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FIXED POINT INDEX FOR G-EQUIVARIANT MULTIVALUED MAPS

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Introduction

The goal of this paper is to extend the construction of the index, which was defined for a class of nonacyclic multivalued maps in [6], to the G-equivariant case (G is a finite group). Our index $\lambda_G(\Phi)$ is an element of the Burnside ring $A(G)$. We use some properties of the Burnside ring to prove several relations between the indices of the map Φ restricted to various sets of fixed points of a G-action.

We partially base on the ideas of Marzantowicz [10]. The congruences we obtain are similar to the results proved by Komiya [8] for single-valued maps.

The organization of this paper is as follows. In the first and second sections we review some of the standard facts on G-actions and multivalued maps. Section 3 contains a sketch of the definition of the index for a broad class of nonacyclic maps. Section 4 presents the construction of the G-chain approximation. In the last section we define the G-index (Def. (5.5)) and prove Komiya-type relations between indices.

1. Finite group actions

Let G be a finite group. If $H \subset G$ is a subgroup, we denote by G/H the space of left cosets Hg.

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Two subgroups H and K of G are *conjugate* if there exists $g \in G$ such that $K = g^{-1}Hg$. The conjugacy class of H is denoted by (H). There is a partial order in the set of conjugacy classes of subgroups defined as follows: $(H) \geq (K)$ if there exist $\overline{H} \in (H)$ and $\overline{K} \in (K)$ such that $\overline{K} \subset \overline{H}$. We denote by S_G a complete set of representatives of conjugacy classes in G.

If K and H are subgroups of G then the set

$$N(K,H) = \{g \in G : g^{-1}Kg \subset H\}$$

is the *normalizer* of K with respect to H.

A G-*set* is a pair (X,ξ), where X is a set and $\xi : G \times X \to X$ a map such that

(i) $\xi(g_1, \xi(g_2, x)) = \xi(g_1g_2, x)$ for $g_1, g_2 \in G$ and $x \in X$,
(ii) $\xi(e,x) = x$ for $x \in X$, where $e \in G$ is the unit.

In the sequel we denote $\xi(g,x)$ by gx.

For each subgroup $H \subset G$ the set G/H is a G-set via the formula $g(\widetilde{g}H) = g\widetilde{g}H$.

A G-*space* is a G-set (X,ξ) for which X is a topological space and ξ is continuous.

For $x \in X$, the subgroup $G_x = \{g \in G : gx = x\}$ of G is the *isotropy group* of X at x. We denote by $\mathrm{Iso}(X)$ the set of all isotropy types in X, i.e. the set of conjugacy classes of isotropy groups. The set $Gx = \{gx : g \in G\}$ is called the *orbit* through x.

For a given subgroup $H \subset G$ we specify several subspaces of the G-space X:

$$\begin{aligned} X_H &= \{x \in X : H = G_x\}, \\ X_{(H)} &= \{x \in X : (H) = (G_x)\}, \end{aligned} \tag{1.1}$$

called the (H)-*orbit bundle* of X,

$$X^H = \{x \in X : H \subset G_x\},$$

the H-*fixed point set* of X, and

$$X^{(H)} = \{x \in X : (H) \leq (G_x)\}.$$

A subset A of a G-space X is G-*invariant* (or a G-*subspace*) if $gy \in A$ for all $g \in G$ and $y \in A$.

Suppose X and Y are G-spaces. A continuous map $f : X \to Y$ is a G-*map* or a G-*equivariant map* if $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$.

A G-*complex* is a simplicial complex X which is a G-set such that for all $g \in G$ the homeomorphism $g : X \to X$ is a simplicial map. A G-complex X is *regular* if the following condition is satisfied: For all $g_0, \ldots, g_n \in G$, if $(v_0, v_1, \ldots, v_n)$ and $(g_0v_0, g_1v_1, \ldots, g_nv_n)$ are two simplices in X, then there exists $g \in G$ such that

$gv_i = g_i v_i$. For any simplicial G-complex X, its second barycentric subdivision turns out to be a regular complex (see [3], III).

In the sequel we always assume G-complexes to be regular. One shows that for X a regular G-complex, X^H is a simplicial subcomplex and $X^{(H)}$ is a G-subcomplex for each subgroup $H \subset G$.

Let $C_*(X)$ denote the oriented chain complex (with any coefficients). If X is a regular G-complex, then $C_*(X)$ is a G-set with the natural G-action given on simplices by $g(v_0, \ldots, v_n) = (gv_0, \ldots, gv_n)$.

2. Multivalued maps

Let X, Y be topological spaces. We say that $\Phi : X \to Y$ is a *multivalued map* if a compact, nonempty subset $\Phi(x)$ of Y is given for each $x \in X$. The *image* of a subset $A \subset X$ under Φ is the set $\Phi(A) = \bigcup_{x \in A} \Phi(x)$. A multivalued map $\Phi : X \to Y$ is *upper semicontinuous* (u.s.c.) provided for any open set $U \subset Y$ its *small pre-image*

$$\Phi^{-1}(U) = \{x \in X \mid \Phi(x) \subset U\}$$

is open in X. It is *lower semicontinuous* (l.s.c.) if for any open set $U \subset Y$ its *large pre-image*

$$\Phi^{+1}(U) = \{x \in X \mid \Phi(x) \cap U \neq \emptyset\}$$

is open in X. Finally, Φ is called *continuous* if it is both u.s.c. and l.s.c.

We call a compact space A *acyclic* if

$$\check{H}_i(A; F) = \begin{cases} 0 & \text{for } i > 0, \\ F & \text{for } i = 0, \end{cases}$$

where $\check{H}(\cdot, F)$ is the Čech homology functor with coefficients in the field F.

(2.1) Definition. An u.s.c. map $\Phi : X \to Y$ is *acyclic* if $\Phi(x)$ is acyclic for all $x \in X$. The class of acyclic maps from X to Y is denoted by $A_1(X, Y)$.

(2.2) Definition. Let $m > 1$ be an integer. We say that $\Phi : X \to Y$ belongs to the class $A_m(X, Y)$ if it is continuous and $\Phi(x)$ has either 1 or m acyclic components for any $x \in X$.

Example. Let $\mathbb{C}$ be the complex plane. We define a multivalued map $\Phi : \mathbb{C} \to \mathbb{C}$ by $\Phi(x) = \{z \in \mathbb{C} : z^m = x\}$. One easily checks that $\Phi \in A_m(\mathbb{C}, \mathbb{C})$.

