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AN ELLIPTIC PROBLEM WITH POINTWISE CONSTRAINT ON THE LAPLACIAN

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Introduction

This paper deals with a class of variational inequalities, coming from variational problems with unilateral constraints. The presence of the constraint modifies the structure of the corresponding functional and increases the topological complexity of its sublevels, giving rise to some phenomena which are typical of nonlinear elliptic equations.

Let Ω be a bounded domain of $\mathbb{R}^n$, λ a real parameter, ψ and h two functions in $H_0^{1,2}(\Omega)$ and in $L^2(\Omega)$ respectively.

Set $K_\psi = \{u \in H_0^{1,2}(\Omega) \mid \Delta u \leq \Delta\psi \text{ (in weak sense)}\}$ (K_ψ is a convex cone with vertex at ψ) and consider the problem

$$P_\psi(h)\begin{cases} u \in K_\psi, \\ \int_\Omega [DuD(v-u) - \lambda u(v-u) + h(v-u)]\,dx \geq 0 \quad \forall v \in K_\psi. \end{cases}$$

The solutions can be obtained as lower critical points (see Definition 1.4) of the functional

$$f_h(u) = \frac{1}{2}\int_\Omega (|Du|^2 - \lambda u^2)\,dx + \int_\Omega hu\,dx$$

constrained on the convex cone K_ψ.

The aim of this paper is to study the solvability of problem $P_\psi(h)$ for a generic pair (ψ, h): we describe the set of pairs (ψ, h) for which $P_\psi(h)$ has

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solutions; for these pairs we analyse the structure of the set of solutions and evaluate the number of solutions under suitable assumptions on the position of the parameter λ with respect to the eigenvalues λ_i of the Laplace operator $-\Delta$ in $H_0^{1,2}(\Omega)$.

The results we obtain (see, for example, Theorem 4.8) exhibit a "folding type behaviour": the set of pairs (ψ, h) such that $P_\psi(h)$ has solution is a convex cone and, if $\lambda_1 < \lambda < \lambda_2$ (λ_1 and λ_2 being the first and second eigenvalues of $-\Delta$), then $P_\psi(h)$ has at least one solution for (ψ, h) on the boundary of this cone, at least two distinct solutions if (ψ, h) lies in its interior (while the functional f_h without the constraint K_ψ has a unique critical point for every $h \in L^2(\Omega)$).

This behaviour makes evident an interesting analogy with a well known result stated by Ambrosetti and Prodi in [2], concerning problems with "jumping" nonlinearity like

$$\Delta u + g(u) = h \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where

$$\lim_{t \to -\infty} g(t)/t < \lambda_1 < \lim_{t \to \infty} g(t)/t < \lambda_2. \tag{2}$$

Notice that, despite the evident analogy of these results, there is a deep difference between our methods and those used in [2], the latter being based on the analysis of singularities that could not be applied in our problem.

Comparing the sublevels of the corresponding functionals, one could see that, roughly speaking, the presence of the constraint K_ψ in problem $P_\psi(h)$ has the same role played in [2] by the condition

$$\lim_{t \to -\infty} g(t)/t < \lambda_1.$$

In [14, 17–19, 21–23] an analogous jumping behaviour was shown for some problems with constraints like

$$\widetilde{K}_\varphi = \{u \in H_0^{1,2}(\Omega) \mid u \geq \varphi \text{ a.e. in } \Omega\} \quad (\varphi \in L^2(\Omega)) \tag{3}$$

in place of K_ψ.

Several papers have been devoted to variational inequalities (see [11, 12, etc.]); they involve unilateral pointwise constraints on the function (like $\widetilde{K}_\varphi$), or on the laplacian (like K_ψ) or also on the gradient: for example, a constraint like

$$\overline{K}_\gamma = \{u \in H_0^{1,2}(\Omega) \mid |Du| \leq \gamma \text{ in } \Omega\} \tag{4}$$

arises in the problem of the elastic-plastic torsion of a bar (see [5, 10]).

Usually, constraints on the function or on the gradient (like $\widetilde{K}_\varphi$ or $\overline{K}_\gamma$) have been used in second order variational inequalities, while constraints on the laplacian (like K_ψ) have been considered in some fourth order variational inequalities (for example for the biharmonic operator: see [6]). However, for the second order variational inequalities we are considering in this paper, the jumping behaviour arises only when we use constraints like $\widetilde{K}_\varphi$ or K_ψ; no analogous folding type result occurs if in problem $P_\psi(h)$ the convex cone K_ψ is replaced by a convex set like $\overline{K}_\gamma$ or $\{u \in H_0^{1,2}(\Omega) \mid (Du, \overline{x}) \le \gamma \text{ in } \Omega\}$, with $\overline{x} \in \mathbb{R}^n$.

