_{j=1}^\infty K_j$ we may choose $g = \{g_j\}_{j=1}^{\infty}$ with each g_j being a C^1 extension of $T|_{K_j}$ to obtain a C^1-representation of T; then $m(T, E, \cdot; K) = m(T, E, \cdot)$, where $m(T, E, y) := \#\{x \in E : T(x) = y\}$ is the usual multiplicity function of T relative to E. In this case we have the following classical area formula:

$$\int_E J(T; x)\, dx = \int_{\mathbb{R}^m} m(T, E, y)\, dH^n(y). \tag{8}$$

The theorem that follows establishes the area formula for approximately differentiable mappings.

THEOREM 4. *Let T be an approximately differentiable mapping from a measurable subset D of $\mathbb{R}^n$ into $\mathbb{R}^m$, $m \geq n$. Then for any measurable subset E of D, the following formula holds:*

$$\int_E J(T; x)\, dx = \int_{\mathbb{R}^m} m(T, E, y; K)\, dH^n(y), \tag{9}$$

where $m(T, E, y; K)$ is any multiplicity function of T relative to E. Furthermore, if $\int_E J(T; x)\, dx < \infty$, then the multiplicity function is uniquely defined H^n-almost everywhere on $\mathbb{R}^m$ and we may simply denote it by $m(T, E, \cdot)$.

PROOF. Let (K, g) be a C^1-representation of T. For any positive integer j we have

$$\int_{K_j \cap E} J(T; x)\, dx = \int_{K_j \cap E} J(g_j; x)\, dx;$$

but by the classical area formula for g_j we know that

$$\int_{K_j \cap E} J(g_j; x)\, dx = \int_{\mathbb{R}^m} m(g_j, K_j \cap E, y)\, dH^n(y),$$

and then, since $m(g_j, K_j \cap E, y) = m^j(T, E, y; K)$, we have

$$\int_{K_j \cap E} J(T; x)\, dx = \int_{\mathbb{R}^m} m^j(T, E, y; K)\, dH^n(y). \tag{10}$$

If we let $j \to \infty$ on both sides of (10), we obtain (9).

Now suppose $\int_E J(T; x)\, dx < \infty$. For any two multiplicity functions $m(T, E, \cdot; K)$, $m(T, E, \cdot; K')$ of T relative to E, corresponding respectively to C^1-representations (K, g) and (K', g'), $(K \cap K', g)$ is a C^1-representation of T, where $K \cap K' := \{K_j \cap K'_j\}_{j=1}^{\infty}$; hence by Theorem 4,

$$\begin{aligned} \int_E J(T; x)\, dx &= \int_{\mathbb{R}^m} m(T, E, y; K \cap K')\, dH^n(y) \\ &= \int_{\mathbb{R}^m} m(T, E, y; K)\, dH^n(y) < \infty. \end{aligned} \tag{11}$$

But $m(T,E,y;K\cap K')\le m(T,E,y;K)$ for $y\in\mathbb{R}^m$ implies together with (11) that $m(T,E,\cdot;K)$ equals $m(T,E,\cdot;K\cap K')$ H^n-almost everywhere on $\mathbb{R}^m$. Similarly, $m(T,E,\cdot;K')$ equals $m(T,E,\cdot;K\cap K')$ H^n-almost everywhere on $\mathbb{R}^m$ and hence $m(T,E,\cdot;K)=m(T,E,\cdot;K')$ H^n-almost everywhere on $\mathbb{R}^m$.

Theorem 4 immediately yields the well-known area formula for Lipschitz mappings:

Corollary 1. *If T is a Lipschitz mapping, then*

$$\int_E J(T;x)\,dx=\int_{\mathbb{R}^m} m(T,E,y)\,dH^n(y),$$

for any measurable subset E of D.

Proof. Let (K,g) be a C^1-representation of T. Since T is Lipschitz, the image $T(D\setminus\bigcup_{j=1}^\infty K_j)$ has H^n-measure zero. Now $m(T,E,y;K)=m(T,E,y)$ except for $y\in T(D\setminus\bigcup_{j=1}^\infty K_j)$, hence $m(T,E,y;K)=m(T,E,y)$ H^n-almost everywhere on $\mathbb{R}^m$. The corollary then follows from Theorem 4.