(2.3) Definition. Let X, Y be two G-spaces. A multivalued map $\Phi : X \to Y$ is *G-equivariant* provided

(i) $\Phi(gx) = g\Phi(x)$ for all $x \in X$ and $g \in G$,
(ii) if $y, gy \in \Phi(x)$ then $y = gy$.

Note that in the case of a single-valued Φ the condition (ii) is superfluous. However, we need the natural fact:

(2.4) Proposition. *If $\Phi : X \to Y$ is G-equivariant, then for each subgroup $H \subset G$,*

(i) $\Phi(X^H) \subset Y^H$,
(ii) $\Phi(X^{(H)}) \subset Y^{(H)}$.

Proof. Let $x \in X^H$, $y \in \Phi(x)$ and $g \in H$. Then

$$y \in \Phi(x) = \Phi(g^{-1}x) = g^{-1}\Phi(x).$$

So $y \in \Phi(x)$ and $gy \in \Phi(x)$ and thus by (2.3)(ii), $y = gy$, i.e. $y \in Y^H$.

We denote by $D(X,Y)$ the set of all compositions of the form

$$X = X_0 \xrightarrow{\Phi_1} X_1 \xrightarrow{\Phi_2} X_2 \to \ldots \to X_{n-1} \xrightarrow{\Phi_n} X_n = Y,$$

where $\Phi_i \in A_k(X_{i-1}, X_i)$ for some $k = 1, 2, \ldots$

We can view elements of $D(X,Y)$ as morphisms. They determine u.s.c. maps. However, two different compositions may determine the same map (cf. [6]).

We say that $\Phi = (\Phi_1, \ldots, \Phi_n) \in D(X,Y)$ is *G-equivariant* if all the X_i are G-spaces and the Φ_i are G-equivariant maps.

We also consider an assumption stronger than acyclicity:

(S) Let $A \subset X$. For each component A_i of A and for each neighbourhood $A_i \subset U \subset X$ there exists a smaller neighbourhood V with $A_i \subset V \subset U$ such that the inclusion $i : V \to U$ induces the trivial homomorphism $i_* : H_*(V, \mathbb{Z}) \to H_*(U, \mathbb{Z})$.

Observe that if A_i has a trivial shape, then it satisfies (S). Denote by A_i^{S} the class of maps $\Phi \in A_i(X,Y)$ satisfying (S), and by $D_{\mathrm{S}}(X,Y)$ their compositions.

3. Chain approximations and index

In this section we sketch the fixed point theory for elements of $D(X,X)$ as given in [6]. We use the chain approximation technique developed in [12]. Let (K,τ) be a compact polyhedron with a fixed triangulation τ. Its nth barycentric subdivision is denoted by τ^n. A subset $U \subset K$ is *polyhedral* if there is an integer l such that τ^l induces a triangulation of the closure $\overline{U}$ of U in K. The kth *closed star* of a subset B in K is defined recursively:

$$\mathrm{St}^1(B,\tau) = \mathrm{St}(B,\tau) = \bigcup\{\sigma \in \tau : \sigma \cap B \neq \emptyset\},$$
$$\mathrm{St}^k(B,\tau) = \mathrm{St}(\mathrm{St}^{k-1}(B,\tau),\tau).$$

A simplex $\sigma \in \tau$ is always assumed to be closed. Let l be a natural number and F a field. We denote by $C_*(K,\tau^l)$ the oriented chain complex $C_*(K,\tau^l;F)$

(see [14]). The *carrier* of $c \in C_*(K,\tau)$, $\operatorname{carr} c$, is the smallest subpolyhedron $X \subset K$ such that $c_* \in C_*(X,\tau)$. We denote by $b : C_*(K,\tau) \to C_*(K,\tau^l)$ the standard barycentric subdivision map and by $\chi : C_*(K,\tau^l) \to C_*(K,\tau)$ any chain mapping induced by a simplicial approximation of the identity $\mathrm{id} : (K,\tau^l) \to (K,\tau)$.

(3.1) Definition. Let $\Phi : (K,\tau) \to (L,\mu)$ be an u.s.c. multivalued map and l, k natural numbers. A chain map $\varphi : C_*(K,\tau^l) \to C_*(L,\mu^k)$ is called an (n,k)-*approximation* of Φ if for each simplex $\sigma \in \tau^l$ there exists a point $y(\sigma) \in K$ such that

$$\sigma \subset \mathrm{St}^n(y(\sigma),\tau^k) \quad \text{and} \quad \operatorname{carr} \varphi\sigma \subset \mathrm{St}^n(\Phi(y(\sigma)),\mu^k).$$

(3.2) Definition. A graded set $A(\Phi) = \{A(\Phi)_j\}_{j\in\mathbb{N}}$, where

$$A(\Phi)_j \subset \hom(C_*(K,\tau^j),C_*(L,\mu^j)),$$

is called an *approximation system* (A-*system*) for Φ if there is an integer $n = n(A)$ such that

(3.2.1) if $\varphi \in A(\Phi)_j$, then $\varphi = \varphi_1 \circ b$, where φ_1 is an (n,j)-approximation of Φ;

(3.2.2) for every $j \in \mathbb{N}$ there exists $j_1 \in \mathbb{N}$ such that for $m \geq l \geq j_1$ and for all $\varphi = \varphi_1 \circ b \subset A(\Phi)_l$ and $\psi = \psi_1 \circ b \in A(\Phi)_m$ and $m_1 \geq l_1$ the diagram

$$\begin{array}{ccc} C_*(K,\tau^{l_1}) & \xrightarrow{\varphi_1} & C_*(L,\mu^l) \\ \Big\uparrow \chi & & \Big\uparrow \chi \\ C_*(K,\tau^{m_1}) & \xrightarrow{\psi_1} & C_*(L,\mu^m) \end{array}$$

is homotopy commutative with a chain homotopy D satisfying the following *smallness* condition: for any simplex $\sigma \in \tau^{m_1}$ there exists a point $z(\sigma) \in K$ such that

$$(*) \qquad \sigma \subset \mathrm{St}^n(z(\sigma),\tau^j) \quad \text{and} \quad \operatorname{carr} D(\sigma) \subset \mathrm{St}^n(\Phi(z(\sigma)),\mu^j).$$