An important tool is the use of supersolutions in order to analyse the properties of the pair (ψ, h) for which $P_\psi(h)$ has solution or to describe the structure of the set of solutions of $P_\psi(h)$ (for example the existence of a minimal solution).

In [14, 17–19], where constraints like $\widetilde{K}_\varphi$ involve only the function, the classical notion of supersolution for the operator $\Delta + \lambda I - h$ has been sufficient (u is said to be a supersolution if $\Delta u + \lambda u - h \le 0$ in weak sense, see [1, 3, etc.]).

In this paper a new notion of supersolution (see Definition 2.1) turns out to be appropriate and useful to handle constraints on the laplacian like K_ψ. It is natural to call them supersolutions with respect to the operator $I + \Delta^{-1}(\lambda I - h)$, since (see Proposition 2.2) u is a supersolution if $u + \Delta^{-1}(\lambda u - h) \ge 0$ (the operator Δ^{-1} is considered in $H_0^{1,2}(\Omega)$).

Notice that, unlike [1] (where monotone iterations are used), we use the supersolutions as "upper fictitious obstacles" (see Lemma 2.3); this property allows us to prove that there exists a minimal solution, that the set of pairs (ψ, h) for which $P_\psi(h)$ has solution is a closed convex cone, etc.

In a different situation, the use of supersolutions as fictitious obstacles to analyse the structure of the set of solutions has been introduced, for example, in [13, 14, 17–20].

This paper is organized as follows: in Section 1 we introduce the problem and characterize its solutions as lower critical points of the functional f_h constrained on the convex cone K_ψ; moreover, we prove the equivalence of problem $P_\psi(h)$ to another variational inequality, which makes evident the pointwise properties of solutions; in Section 2 we introduce the supersolutions and state their main properties, which are used in Section 3 to analyse the solvability of $P_\psi(h)$ for a generic pair (ψ, h) and to describe the properties of the set of solutions; in Section 4 we obtain the alternatives exhibiting the jumping behaviour for $\lambda_1 < \lambda < \lambda_2$; in Section 5 they are extended to the case $\lambda = \lambda_2$, while Section 6 is devoted to the case $\lambda = \lambda_1$.

1. The problem, the variational setting and preliminary remarks

Let Ω be a bounded domain of $\mathbb{R}^n$, λ a real number, ψ and h two functions that we assume, for simplicity, in $H_0^{1,2}(\Omega)$ and $L^2(\Omega)$ respectively. We consider the following problem:

Definition 1.1. Let

$$K_\psi = \left\{ u \in H_0^{1,2}(\Omega) \;\middle|\; \int_\Omega DuDw\,dx \ge \int_\Omega D\psi Dw\,dx \ \forall w \in C_0^\infty(\Omega),\ w \ge 0 \right\};$$

we say that u is a solution of problem $P_\psi(h)$ if

$$\begin{cases} u \in K_\psi, \\ \int_\Omega [DuD(v-u) - \lambda u(v-u) + h(v-u)]\,dx \geq 0 \quad \forall v \in K_\psi. \end{cases}$$

Remark. Notice that, if we can apply the Gauss–Green formula, the inequality of problem $P_\psi(h)$ becomes

$$\int_\Omega [u + \Delta^{-1}(\lambda u - h)]\Delta(v-u)\,dx \leq 0 \quad \forall v \in K_\psi,$$

whose pointwise meaning would be

$$\begin{cases} u + \Delta^{-1}(\lambda u - h) = 0 & \text{a.e. where } \Delta u < \Delta\psi, \\ u + \Delta^{-1}(\lambda u - h) \geq 0 & \text{a.e. where } \Delta u = \Delta\psi, \end{cases}$$

or, equivalently

$$\begin{cases} u \geq -\Delta^{-1}(\lambda u - h) \quad \text{a.e. in } \Omega, \\ u > -\Delta^{-1}(\lambda u - h) \Rightarrow \Delta u = \Delta\psi. \end{cases}$$

This remark is made precise in the following lemma.