A mapping T from an open set $D\subset\mathbb{R}^n$ into $\mathbb{R}^m$, $m\ge n$, is called a *Sobolev mapping* if each coordinate function $f^i, i=1,\dots,m$, is in $W^k_{p_i}(D)$ with $\sum_{l=1}^n 1/p_{i_l}\le 1$ for all $1\le i_1<\dots<i_n\le m$. This class of mappings is introduced in [7] generalizing that introduced in [17] when $n=2$ and the mappings are continuous.

Corollary 2. *If T is a Sobolev mapping from an open set $D\subset\mathbb{R}^n$ into $\mathbb{R}^m$, $m\ge n$, then for each measurable subset E of D there is a multiplicity function $m(T,E,\cdot)$ of T relative to E which is uniquely defined H^n-almost everywhere on $\mathbb{R}^m$ such that*

$$\int_E J(T;x)\,dx=\int_{\mathbb{R}^m} m(T,E,y)\,dH^n(y).$$

It is now clear that the same approach can be used to handle changes of variables by approximately differentiable mappings. We will not do it here, but refer the reader to [6] and [9].

3. Some strong Lusin properties and interpolation of Sobolev spaces

In [3] Calderón and Zygmund introduce the classes $T^p_a(x)$ and $t^p_a(x)$ of functions, where $1\le p\le\infty$ and $a\ge np^{-1}$. A function f defined on $\mathbb{R}^n$ belongs to $T^p_a(x)$ if there exists a polynomial $p(y)$ of degree strictly less than a such that

$$\left\{\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)-p(y)|^p\,dy\right\}^{1/p}\le Mr^a,\quad 0<r<\infty, \tag{1}$$

while f belongs to $t_a^p(x)$ if there exists a polynomial $q(y)$ of degree at most a such that

$$\left\{\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)-q(y)|^p\,dy\right\}^{1/p}\le Mr^a,\quad 0<r<\infty,$$

and, in addition, the expression on the left is of order smaller than r^a as $r\to 0$. Under the condition of uniform boundedness of M and of the coefficients of $p(y)$ or $q(y)$ for x in a closed set F, Calderón and Zygmund obtain in [3] certain regularity properties of f on F. In particular, they show that functions in $W_p^k(\mathbb{R}^n)$ have the Lusin property of order k. We consider instead in [14] functions related to the definitions of $T_k^1(x)$ and $t_k^1(x)$ with k a positive integer and use the behavior of these functions to characterize various classes of functions. To introduce such functions, for a measurable set A with $|A|>0$ we shall denote by $m(f;A)$ the integral mean of a function f over A, i.e. $m(f;A):=\frac{1}{|A|}\int_A f(y)\,dy$, for notational simplicity. For a positive integer k and a function u defined on $\mathbb{R}^n$, we let for $x\in\mathbb{R}^n$,

$$\widetilde{M}_k(u;x):=\inf\sup_{r>0} r^{-k}m(|u-p|;B(x,r)),$$

where the infimum is taken over all polynomials p of degree strictly less than k. If $\widetilde{M}_k(u;x)<\infty$ there is a unique polynomial $p(y)$ of degree strictly less than k such that

$$\widetilde{M}_k(u;x):=\sup_{r>0} r^{-k}m(|u-p|;B(x,r));$$

the polynomial $p(y)$ can then be expressed as

$$p(y)=\sum_{|\alpha|<k}\frac{1}{\alpha!}u_\alpha(x)(y-x)^\alpha. \tag{2}$$

We denote by $D^{(k)}$ the space of all those functions u for which $\widetilde{M}_k(u;x)<\infty$ for almost all x. Let now $u\in D^{(k)}$. Then the functions u_α, $|\alpha|<k$, are defined almost everywhere on $\mathbb{R}^n$; it is shown in [3] that for almost all x for which $\widetilde{M}_k(u;x)<\infty$ there is a unique polynomial $q(y)$ of degree at most k such that $\lim_{r\to 0} r^{-k}m(|u-q|;B(x,r))=0$. The polynomial $q(y)$ may then be written as