The above definition looks a little sophisticated, but it allows us to define the index properly. Let $U \subset K$ be an open polyhedral subset and let $\Phi : \overline{U} \to K$ be an u.s.c. map such that $x \notin \Phi(x)$ for $x \in \partial U$. Let $A(\Phi)$ be an A-system for Φ. Then the index $\mathrm{ind}_A(K,\Phi,U) \in F$ is defined as follows: Denote by

$$p_U : C_*(K,\tau^k) \to C_*(\overline{U},\tau^k)$$

the natural linear projection. Let $\varphi \in A(\Phi)_k$. Then the local Lefschetz number is defined by the formula

$$\lambda(p_U \circ \varphi) = \sum_{i=0}^{\dim K} (-1)^i \operatorname{tr}(p_U \circ \varphi)_i.$$

It is proved in [12] that for k_0 sufficiently large the above element of F is independent of the choice of $\varphi \in A(\Phi)_k$ $(k \geq k_0)$, since all the approximations are small homotopic (i.e. they satisfy (3.2.2)(∗)).

(3.3) Definition.

$$\operatorname{ind}_A(K, \Phi, U) := \lambda(p_U \circ \varphi) \quad \text{for } \varphi \in A(\Phi)_k.$$

This index satisfies all the standard properties of a fixed point index (although it may depend on the choice of an A-system for Φ in general). For detailed proofs see [12].

Therefore the existence of an index theory for any class of u.s.c. maps reduces to the existence of an A-system. For example, if Φ is a single-valued continuous map, then the set of all chain maps induced by simplicial approximations of Φ forms an A-system and by the uniqueness theorem it gives the classical Hopf fixed point index. In [12] the existence of A-systems for acyclic maps was also proved. Moreover, all A-systems for such maps are equivalent (see [12]). The existence of an A-system for elements of $D(X, X)$ is the main result of [6].

This index theory can be generalized to the more general situations where X is a compact ANR-space by using r-domination arguments (see [6] for detailed proofs).

4. Equivariant chain approximations

In this section we shall prove that a G-equivariant set-valued map from $A_i(X, Y)$ has equivariant chain approximations.

We adapt the proof from [6]. We start by recalling some notation (cf. [3]). Let G be a finite group and X, Y compact G-spaces.

(4.1) Definition. An open covering $\alpha \in \operatorname{Cov} X$ is a *G-covering* if

(i) $U \in \alpha$ implies that $gU \in \alpha$ for each $g \in G$,
(ii) $U \cap gU \neq \emptyset \Rightarrow U = gU$, for each $U \in \alpha$ and $g \in G$.

(4.2) Definition. A G-covering α of X is *regular* if for each subgroup $H \subset G$ the following condition holds: If $U_0 \cap \ldots \cap U_n \neq \emptyset \neq h_0 U_0 \cap \ldots \cap h_n U_n$ for some $U_i \in \alpha$ and $h_i \in H$, then there exists $h \in H$ such that $hU_i = h_i U_i$ for $i = 0, 1, \ldots, n$.

Recall that the *nerve* $N(\alpha)$ of the covering $U \in \operatorname{Cov} X$ is a simplicial complex with all $U \in \alpha$ as vertices. $(U_0, \ldots, U_n)$ forms a simplex in $N(\alpha)$ if $U_0 \cap \ldots \cap U_n \neq \emptyset$. So U being a G-covering implies that $N(\alpha)$ is a G-complex, and U being regular implies that $N(\alpha)$ is a regular G-complex. Denote the family of all finite regular G-coverings of X by $\operatorname{Cov}_G X$.

(4.3) PROPOSITION (see [3]). *If X is a compact G-space then* $\operatorname{Cov}_G X$ *is a cofinal family in* $\operatorname{Cov} X$.

Let $\alpha, \beta \in \operatorname{Cov}_G X$ and let α be a refinement of β. Then there exists a natural map $\Pi^\alpha_\beta : \alpha \to \beta$ which is equivariant, i.e. $U \subset \Pi^\alpha_\beta(U)$ and $\Pi^\alpha_\beta(gU) = g\Pi^\alpha_\beta(U)$. We denote by $N^{(n)}(\alpha)$ the n-skeleton of $N(\alpha)$ and by $C_*(N^{(n)}(\alpha))$ the complex of oriented chains with coefficients in a field F. The *Kronecker index* of a 0-chain $c = \sum c_i\sigma_i \in C_0(N^{(n)}(\alpha))$ is the sum $\sum c_i$.

(4.4) DEFINITION (see [6]). Let $\alpha, \overline{\alpha} \in \operatorname{Cov} X$, $\beta, \overline{\beta} \in \operatorname{Cov} Y$ and $\Phi \in A_m(X, Y)$. A chain map

$$\varphi : C_*(N^{(n)}(\overline{\alpha})) \to C_*(N^{(n)}(\overline{\beta}))$$

is an (α, β)-*approximation* of Φ if

(i) φ multiplies the Kronecker index by m,

(ii) for each simplex $\sigma \in N^{(n)}(\overline{\alpha})$ there exists a point $p(\sigma) \in X$ such that

$$\operatorname{supp} \sigma \subset \operatorname{St}(p(\sigma), \alpha), \quad \operatorname{supp} \varphi(\sigma) \subset \operatorname{St}(\Phi(p(\sigma)), \beta),$$

(iii) for any vertex $v \in C_0(N(\overline{\alpha}))$,

$$\operatorname{supp} \varphi(v) \cap \operatorname{St}(C_j, \beta) \neq \emptyset,$$

where the C_j are connected components of the set $\Phi(p(v))$.

The following theorem is an analogue of the classical simplicial approximation theorem.

(4.5) THEOREM ([6], 4.3). *Let X, Y be compact spaces, $\Phi \in A_m(X, Y)$ and $\alpha \in \operatorname{Cov} X$, $\beta \in \operatorname{Cov} Y$. For each $n \in \mathbb{N}$ there exist a refinement $\overline{\alpha}$ of α and an (α, β)-approximation $\varphi : C_*(N^{(n)}(\overline{\alpha})) \to C_*(N^{(n)}(\beta))$ of Φ.*

Our aim is to obtain a G-equivariant version of (4.5). We start with a technical result.