Lemma 1.2. *Assume $\psi \in H_0^{1,2}(\Omega)$, $k \in L^2(\Omega)$ and set*

$$\overline{K} = \{u \in H_0^{1,2}(\Omega) : u \geq \Delta^{-1}k \ \text{a.e. in } \Omega\}.$$

Then a function $u \in H_0^{1,2}(\Omega)$ solves the problem

$$\begin{cases} u \in K_\psi, \\ \int_\Omega DuD(v-u)\,dx + \int_\Omega k(v-u)\,dx \geq 0 \quad \forall v \in K_\psi, \end{cases} \tag{5}$$

if and only if it is a solution of the variational inequality

$$\begin{cases} u \in \overline{K}, \\ \int_\Omega DuD(w-u)\,dx - \int_\Omega D\psi D(w-u)\,dx \geq 0 \quad \forall w \in \overline{K}. \end{cases} \tag{6}$$

Proof. Suppose that $u \in K_\psi$ solves problem (5). If, for every $\delta \in C_0^\infty(\Omega)$, $\delta \geq 0$ in Ω, we take $v = u - \Delta^{-1}\delta$, then $v \in K_\psi$ and the inequality (5) implies

$$\int_\Omega (u - \Delta^{-1}k)\delta\,dx \geq 0.$$

Therefore $u \in \overline{K}$. Now, if w is in $\overline{K}$, then

$$\begin{aligned} \int_\Omega DuD(w-u)\,dx &- \int_\Omega D\psi D(w-u)\,dx \\ &= \int_\Omega (Du - D\psi)D(w-u)\,dx \geq \int_\Omega (Du - D\psi)D(\Delta^{-1}k - u)\,dx, \end{aligned} \tag{7}$$

where the last inequality is true because $\Delta u \leq \Delta\psi$ (in weak sense) and $w \geq \Delta^{-1}k$. The last integral in (7) is equal to

$$\int_\Omega DuD(\psi - u)\,dx + \int_\Omega k(\psi - u)\,dx,$$

which is nonnegative by assumption (notice that $\psi \in K_\psi$).

Conversely, let $u \in \overline{K}$ be a solution of problem (6). If, for every $\alpha \in C_0^\infty(\Omega)$, $\alpha \geq 0$ in Ω, we take $w = u + \alpha$, then $w \in \overline{K}$ and inequality (6) implies

$$\int_\Omega (Du - D\psi)D\alpha\,dx \geq 0.$$

Therefore $u \in K_\psi$. Now, if v is in K_ψ, then

$$\begin{aligned}\text{(8)}\quad &\int_\Omega DuD(v - u)\,dx + \int_\Omega k(v - u)\,dx \\ &= \int_\Omega D(u - \Delta^{-1}k)D(v - u)\,dx \geq \int_\Omega D(u - \Delta^{-1}k)D(\psi - u)\,dx,\end{aligned}$$

where the last inequality is true because $u \geq \Delta^{-1}k$ and $\Delta(\psi - u) \geq \Delta(v - u)$ (in weak sense). The last integral in (8) is equal to

$$\int_\Omega DuD(\Delta^{-1}k - u)\,dx - \int_\Omega D\psi D(\Delta^{-1}k - u)\,dx,$$

which is nonnegative by assumption because $\Delta^{-1}k \in \overline{K}$. $\square$

Notations. Let $\lambda_1 < \lambda_2 < \ldots$ be the eigenvalues of the operator $-\Delta$ in $H_0^{1,2}(\Omega)$ and e_1 the positive eigenfunction corresponding to the first eigenvalue and such that $\int_\Omega e_1^2\,dx = 1$. Moreover, let X_1 and X_2 be the vector spaces spanned by the eigenfunctions corresponding to λ_1 and λ_2 respectively and set $X_3 = (X_1 \oplus X_2)^\perp$. Finally, let Π_1, Π_2 and Π_3 be the projections on the spaces X_1, X_2 and X_3 respectively.

Definition 1.3. Let X be a set and $V \subseteq X$. We define the *indicator function* of the set V as the function $I_V : X \to \mathbb{R} \cup \{\infty\}$ such that

$$I_V(u) = \begin{cases} 0 & \text{if } u \in V \\ \infty & \text{if } u \in X \setminus V. \end{cases}$$

Let $h \in L^2(\Omega)$ and $\psi \in H_0^{1,2}(\Omega)$; we denote by $f_{h,\psi} : L^2(\Omega) \to \mathbb{R} \cup \{\infty\}$ the functional $f_{h,\psi} = f_h + I_{K_\psi}$, where

$$f_h(u) = \begin{cases} \frac{1}{2}\int_\Omega (|Du|^2 - \lambda u^2)\,dx + \int_\Omega hu\,dx & \text{if } u \in H_0^{1,2}(\Omega), \\ \infty & \text{if } u \in L^2(\Omega) \setminus H_0^{1,2}(\Omega), \end{cases}$$

and I_{K_ψ} is the indicator function of the set K_ψ.