$$\sum_{|\alpha|\le k}\frac{1}{\alpha!}u_\alpha(x)(y-x)^\alpha$$

without ambiguity because the $u_\alpha(x)$ with $|\alpha|<k$ will be the same as those appearing in (2). For such x we let

$$\widetilde{m}_k(u;x):=\sup_{r>0} r^{-k}m(|u-q|;B(x,r)).$$

Otherwise we let $\widetilde{m}_k(u;x):=\infty$. Thus, for $u\in D^{(k)}$, $\widetilde{m}_k(u;\cdot)$ is defined and u_α, $|\alpha|\le k$, are all defined and finite almost everywhere. We know that u_α is measurable for $|\alpha| \le k$ (see [3] or [23]). Each u_α is called an L^1-*derivative* of u. Since both $\widetilde{m}_k(u;\cdot)$ and $\widetilde{M}_k(u;\cdot)$ are approximately lower semicontinuous wherever all u_α, $|\alpha| \le k$, are approximately continuous, they are both measurable. We prove in [14] the following two theorems:

THEOREM 5. *If $u \in L^p$, $1 < p < \infty$, then $u \in W_p^k$ if and only if $u \in D^{(k)}$ and $\widetilde{M}_k(u;\cdot) \in L^p$.*

THEOREM 6. *Let $u \in L^p$, $1 \le p < \infty$. The following statements are equivalent:*

1) *$u \in W_p^k$.*

2) *$u \in D^{(k)}$ with each L^1-derivative $u_\alpha \in L^p$ and there is a positive constant C such that*

$$|\{x \in \mathbb{R}^n : \widetilde{m}_k(u;x) > \lambda\}| \le C/\lambda^p,$$
$$\lim_{\lambda\to\infty} \lambda^p |\{x \in \mathbb{R}^n : \widetilde{m}_k(u;x) > \lambda\}| = 0.$$

3) *There is a positive constant C such that for each $\lambda > 0$ there is a closed set F_λ with the property that $|\mathbb{R}^n \setminus F_\lambda| \le C\lambda^{-p}$, $\lim_{\lambda\to\infty} \lambda^p|\mathbb{R}^n \setminus F_\lambda| = 0$, and $u|_{F_\lambda} \in t^k(F_\lambda)$ with $\|D^\alpha(u|_{F_\lambda})\|_{L^p} \le C$ and $N(u;t^k;F_\lambda) \le \lambda$.*

When $k = 1$, we denote in [14] the function $\widetilde{M}_1(u;\cdot)$ by $Q(u;\cdot)$ and call it the *maximal mean steepness* of u. If Q_w^p is the class of all those functions u with $Q(u;\cdot) \in L_w^p$, then we show in [14] that BV functions are in Q_w^1 and $W_p^1 \subset Q_w^p$. We also show in [14] that $Q_w^p \cap L^p$ enjoys a Lusin-type property which is stronger than the Lusin property of order 1 but weaker than the strong $(1,p)$-Lusin property:

THEOREM 7. *If $u \in Q_w^p \cap L^p$, $1 \le p < \infty$, then for any $\lambda > 0$ there is a Lipschitz function g with $\|g\|_{\mathrm{Lip}} \le \lambda$ and $|\{x \in \mathbb{R}^n : u(x) \ne g(x)\}| \le C\lambda^{-p}$, where C is some positive constant depending only on n, p, and u, and*

$$\|g\|_{\mathrm{Lip}} := \sup_x |g(x)| + \sup_{x\ne y} \frac{|g(y)-g(x)|}{|y-x|}.$$

We now consider another form of the Lusin property for functions in $W_{\mathrm{loc}}^k(D)$ which is suggested by the K-method in interpolation theory. For a real function u defined on D and $\lambda \ge 0$, $t \ge 0$ let

$$\mu(u;\lambda) := |\{x \in \mathbb{R}^n : |u(x)| > \lambda\}|, \quad u^*(t) := \sup\{\lambda : \mu(u;\lambda) > t\}.$$

The function u^* is called the *nonincreasing rearrangement* of u. It is well known that $\mu(u;u^*(t)) \le t$ (see, for example, [23, p. 26]).