(4.6) LEMMA. *Let X, Y be two compact G-spaces and $\Phi \in A_m(X, Y)$ a G-equivariant map. For any finite G-coverings $\alpha_0 \in \operatorname{Cov}_G X$ and $\beta_0 \in \operatorname{Cov}_G Y$ and $n \in \mathbb{N}$ there exist sequences of coverings $\alpha_i \in \operatorname{Cov}_G X$ and $\beta_i \in \operatorname{Cov}_G Y$ with*

$$\alpha_{n+1} \geq \alpha_n \geq \ldots \geq \alpha_0, \quad \beta_{n+1} \geq \beta_n \geq \ldots \geq \beta_0$$

such that for each simplex $s \in N(\alpha_i)$ there exist a point $a(s) \in X$ and a covering $\beta_{i-1}(s) \in \operatorname{Cov}_G Y$ $(\beta_i \geq \beta_{i-1}(s) \geq \beta_{i-1})$ with the following properties:

(i) $\operatorname{supp} s \subset \operatorname{St}(a(s), \alpha_{i-1})$,

(ii) $a(gs) = g(a(s))$,

(iii) $\Phi(\operatorname{St}(\operatorname{supp} s, \alpha_i)) \subset \operatorname{St}(\Phi(a(s)), \beta_{i-1}(s))$,

(iv) *if* $C_j(a(s))$ *are the components of* $\Phi(a(s))$*, then the sets* $\mathrm{St}^2(C_j(a(s)),$ $\beta_{i-1}(s))$ *are pairwise disjoint,*

(v) $\Phi(y) \cap \mathrm{St}(C_j(a(s)), \beta_{i-1}(s)) \neq \emptyset$ *for all* $y \in \mathrm{St}(\operatorname{supp} s, \alpha_i)$*,*

(vi) $$\Pi^{\beta_{i-1}(s)}_{\beta_{i-1}}{}_* : \check{H}_*(N(\beta_{i-1}(s)|_{\mathrm{St}^2(C_j(a(s)),\beta_{i-1}(s))}) \to \check{H}_*(N(\beta_{i-1})|_{\mathrm{St}(\Phi(a(s)),\beta_{i-1})})$$

is a zero homomorphism of reduced homology spaces.

PROOF. Let $n = 0$. For each $x \in X$ every component C_j of $\Phi(x)$ is acyclic, so by continuity of the Čech homology functor there exists $\beta \geq \beta_0(x) \in \mathrm{Cov}_G X$ such that $\mathrm{St}^2(C_j, \beta_0(x))$ are pairwise disjoint and

$$\Pi^{\beta}_{\beta_0} : \check{H}_*(N(\beta)|_{\mathrm{St}^2(C_j,\beta)}) \to \check{H}_*(N(\beta_0)|_{\mathrm{St}(\Phi(x),\beta_0)})$$

are trivial homomorphisms (cf. [6], 1.3).

Since Φ is continuous, there exists a neighbourhood U_x of x such that

(i) $\Phi(U_x) \subset \mathrm{St}(\Phi(x), \beta)$,

(ii) $\Phi(y) \cap \mathrm{St}(C_j, \beta) \neq \emptyset$ for each $y \in U_x$.

Observe that the above property is nothing new whenever $\Phi(x)$ is acyclic, so the l.s.c. assumption is superfluous in that case.

Without loss of generality we can assume that the covering $\{U_x\}_{x\in X}$ is a regular G-covering and refines α_0. Now we choose a finite G-subcovering $\{U_{x_i}\}_{i=1}^k$ (with the property that if $x = x_i$, then $gx_i = x_l$ for some $l = 1, \ldots, k$). Let α_1 be a finite regular G-covering of X which is a star-refinement of $\{U_{x_i}\}$. For a simplex $s \in N(\alpha_1)$ we define $a(s) := x_i$ where $\operatorname{supp} s \subset U_{x_i}$, and $a(gs) := gx_i$.

Now set $\beta_0(s) := \beta_0(x_i)$ and let β_1 be a common G-regular refinement of all $\beta_0(x_i)$. The same procedure works inductively for any n.

(4.7) THEOREM. *Let* X, Y *be two compact* G*-spaces,* $\Phi \in A_m(X, Y)$ *a* G*-map, and* $\alpha \in \mathrm{Cov}_G X$*,* $\beta \in \mathrm{Cov}_G Y$*. For every* $n \in \mathbb{N}$ *there exist a refinement* $\overline{\alpha} \in \mathrm{Cov}_G X$ *of* α *and a* G*-equivariant* (α, β)*-approximation*

$$\varphi : C_*(N^{(n)}(\overline{\alpha})) \to C_*(N^{(n)}(\beta)) \quad \textit{of } \Phi.$$

PROOF. We take the sequences (α_i, β_i) from Lemma (4.6) with $\alpha_0 = \alpha$, $\beta_0 = \beta$ and define $\overline{\alpha} = \alpha_{n+1}$.

The desired chain map φ is constructed inductively. Since the proof is similar to [6], 4.3, we only present the first two steps.

$k := 0$: Let s_0 be a vertex of $N(\overline{\alpha})$. By (4.6) we have a point $a(s_0) \in X$. For $\Phi(a(s_0))$ connected we define $\varphi_0 s_0 := m\overline{a}$, where $\overline{a}$ is an arbitrary vertex of $N(\beta_{n+1})$ with $\operatorname{supp} \overline{a} \subset \mathrm{St}(\Phi(a(s_0)), \beta(s_0))$.

If $\Phi(a(s_0))$ consists of m components, then

$$\varphi_0 s_0 := a_1 + \ldots + a_m,$$

where the a_i are vertices of $N(\beta_{n+1})$ such that $\operatorname{supp} a_i \subset \mathrm{St}(C_i(a(s_0)), \beta_n(s_0))$. For a vertex gs_0 in the same orbit we have the same situation with $\Phi(a(gs_0)) = \Phi(ga(s_0))$ and we define $\varphi_0 gs_0 := mga$ or

$$\varphi_0 gs_0 := ga_1 + \ldots + ga_m,$$

respectively. Then we extend it to a linear G-map $\varphi_0 : C_0(N(\alpha_{n+1})) \to C_0(N(\beta_{n+1}))$.