Let $f_h'(u)$ be the differential of f_h in u, that is,

$$f_h'(u)[w] = \int_\Omega [DuDw - \lambda uw + hw]\,dx \quad \forall u, w \in H_0^{1,2}(\Omega).$$

Let H be a Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let $W \subset H$ and consider $f : W \to \mathbb{R} \cup \{\infty\}$. We define the *domain* of f to be the set

$$\mathcal{D}(f) = \{u \in W \mid f(u) < \infty\}.$$

Definition 1.4 (see [4, 8, 9]). Let $u \in \mathcal{D}(f)$. We define the *subdifferential* of f at u to be the set $\partial^- f(u)$ consisting of all α in H such that

$$\liminf_{v \to u} \frac{f(v) - f(u) - (\alpha, v - u)}{\|v - u\|} \geq 0.$$

If $\partial^- f(u) \neq \emptyset$, then we define the *subgradient* of f at u, denoted by $\operatorname{grad}^- f(u)$, to be the element of $\partial^- f(u)$ having minimal norm (it is easy to check that $\partial^- f(u)$ is a closed and convex subset of H).

Lastly, we say that u is a *lower critical point* for f if $0 \in \partial^- f(u)$, that is, if $\operatorname{grad}^- f(u) = 0$.

Remark 1.5. The functional $f_{h,\psi}$ is lower semicontinuous in the metric of $L^2(\Omega)$ and its domain is $D(f_{h,\psi}) = K_\psi$.

Furthermore, it is easy to verify that

$$f_h(v) = f_h(u) + f_h'(u)[v - u] + \frac{1}{2}\|v - u\|^2_{H_0^{1,2}} - \frac{\lambda}{2}\|v - u\|^2_{L^2} \quad \forall u, v \in H_0^{1,2}(\Omega).$$

Hence $\alpha \in \partial^- f_{h,\psi}(u)$ if and only if

$$f_h'(u)[v - u] \geq (\alpha, v - u) \quad \forall v \in K_\psi$$

and

$$f_h(v) \geq f_h(u) + (\alpha, v - u) + \frac{1}{2}\|v - u\|^2_{H_0^{1,2}} - \frac{\lambda}{2}\|v - u\|^2_{L^2} \qquad \forall u, v \in H_0^{1,2}(\Omega),\ \forall \alpha \in \partial^- f(u).$$

We get immediately the following result:

Proposition 1.6. *The function u is a solution of problem $P_\psi(h)$ if and only if u is a lower critical point for $f_{h,\psi}$.*

Remark 1.7. If $\lambda < \lambda_1$, then there exists a unique solution to problem $P_\psi(h)$ for every $h \in L^2(\Omega)$ and $\psi \in H_0^{1,2}(\Omega)$.

In fact, the functional $f_{h,\psi}$ introduced in Definition 1.3 is coercive and strictly convex if $\lambda < \lambda_1$; thus it has only one lower critical point: its unique minimum point.

2. Supersolutions as fictitious obstacles

In this section we introduce the notion of supersolution for our problem. Then (see Lemma 2.3) we point out a useful and simple connection with the solutions of problem $P_\psi(h)$.

In the next sections the results obtained here will be used to get information about the set of data for which there exist solutions and about their multiplicity.

Definition 2.1. We say that a function $u \in H_0^{1,2}(\Omega)$ is a *supersolution for the operator* $I + \Delta^{-1}(\lambda I - h)$ if

$$\int_\Omega (DuDw - \lambda uw + hw)\,dx \geq 0 \quad \forall w \in K_0.$$

Remark. It is evident that every solution of problem $P_\psi(h)$ is a supersolution for the operator $I + \Delta^{-1}(\lambda I - h)$.

Let us point out that this definition of supersolution is rather different from the usual one (used, for example, in [18, 19]) because it makes use of test functions w such that $\Delta w \leq 0$ in weak sense, instead of the more general functions w such that $w \geq 0$. The next proposition suggests why we use the name "supersolutions for $I + \Delta^{-1}(\lambda I - h)$" for the ones introduced in Definition 2.1, while it is natural to call the other ones "supersolutions for the operator $\Delta + \lambda I - h$".