Theorem 8. *There is a positive constant $C = C(n,k)$ such that for $u \in W^k_{\mathrm{loc}}(D)$ and $t > 0$, there exist $u_t \in C^k(\mathbb{R}^n)$ and a closed subset F_t of D so that*

(i) $|D \setminus F_t| \le 2t$;
(ii) $u_\alpha(x) = D^\alpha u_t(x)$ *for* $x \in F_t, |\alpha| \le k$; *and*
(iii) $\|u_t\|_{W^k_\infty} \le C(\sum_{|\alpha|\le k} M_0 u_\alpha)^*(t)$.

Proof. When $(\sum_{|\alpha|\le k} M u_\alpha)^*(t) = \infty$, the theorem follows from the fact that functions in $W^k_{\mathrm{loc}}(D)$ have the Lusin property of order k. We now assume $(\sum_{|\alpha|\le k} M u_\alpha)^*(t) < \infty$. For $u \in W^k_{\mathrm{loc}}(D)$ and $t > 0$, let

$$W_t := \bigg\{x \in D : \sum_{|\alpha|\le k} M_0 u_\alpha(x) \le \bigg(\sum_{|\alpha|\le k} M_0 u_\alpha\bigg)^*(t)\bigg\}.$$

Then

$$|D \setminus W_t| = \bigg|\bigg\{x \in D : \sum_{|\alpha|\le k} M_0 u_\alpha(x) > \bigg(\sum |\alpha| \le k M_0 u_\alpha\bigg)^*(t)\bigg\}\bigg| \le t.$$

Since we have

$$r^{-k} m(|u - \pi_k(u;x,\cdot)|; B(x,r)) \le \frac{1}{|B(x,r)|}\int_{B(x,r)} \frac{|u(y) - \pi_k(u;x,y)|}{|y-x|^k}\,dy,$$

by Lemma 1 we know that

$$\lim_{r\to 0} r^{-k} m(|u - \pi_k(u;x,\cdot)|; B(x,r)) = 0$$

for almost all $x \in \mathbb{R}^n$. Hence by the Egorov theorem for a continuous parameter [8, (10.2.64), p. 124] there is a closed subset F_t of W_t with $|D \setminus F_t| \le 2t$ such that

$$\lim_{r\to 0} r^{-k} m(|u - \pi_k(u;x,\cdot)|; B(x,r)) = 0$$

uniformly on compact subsets of F_t. Furthermore, we choose F_t so that

$$\sup_{r>0} r^{-k} m(|u - \pi_k(u;x,\cdot)|; B(x,r)) \le 2\sum_{|\alpha|\le k} M_0 u_\alpha(x) \le 2\bigg(\sum_{|\alpha|\le k} M_0 u_\alpha\bigg)^*(t),$$

$$|u_\alpha(x)| \le \bigg(\sum_{|\alpha|\le k} M_0 u_\alpha\bigg)^*(t)$$

for $x \in F_t, |\alpha| \le k$. We can do this because of formula (1) in Lemma 1 and the fact that $|u_\alpha(x)| \le M u_\alpha(x)$ for almost all $x \in D$. We may now apply Lemma 3 to conclude that $u|_{F_t} \in t^k(F_t)$ and

$$N(u|_{F_t}; t^k; F_t) \le 2C(n,k)\bigg(\sum_{|\alpha|\le k} M_0 u_\alpha\bigg)^*(t).$$

By Lemma 2, there is a positive constant C depending only on n and k, still denoted by $C(n,k)$, such that $u|_{F_t}$ can be extended to a function $u_t \in C^k(\mathbb{R}^n)$ with the properties stated in the conclusion of the theorem.