$k := 1$: Let s be a 1-simplex in $N(\overline{\alpha})$ (the first one of a given orbit). Then $\partial s = s_1 - s_0$. Since the points $a(s_0)$ and $a(s_1)$ belong to $\mathrm{St}(\operatorname{supp} s, \alpha_n)$, we have

$$\Phi(a(s_0)) \cup \Phi(a(s_1)) \subset \mathrm{St}(\Phi(a(s)), \beta_{n-1}(s)))$$

by (4.6)(iii). Let

$$\varphi_0 \partial s = \sum a_i - \sum b_i, \quad a_i, b_i \in C_0(N(\beta_{n+1})).$$

If $\Phi(a(s))$ is connected, then by (4.6)(vi),

$$\Pi^{\beta_{n+1}}_{\beta_{n-1}} \Big(\sum (a_i - b_i) \Big) = \sum \partial c_i, \quad \text{where } c_i \in C_1(N(\beta_{n-1})).$$

If $\Phi(a(s)) = \bigcup_{i=1}^m C_i(a(s))$, then

$$\operatorname{supp}(a_i - b_i) \subset \mathrm{St}(C_i(a(s)), \beta_{n-1}(s))$$

for each pair a_i, b_i. Thus

$$\Pi^{\beta_{n+1}}_{\beta_{n-1}} (a_i - b_i) = \partial c_i, \quad \text{where } \operatorname{supp} c_i \subset \mathrm{St}(C_i(a(s)), \beta_{n-1}).$$

Now we can define $\varphi_1 s := \sum c_i$. For 1-simplices from the same orbit we define φ_1 by equivariance: $\varphi_1 gs := \sum gc_i$. This definition is correct provided (4.6)(vi) is satisfied uniformly for all gs. We can assume this is the case by choosing sufficiently fine refinements.

We obtain a commutative diagram

$$\begin{tikzcd} C_0(N(\alpha_{n+1})) \arrow[r, "\varphi_0"] & C_0(N(\beta_{n+1})) \arrow[r, "\Pi^{\beta_{n+1}}_{\beta_{n-1}}"] & C_0(N(\beta_{n-1})) \\ C_1(N(\alpha_{n+1})) \arrow[u, "\partial"] \arrow[rr, "\varphi_1"] & & C_1(N(\beta_{n-1})) \arrow[u, "\partial"] \end{tikzcd}$$

where $\Pi^{\beta_{n+1}}_{\beta_{n-1}}$ is also G-equivariant. Therefore

$$\varphi_1 : C_*(N^{(1)}(\alpha_{n+1})) \to C_*(N^{(1)}(\beta_{n-1}))$$

has been defined (on 0-chains $\varphi_1 c := (\Pi \circ \varphi_0)c$). This procedure is now continued inductively and in the nth step one obtains the desired approximation which is G-equivariant by definition.

Now let (K, τ) be a compact polyhedron with a fixed triangulation τ. We associate a covering $\alpha(\tau)$ with τ:

$$\alpha(\tau) := \{\xi : \xi = \operatorname{Int} \operatorname{St}(\upsilon_i, \tau)\},$$

where the υ_i are vertices of τ. There are simplicial maps $\Theta : (K, \tau) \to N(\alpha(\tau))$ and $\lambda : N(\alpha(\tau)) \to (K, \tau)$ defined on vertices by $\Theta(\upsilon) := \operatorname{St}(\upsilon, \tau)$ and $\lambda(\operatorname{St}(\upsilon, \tau)) := \upsilon$. These maps define a canonical simplicial isomorphism between the complexes (K, τ) and $N(\alpha(\tau))$. Moreover,

$$\operatorname{carr} s \subset \operatorname{supp} \Theta s \quad \text{and} \quad \operatorname{supp} \sigma \subset \operatorname{St}(\operatorname{carr} \lambda\sigma, \alpha(\tau)).$$

Let $(\Phi_1, \ldots, \Phi_k) \in D(K, L)$. Let τ be a triangulation of K and μ a triangulation of L. Define $A_j(\Phi_1, \ldots, \Phi_k)$ to be the set of chain maps $\varphi : C_*(K, \tau^j) \to C_*(L, \mu^j)$ which are of the form $\varphi = \lambda \circ \varphi_k \circ \ldots \circ \varphi_1 \circ \Theta \circ b$, where b is the standard subdivision map. The graded set $\{A_j(\Phi_1, \ldots, \Phi_k)\}_j$ is an A-system for the map Φ determined by $(\Phi_1, \ldots, \Phi_k)$ (see [6], 5.3).

(4.8) Theorem. *Assume that K, L are compact G-polyhedra, and let $\Phi = (\Phi_1, \ldots, \Phi_k) \in D(K, L)$. If all the spaces in the sequence*

$$X_0 = L \xrightarrow{\Phi_1} X_1 \to \ldots \to X_{k-1} \xrightarrow{\Phi_k} K$$

are G-spaces, and the maps Φ_i are G-equivariant, then the above-defined A-system for Φ contains G-equivariant chain maps in each $A_i(\Phi_1, \ldots, \Phi_k)$.

Proof. It is enough to observe that the canonical maps Θ, λ, b are equivariant if K, L are G-complexes and use (4.7).

(4.9) Remark. Observe that if $\Phi \in D_{\mathrm{S}}(X, Y)$ then it admits chain approximations with integral coefficients. The same proof works.

5. Index of equivariant multivalued maps

Let G be a finite group.

(5.1) Definition (see [4], [5]). Let $B(G)$ be the semiring of all finite G-sets (up to isomorphism) with disjoint union as addition and cartesian product as multiplication. The *Burnside ring* $A(G)$ of G is the universal ring of $B(G)$ in the sense of Grothendieck.

The additive structure of $A(G)$ is the free abelian group generated by the G-sets of the form $[G/H]$, where (H) runs through the elements of S_G. Let H be a subgroup of G and S, T finite G-sets. Denoting by $|X|$ the cardinality of the set X we have

$$|(S + T)^H| = |S^H| + |T^H|, \quad |(S \times T)^H| = |S^H||T^H|.$$

Therefore the map $S \to |S^H|$ extends to a homomorphism $\chi^H : A(G) \to \mathbb{Z}$. Since for conjugate subgroups the above homomorphisms are the same, we can define

$$\chi = (\chi^H) : A(G) \to \prod_{(H)\in S_G} \mathbb{Z}.$$

(5.2) Theorem (see [4], [5]). *The map χ is an injective ring homomorphism.*

Let us recall the notion of the regular representation reg_H^F of the group G over a field F. As a linear space, reg_H^F has a basis $\{e_{[g]}\}$ indexed by elements of the G-set G/H. A linear G-action is given by $\overline{g}e_{[g]} = e_{[\overline{g}g]}$. Let $M = k{*}\mathrm{reg}_H^F$. We denote by $M^{[K]}$ the subspace spanned by those elements $e_{[g]}$ for which $[g] \in (G/H)^K$.