Proposition 2.2. *The function u is a supersolution for the operator $I + \Delta^{-1}(\lambda I - h)$ (in the sense of Definition* 2.1*) if and only if $u + \Delta^{-1}(\lambda u - h) \geq 0$ a.e. in Ω.*

Proof. If u is a supersolution, then Definition 2.1 (with $w = -\Delta^{-1}\varphi$) implies

$$\int_\Omega [u + \Delta^{-1}(\lambda u - h)]\varphi \geq 0 \quad \forall \varphi \in L^2(\Omega) \text{ such that } \varphi \geq 0;$$

so $u + \Delta^{-1}(\lambda u - h) \geq 0$ a.e. in Ω.

Conversely, if $u + \Delta^{-1}(\lambda u - h) \geq 0$, then multiplying by Δw for $w \in C_0^\infty(\Omega)$ such that $\Delta w \leq 0$, we get

$$\int_\Omega [u + \Delta^{-1}(\lambda u - h)]\Delta w\,dx \leq 0,$$

which implies

$$\int_\Omega (DuDw - \lambda uw + hw)\,dx \geq 0.$$

Hence it suffices to remark that the last inequality can be extended to all $w \in K_0$ by density arguments. $\square$

Lemma 2.3 and Proposition 2.4 below exhibit an important property of supersolutions: a constraint like $\{u \in L^2(\Omega) \mid u \le \overline{u}\}$ is in a certain sense a fictitious obstacle if $\overline{u}$ is a supersolution for $I + \Delta^{-1}(\lambda I - h)$ according to Definition 2.1.

Lemma 2.3. *Let $\overline{u} \in K_\psi$ be a supersolution for the operator $I + \Delta^{-1}(\lambda I - h)$ with $\lambda \ge 0$; set $K = \{u \in K_\psi \mid u \le \overline{u}\}$ and assume that w is a lower critical point for $f_h + I_K$. Then w is a solution of problem $P_\psi(h)$.*

Proof. Let us remark, first of all, that $\overline{u} \ge -\Delta^{-1}(\lambda\overline{u} - h)$ a.e., because $\overline{u}$ is a supersolution. Moreover, $\lambda w - h \le \lambda\overline{u} - h$, because $w \in K$ and $\lambda \ge 0$, so we obtain

$$-\Delta^{-1}(\lambda w - h) \le -\Delta^{-1}(\lambda\overline{u} - h) \le \overline{u}. \tag{9}$$

The function w satisfies

$$\int_\Omega [DwD(v - w) - \lambda w(v - w) + h(v - w)]\, dx \ge 0 \quad \forall v \in K;$$

therefore, if we put

$$\widetilde{f}(u) = \frac{1}{2}\int_\Omega |Du|^2\, dx + \int_\Omega [h - \lambda w]u\, dx,$$

then w is a lower critical point for $\widetilde{f} + I_K$. The functional $\widetilde{f}$ is strictly convex, lower semicontinuous and coercive; so there exists only one minimum point for $\widetilde{f}$ on K_ψ; let us call it $\widetilde{w}$.

The function $\widetilde{w}$ satisfies

$$\int_\Omega D\widetilde{w}D(v - \widetilde{w})\, dx - \int_\Omega (\lambda w - h)(v - \widetilde{w})\, dx \ge 0 \quad \forall v \in K_\psi. \tag{10}$$

The functional $\widetilde{f} + I_K$ admits only one lower critical point (its unique minimum point), because it is strictly convex; so, if we show that $\widetilde{w} \le \overline{u}$, then we have $\widetilde{w} = w$ and (10) gives us the desired conclusion.

Applying Lemma 1.2 with $k = h - \lambda w$, we see that $\widetilde{w}$ is a lower critical point for the functional

$$F(u) = \frac{1}{2}\int_\Omega |Du|^2\, dx - \int_\Omega D\psi Du\, dx$$

constrained on the set

$$\overline{K} = \{u \in H_0^{1,2}(\Omega) \mid u \ge \Delta^{-1}(h - \lambda w) \text{ a.e. in } \Omega\}.$$

The function $\overline{u}$ is in $\overline{K}$ by (9) and it satisfies $\Delta\overline{u} \le \Delta\psi$ (in weak sense) by assumption; thus it is a supersolution for the operator F' (in the usual sense: see, for example, [18, 19]).