We now state two lemmas that we need in applying Theorem 8 to interpolation of Sobolev spaces.

Lemma 5 (Hardy–Littlewood). *If f and g are finite almost everywhere on a σ-finite measure space (X,μ), then*

$$\int_X |fg|\,d\mu \le \int_0^\infty f^*(s)g^*(s)\,ds.$$

Lemma 6. *There is a positive constant C depending only on n such that*

$$(M_0 f)^*(t) \le \frac{C}{t}\int_0^t f^*(s)\,ds,$$

for every locally integrable function f on an open subset D of $\mathbb{R}^n$ and every $t>0$.

The proof of Lemma 5 can be found in [1, p. 44], while the proof of Lemma 6 follows that of the first inequality of [1, Theorem III.3.8, p. 122].

We now prove the following theorem which is first given in [4] under certain restriction on D:

Theorem 9. *The Sobolev space $W_p^k(D)$, $1<p<\infty$, is an interpolation space between the Sobolev spaces $W_1^k(D)$ and $W_\infty^k(D)$.*

Proof. By the K-method in interpolation of Banach spaces (see [4, §5], [1, Chapter 5]), this amounts to showing that there are positive numbers a and b such that

$$\begin{aligned}(3)\quad a\sum_{|\alpha|\le k}\int_0^t (D^\alpha u)^*(s)\,ds &\le K(u,t,W_1^k(D),W_\infty^k(D))\\ &\le b\sum_{|\alpha|\le k}\int_0^t (D^\alpha u)^*(s)\,ds\end{aligned}$$

for every $u \in W_1^k(D)+W_\infty^k(D)$, where the K-functional K is defined by

$$\begin{aligned}K(u,t,W_1^k(D),W_\infty^k(D)) = \inf\{&\|u_1\|_{W_1^k} + t\|u_2\|_{W_\infty^k} : u=u_1+u_2,\\ &u_1\in W_1^k(D),\ u_2\in W_\infty^k(D)\}.\end{aligned}$$

As is shown in [4], the left inequality in (3) follows immediately from the definition of the K-functional. It remains to establish the right inequality in (3). Let

u be in $W_1^k(D) + W_\infty^k(D)$. Then $u \in W_{\mathrm{loc}}^k(D)$. For any $t > 0$, choose a closed subset F_t of D and a C^k function u_t on $\mathbb{R}^n$ as in Theorem 8. Then

$$\begin{aligned}K(u,t,W_1^k(D),W_\infty^k(D)) &\le \|u-u_t\|_{W_1^k} + t\|u_t\|_{W_\infty^k}\\ &\le \sum_{|\alpha|\le k}\|D^\alpha u\|_{L^1(D\setminus F_t)} + Ct\bigg(\sum_{|\alpha|\le k} M_0 u_\alpha\bigg)^*(t),\end{aligned}$$

where $C = C(n,k)$ is the constant in Theorem 8. If we let g be the indicator function of the set $D \setminus F_t$, then by Lemma 5 we have

$$\begin{aligned}\|D^\alpha u\|_{L^1(D\setminus F_t)} &= \int_D |D^\alpha u(x) g(x)|\,dx \le \int_0^\infty (D^\alpha u)^*(s) g^*(s)\,ds\\ &= \int_0^{|D\setminus F_t|}(D^\alpha u)^*(s)\,ds \le 2\int_0^t (D^\alpha u)^*(s)\,ds;\end{aligned}$$

while from Lemma 6 there is a positive number C_1 depending only on n such that

$$t(M_0 D^\alpha u)^*(t) \le C_1 \int_0^t (D^\alpha u)^*(s)\,ds.$$

Thus,

$$K(u,t,W_1^k(D),W_\infty^k(D)) \le \sigma(2+CC_1)\sum_{|\alpha|\le k}\int_0^t (D^\alpha u)^*(s)\,ds,$$

where we have used the fact that $(\sum_{|\alpha|\le k} M_0 D^\alpha u)^*(s) \le \sum_{|\alpha|\le k}(M_0 D^\alpha u)^*(s/\sigma)$, with $\sigma := \#\{\alpha : |\alpha| \le k\}$. We complete the proof by letting $b = 2 + CC_1$.