(5.3) Theorem ([10]). *Let M be the direct sum of a finite number of the spaces reg_H^F, and let $f : M \to M$ be a G-equivariant homomorphism such that $f(M^{[K]}) \subset M^{[K]}$. Then*

$$\operatorname{tr} f \equiv 0 \bmod |G/H|, \quad \operatorname{tr}(f|_{M^{[K]}}) = \frac{|N(K,H)|}{|G|}\operatorname{tr} f.$$

Proof. It is enough to calculate the trace of f restricted to one component reg_H^F. Let

$$f(e_{[g]}) = \sum_{[\widehat{g}]\in G/H} c_{g,\widehat{g}}e_{[\widehat{g}]}.$$

For each $[\overline{g}] \in G/H$ there is $h \in G$ such that $[\overline{g}] = [hg]$, therefore

$$\begin{aligned} f(e_{[\overline{g}]}) &= f(e_{[hg]}) = f(he_{[g]}) = hf(e_{[g]}) \\ &= h\biggl(\sum_{[\widehat{g}]\in G/H} c_{g,\widehat{g}}e_{[\widehat{g}]}\biggr) = \sum_{[\widehat{g}]\in G/H} c_{g,\widehat{g}}e_{[h\widehat{g}]}. \end{aligned}$$

Thus the coefficient $c_{g,g}$ is equal to $c_{\overline{g},\overline{g}}$. Since the basis of reg_H^F consists of $|G/H|$ elements, we have $\operatorname{tr} f \equiv 0 \bmod |G/H|$. Now we find the dimension of the space $M^{[K]}$. We have

$$\begin{aligned} g \in N(K,H) &\Leftrightarrow g^{-1}Kg \subseteq H \Leftrightarrow KgH \subseteq gH \\ &\Leftrightarrow gH \in (G/H)^K \Leftrightarrow [g] \in (G/H)^K \\ &\Leftrightarrow e_{[g]} \text{ is an element of the basis of } M^{[K]}. \end{aligned}$$

(Incidentally, note the relation $|N(K,H)|/|H| = |(G/H)^K|$.) Thus the subspace $M^{[K]}$ is spanned by $k \cdot N(K,H)/|H|$ elements from the basis of M. The coefficients $c_{g,g}$ of the matrix of f corresponding to the basis elements from a given space reg_H^F are equal. By summing these diagonal coefficients we obtain

$$\frac{|N(K,H)|}{|H|}\operatorname{tr} f = |G/H| \operatorname{tr}(f|_{M^{[K]}}),$$

which proves the second assertion of the theorem.

(5.4) COROLLARY. *If $M_* = \bigoplus M_i$, where the M_i are as in (5.3), then we have similar relations for the Lefschetz numbers:*

(i) $\lambda(f_*, M_*) \equiv 0 \bmod |G/H|$,
(ii) $\lambda(f_*|_{M_*^{[K]}}, M_*^{[K]}) = \frac{|N(K,H)|}{|G|}\lambda(f_*, M_*)$, *where* $M_*^{[K]} = \bigoplus M_i^{[K]}$ *and* $f_* = \bigoplus f_i : M_* \to M_*$ *is a graded G-equivariant map.*

Now let (K, τ) be a compact G-polyhedron, and U an invariant open G-subset of K. Let $\Phi \in D(\overline{U}, K)$ be G-equivariant and such that for any subgroup $H \subset G$ we have $x \notin \Phi(x)$ for $x \in \partial U^H \cup \partial U^{(H)}$. By (4.8) we know that for sufficiently large $j \geq j_0$ there are G-equivariant chain maps

$$\varphi : C_*(\overline{U}, \tau^j) \to C_*(K, \tau^j).$$

Since the linear projection

$$p_U : C_*(K, \tau^j) \to C_*(\overline{U}, \tau^j)$$

is G-equivariant, we can assume that the map $\psi = p_U \circ \varphi$ defining the index

$$\operatorname{ind}_A(K, \Phi, U) = \lambda(p_U \circ \varphi)$$

in (3.3) is G-equivariant. Assume for simplicity that the coefficient field F is $\mathbb{Q}$. Let L, N, H be subgroups of G. Observe that the G-endomorphism

$$\psi = p_U \circ \varphi : C_*(\overline{U}, \tau^j) \to C_*(\overline{U}, \tau^j)$$

maps the subspace $C_*(\overline{U}^{(H)})$ into itself, and also the subset $C_*(\bigcup_{(L)>(H)} \overline{U}^{(L)})$ into itself. Therefore we obtain a quotient map

$$\psi_{(H)} : C_*(\overline{U}^{(H)})/C_*\biggl(\bigcup_{(L)>(H)} \overline{U}^{(L)}\biggr) \to C_*(\overline{U}^{(H)})/C_*\biggl(\bigcup_{(L)>(H)} \overline{U}^{(L)}\biggr).$$

Note that

$$C_*\biggl(\overline{U}^{(H)}, \bigcup_{(L)>(H)} \overline{U}^{(L)}\biggr) = C_*(\overline{U}^{(H)})/C_*\biggl(\bigcup_{(L)>(H)} \overline{U}^{(L)}\biggr).$$

We can draw the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & C_i(\bigcup_{(L)>(H)} \overline{U}^{(L)}) \arrow[r] & C_i(\overline{U}^{(H)}) \arrow[r] & C_i(\overline{U}^{(H)}, \bigcup_{(L)>(H)} \overline{U}^{(L)}) \arrow[r] & 0 \\
0 \arrow[r] & C_i(\bigcup_{(L)>(H)} \overline{U}^{(L)}) \arrow[r] \arrow[u, "\psi_2"'] & C_i(\overline{U}^{(H)}) \arrow[r] \arrow[u, "\psi_1"'] & C_i(\overline{U}^{(H)}, \bigcup_{(L)>(H)} \overline{U}^{(L)}) \arrow[r] \arrow[u, "\psi_{(H)}"'] & 0
\end{tikzcd}$$

where ψ_1, ψ_2 are restrictions of ψ.

By the additivity of the trace function, $\operatorname{tr}\psi_1 = \operatorname{tr}\psi_2 + \operatorname{tr}\psi_{(H)}$. Therefore we obtain the following equation for the Lefschetz numbers:

$$\lambda(\psi_{(H)}) = \lambda(\psi_1) - \lambda(\psi_2).$$

Observe that ψ_1, ψ_2 define the indices of the restrictions of the map ϕ to the sets $\overline{U}^{(H)}$ and $\bigcup_{(L)>(H)} \overline{U}^{(L)}$, respectively (cf. (3.3)). Therefore

$$\begin{aligned}\lambda(\psi_{(H)}) = {} & \operatorname{ind}_A(K^{(H)}, \Phi, \operatorname{Int}(\overline{U}^{(H)})) \\ & - \operatorname{ind}_A\Big(\bigcup_{(L)>(H)} K^{(L)}, \Phi, \operatorname{Int}\Big(\bigcup_{(L)>(H)} \overline{U}^{(L)}\Big)\Big).\end{aligned}$$

We can now define an element $\lambda_G(\Phi) \in A(G) \otimes \mathbb{Q}$:

(5.5) Definition.

$$\lambda_G(\Phi) := \sum_{H\in S_G} \frac{\lambda(\psi_{(H)})}{|G/H|}[G/H].$$

Denote by $\psi^H : C_*(\overline{U}^H) \to C_*(\overline{U}^H)$ the restriction of ψ to this subspace.

(5.6) Theorem (cf. [10], 2.1). *Let $\Phi \in D_{\mathrm{S}}(\overline{U}, K)$ and $H \subset G$ a subgroup. Then*

(i) $\lambda_G(\Phi) \in A(G)$,
(ii) $\chi^H(\lambda_G(\Phi)) = \lambda(\psi^H) = \operatorname{ind}_A(K^H, \Phi, \operatorname{Int}\overline{U}^H)$.

Proof. Let $T \subset G$ be a subgroup. Observe that $\overline{U}^{(T)} - \bigcup_{(L)>(T)} \overline{U}^{(L)}$ is a G-space with only one orbit type (T). Thus the space $C_i(\overline{U}^{(T)}, \bigcup_{(L)>(T)} \overline{U}^{(T)})$ is of the form $\bigoplus_{G\sigma} \operatorname{reg}_T^{\mathbb{Q}}$, where the sum runs over the orbits of i-dimensional simplices. Moreover, the space

$$C_i\Big((\overline{U}^{(T)})^H, \Big(\bigcup_{(L)>(T)} \overline{U}^{(L)}\Big)^H; \mathbb{Q}\Big)$$

is generated by those simplices from each orbit $G\sigma$ which belong to $(G/G_\sigma)^H = (G/T)^H$. Now we apply Corollary (5.4) with

$$M_i = C_i\Big(\overline{U}^{(T)}, \bigcup_{(L)>(T)} \overline{U}^{(L)}; \mathbb{Q}\Big),$$

$$M_i^{[H]} = C_i\Big((\overline{U}^{(T)})^H, \Big(\bigcup_{(L)>(T)} \overline{U}^{(L)}\Big)^H; \mathbb{Q}\Big).$$

The maps f_i are defined by ψ. Since $\Phi \in D_{\mathrm{S}}$, we can assume that ψ is given by an integer matrix.

By (5.4)(i), $\lambda(\psi_{(T)}) \equiv 0 \bmod |G/T|$. Therefore $\lambda_G(\Phi) \in A(G)$.

By (5.4)(ii) we have

$$\lambda(\psi^H_{(T)}) = \frac{|N(H,T)|}{|G|}\lambda(\psi_{(T)}).$$

On the other hand,

$$\chi^H([G/T]) = |(G/T)^H| = \frac{|N(H,T)|}{|T|}.$$

Therefore

$$\chi^H(\lambda_G(\Phi)) = \chi^H\bigg(\sum_{T\in S_G}\frac{\lambda(\psi_{(T)})}{|G/T|}[G/T]\bigg) = \sum_{T\in S_G}\frac{\lambda(\psi_{(T)})}{|G/T|}\cdot\frac{|N(H,T)|}{|T|}$$
$$= \sum_{T\in S_G}\lambda(\psi_{(T)}^H) = \lambda(\psi^H) = \mathrm{ind}_A(K^H,\Phi,\mathrm{Int}\,\overline{U}^H).$$

This ends the proof.

(5.7) Corollary. *If* $|G/H| \equiv 0 \bmod r$ *for each subgroup* $H\subset G$ *such that* $\overline{U}_{(H)} \neq \emptyset$, *then* $\mathrm{ind}_A(K,\Phi,U) \equiv 0 \bmod r$.

Proof. From (5.6) we know that $\lambda_G(\Phi)\in A(G)$ and $\lambda(\psi_{(H)})/|G/H| \in \mathbb{Z}$. On the other hand,

$$\chi^e(\lambda_G(\Phi)) = \lambda(\psi^e) = \mathrm{ind}(K,\Phi,U)$$

and, by definition of χ^e, $\chi^e([G/H]) = |G/H|$. Therefore

$$\mathrm{ind}_A(K,\Phi,U) = \lambda(\psi) = \chi^e(\lambda_G(\Phi))$$
$$= \sum_{H\in S_G}\frac{\lambda(\psi_{(H)})}{|G/H|}\chi^e([G/H]) \equiv 0 \bmod r.$$

(5.8) Corollary. *If* G *is a* p*-group, then*

$$\mathrm{ind}_A(K,\Phi,U) \equiv \mathrm{ind}_A(K^G,\Phi,\overline{U}^G) \bmod p.$$

Proof. For G a p-group we have the following relation in $A(G)$:

$$\chi^G(\alpha) \equiv \chi^e(\alpha) \bmod p \quad \text{for each } \alpha\in A(G)$$

(see [13], Th. 10.3). Therefore

$$\mathrm{ind}_A(K,\Phi,U) = \lambda(\psi) = \chi^e(\lambda_G(\psi))$$
$$\equiv \chi^G(\lambda_G(\psi)) \bmod p$$
$$= \lambda(\psi^G) = \mathrm{ind}_A(K^G,\Phi,\overline{U}^G).$$

The above corollaries correspond to relations given in [10] for Lefschetz numbers of single-valued maps.