Finally, we remark that the strong (k,p)-Lusin property of functions in $W_p^k(D)$ follows from Theorem 8. Indeed, for $u \in W_p^k(D)$ and $t > 0$, choose F_t and u_t as in Theorem 8. Then

$$\|u-u_t\|_{W_p^k} \le A\bigg[t\bigg(\sum_{|\alpha|\le k} M_0 u_\alpha\bigg)^*(t)\bigg]^{1/p} \le A\bigg[C\sum_{|\alpha|\le k}\int_0^t u_\alpha^*(s)\,ds\bigg]^{1/p},$$

where A is a positive number depending only on n, k and p and C is the constant in Lemma 6. The strong (k,p)-Lusin property for u then follows.

References

[1] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, 1988.

[2] B. Bojarski and P. Hajłasz, *Pointwise inequalities for Sobolev functions and some applications*, Studia Math. **106** (1993), 77–92.

[3] A. P. Calderón and A. Zygmund, *Local properties of solutions of elliptic partial differential equations*, Studia Math. **20** (1961), 171–225.

[4] R. A. DeVore and K. Scherer, *Interpolation of linear operators on Sobolev spaces*, Ann. of Math. **109** (1979), 583–599.

[5] H. Federer, *Geometric Measure Theory*, Springer-Verlag, 1969.

[6] M. GIAQUINTA, G. MODICA AND J. SOUČEK, *Area and the area formula*, Rend. Sem. Mat. Fis. Milano **62** (1992), 53–87.

[7] C. GOFFMAN AND F. C. LIU, *Discontinuous mappings and surface area*, Proc. London Math. Soc. (3) **20** (1970), 237–248.

[8] H. HAHN AND A. ROSENTHAL, *Set Functions*, The University of New Mexico Press, 1947.

[9] P. HAJŁASZ, *Change of variables formula under minimal assumptions*, Colloq. Math. **64** (1993), 93–101.

[10] F. C. LIU, *On a theorem of Whitney*, Bull. Inst. Math. Acad. Sinica **1** (1973), 63–70.

[11] ———, *Essential multiplicity function for a.e. approximately differentiable mappings*, C. K. S. Memor. Vol., Acad. Sinica, 1976, pp. 69–73.

[12] ———, *A Luzin type property of Sobolev functions*, Indiana Univ. Math. J. **26** (1977), 645–651.

[13] F. C. LIU AND W. S. TAI, *Approximate Taylor polynomials and differentiation of functions*, Topol. Methods Nonlinear Anal. **3** (1994), 189–196.

[14] ———, *Maximal mean steepness and Lusin type properties*, Ricerche Mat. **43** (1994), 365–384.

[15] J. MICHAEL, *The equivalence of two areas for nonparametric discontinuous surfaces*, Illinois J. Math. **7** (1963), 59–78.

[16] J. MICHAEL AND W. P. ZIEMER, *A Lusin type approximation of Sobolev functions by smooth functions*, Contemp. Math., vol. 42, Amer. Math. Soc., 1985, pp. 135–167.

[17] C. B. MORREY, *A class of representations of manifolds, I*, Amer. J. Math. **55** (1933), 683–707.

[18] H. RADEMACHER, *Ueber partielle und totale Differenzierbarkeit I*, Math. Ann. **79** (1919), 340–359.

[19] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[20] E. M. STEIN AND A. ZYGMUND, *On the differentiability of functions*, Studia Math. **23** (1964), 247–283.

[21] W. STEPANOFF, *Sur les conditions de l'existence de la différentielle totale*, Rec. Math. Soc. Moscou **32** (1925), 511–526.

[22] H. WHITNEY, *On totally differentiable and smooth functions*, Pacific J. Math. **1** (1951), 143–159.

[23] W. P. ZIEMER, *Weakly Differentiable Functions*, Springer-Verlag, 1989.

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