The following formula has been obtained by Komiya [8] for single-valued maps.

(5.9) Corollary. *For each* $L\in S_G$ *we have*

$$\mathrm{ind}_A(K^L,\Phi,\mathrm{Int}\,\overline{U}^L) = \sum_{(H)\geq(L)}\frac{|N(L,H)|}{|H|}a_{(H)}(\Phi),$$

where the $a_{(H)}(\Phi)$ *are integers.*

Proof. By (5.6) we have

$$\operatorname{ind}_A(K^L, \Phi, \operatorname{Int} \overline{U}^L) = \chi^L(\lambda_G(\Phi)) = \sum_{H \in S_G} \frac{\lambda(\psi_{(H)})}{|G/H|} \cdot \frac{|N(L,H)|}{|H|}.$$

Moreover, $\lambda(\psi_{(H)})/|G/H| \in \mathbb{Z}$ and $(G/H)^L = \emptyset$ if $(H) \geq (L)$ does not hold. By setting $a_{(H)}(\Phi) = \lambda(\psi_{(H)})/|G/H|$ we obtain the desired formula.

In order to obtain further congruences we apply Möbius inversion. Let $(P, \leq)$ be a partially ordered set. For $x, y \in P$ an *interval* $[x, y]$ is the set all elements $w \in P$ such that $x \leq w \leq y$. The set P is *locally finite* if the number of elements in any interval is finite. There is a unique *Möbius function* μ defined on all pairs (x, y) such that $x \leq y$ and satisfying

$$\mu(x,x) = 1 \quad \text{for all } x \in P,$$
$$\mu(x,y) = -\sum_{x \leq z < y} \mu(x,z) = -\sum_{x < z \leq y} \mu(z,y) \quad \text{if } x < y.$$

A function $F : P \to \mathbb{R}$ is *summable* if for each $x \in P$ the number of nonzero components in the sum $G(x) = \sum_{y\colon y \leq x} F(y)$ is finite.

(5.10) Theorem (see e.g. [9]). *Let $P = (P, \leq)$ be a locally finite partially ordered set and $F_= : P \to \mathbb{R}$ a summable function. Define*

$$F_{\geq}(x) = \sum_{y\colon y \geq x} F_=(y).$$

Then

$$F_=(x) = \sum_{y\colon y \geq x} F_{\geq}(y)\mu(x,y),$$

where μ is the Möbius function.

(5.11) Proposition. *Let G be a finite abelian group. Then*

$$\sum_{L: H \subset L} \mu(H,L) \operatorname{ind}_A(K^L, \Phi, \operatorname{Int} \overline{U}^L) \equiv 0 \bmod |G/H|$$

for each $H \subset G$, where μ is the Möbius function on S_G.

Proof. Since G is abelian,

$$N(L,H) = G, \quad (H) = H, \quad H \leq L \Leftrightarrow H \subset L.$$

So by (5.9) we have

$$\sum_{H \geq L} |G/H| a_{(H)}(\Phi) = \operatorname{ind}_A(K^L, \Phi, \operatorname{Int} \overline{U}^L)$$

for each $L \subset G$. Applying (5.10) we obtain

$$a_{(H)}(\Phi)|G/H| = \sum_{L: H \subset L} \mu(H,L) \operatorname{ind}_A(K^L, \Phi, \operatorname{Int} \overline{U}^L),$$

and thus (5.11) must hold.

Example 1. Let $G = Z_m$ be a cyclic group of order m. Then $S_{Z_m} = \{Z_a : a \,|\, m\}$ and $\mu(Z_a, Z_b) = \mu(b/a)$ for $a \,|\, b$, where $\mu(b/a)$ is the classical Möbius function, i.e.

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 \dots p_k,\ p_i \text{ different primes}, \\ 0 & \text{otherwise}. \end{cases}$$

By (5.11) we have the formula

$$\sum_{b:a|b|m} \mu(b/a) \operatorname{ind}_A(K^{Z_b}, \Phi, \operatorname{Int} \overline{U}^{Z_b}) \equiv 0 \bmod m/a$$

for each a dividing m. Here the sum runs over all b such that $a \,|\, b$ and $b \,|\, m$.

Example 2. Let $m = p^k$ and $a = p^n$ be powers of a prime p. Then the above congruences reduce to

$$\operatorname{ind}_A(K^{Z_{p^n}}, \Phi, \operatorname{Int} \overline{U}^{Z_{p^n}}) - \operatorname{ind}_A(K^{Z_{p^{n+1}}}, \Phi, \operatorname{Int} \overline{U}^{Z_{p^{n+1}}}) \equiv 0 \bmod p^{k-n}.$$

Example 3. Taking $m = 12$, $a = 1$, we obtain

$$\begin{aligned} \operatorname{ind}_A(K, \Phi, U) - {}& \operatorname{ind}_A(K^{Z_2}, \Phi, \operatorname{Int} \overline{U}^{Z_2}) \\ & - \operatorname{ind}_A(K^{Z_3}, \Phi, \operatorname{Int} \overline{U}^{Z_3}) \\ & + \operatorname{ind}_A(K^{Z_6}, \Phi, \operatorname{Int} \overline{U}^{Z_6}) \equiv 0 \bmod 12. \end{aligned}$$

Remarks.

(1) Let us point out that all the above results remain true if we consider $\Phi \in D_{\mathrm{S}}(U, X)$, where X is a compact G-ANR. The proofs are by a standard reduction to the G-polyhedral case (cf. [11]) and therefore are omitted.

(2) The results of Komiya [8] are given for G a compact Lie group. Our method of proof, based on simplicial techniques, is effective only for a finite group. But even in the case of single-valued maps it is alternative to [8].

(3) We were able to prove all the congruences only for maps $\Phi \in D_{\mathrm{S}}(U, X)$. It is still an open question whether they are true for $\Phi \in D(U, X)$. They should hold at least for $\mathbb{Z}$-acyclic maps because of the uniqueness of index (see [1]).

(4) Similar congruences for iterates were proved in [2].

(5) In [7] the G-chain approximation technique was developed for a larger class of maps with multiplicity in the case of $G = \mathbb{Z}_2$ in order to obtain Borsuk–Ulam type theorems.

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