Therefore, as stated in [18], the functional $F + I_{\overline{K}}$ has a lower critical point, which we call w'; furthermore, this point satisfies $w' \le \overline{u}$; but $F + I_{\overline{K}}$ has only one critical point, because it is strictly convex, so $\widetilde{w} = w'$. This implies that $\widetilde{w} \le \overline{u}$ and so $\widetilde{w} = w$, which completes the proof. □

Proposition 2.4. *Let $\overline{u} \in K_\psi$ be a supersolution for the operator $I + \Delta^{-1}(\lambda I - h)$; then problem $P_\psi(h)$ has a solution w such that $w \le \overline{u}$ a.e.*

For the proof it is sufficient to apply the previous lemma, with w a minimum point of the functional $f_h + I_K$ (notice that $K \ne \emptyset$ because $\overline{u} \in K$ and moreover $f_h + I_K$ has a minimum because K is bounded in $L^2(\Omega)$ and so the sublevels of $f_h + I_K$ are bounded in $H_0^{1,2}(\Omega)$).

Lemma 2.5. *Let $\lambda \ge 0$; if u_1 and u_2 are supersolutions for the operator $I + \Delta^{-1}(\lambda I - h)$, then so is $u_1 \wedge u_2$.*

Proof. It suffices to remark that

$$u_1 \ge -\Delta^{-1}(\lambda u_1 - h) \ge -\Delta^{-1}(\lambda(u_1 \wedge u_2) - h) \quad \text{a.e. in } \Omega,$$
$$u_2 \ge -\Delta^{-1}(\lambda u_2 - h) \ge -\Delta^{-1}(\lambda(u_1 \wedge u_2) - h) \quad \text{a.e. in } \Omega,$$

because u_1 and u_2 are supersolutions and $\lambda \ge 0$. Therefore

$$u_1 \wedge u_2 \ge \Delta^{-1}(\lambda(u_1 \wedge u_2) - h) \quad \text{a.e. in } \Omega,$$

that is, $u_1 \wedge u_2$ is a supersolution, by Proposition 2.2. □

Theorem 2.6. *If u_1 and u_2 are solutions of problem $P_\psi(h)$, then there exists a solution u such that $u \le u_1 \wedge u_2$.*

Proof. The functions u_1 and u_2 are supersolutions for the operator $I + \Delta^{-1}(\lambda I - h)$, because they are solutions of problem $P_\psi(h)$ and so, by Lemma 2.5, also the function $u_1 \wedge u_2$ is a supersolution.

By Proposition 2.4, it suffices that $u_1 \wedge u_2 \in K_\psi$, which is stated in the next proposition (that we prove for sake of completeness). □

Proposition 2.7. *Let u_1 and u_2 be in K_ψ; then also $u_1 \wedge u_2 \in K_\psi$.*

Proof. Set $\overline{u} = u_1 \wedge u_2$, $\pi(v) = v \wedge u_1$ and let $F : H_0^{1,2}(\Omega) \to \mathbb{R}$ be defined by

$$F(u) = \frac{1}{2}\int_\Omega |Du|^2\,dx - \int_\Omega D\psi Du\,dx.$$

Then

$$F(u) \ge F(v) + F'(v)[u - v] \quad \forall u, v \in H_0^{1,2}$$

because the functional F is convex.

So, if $w \in C_0^\infty(\Omega)$, we have

$$F(\overline{u} + w) \ge F(\pi(\overline{u} + w)) + F'(\pi(\overline{u} + w))[\overline{u} + w - \pi(\overline{u} + w)],$$
$$F(\pi(\overline{u} + w)) \ge F(\overline{u}) + F'(\overline{u})[\pi(\overline{u} + w) - \overline{u}].$$

Now, if $w \geq 0$, it follows that

$$F'(\pi(\overline{u}+w))[\overline{u}+w-\pi(\overline{u}+w)] = F'(u_1)[\overline{u}+w-\pi(\overline{u}+w)]$$

because $\pi(\overline{u}+w) = u_1$ where $\overline{u}+w-\pi(\overline{u}+w) \neq 0$.

So, since $u_1 \in K_\psi$ and $\overline{u}+w-\pi(\overline{u}+w) \geq 0$, it follows that

$$F'(\pi(\overline{u}+w))[\overline{u}+w-\pi(\overline{u}+w)] \geq 0. \tag{11}$$

Analogously

$$F'(\overline{u})[\pi(\overline{u}+w)-\overline{u}] = F'(u_2)[\pi(\overline{u}+w)-\overline{u}] \geq 0. \tag{12}$$

Finally, the inequalities (11) and (12) imply that $F(\overline{u}+w) \geq F(\overline{u})$, which yields $F'(\overline{u})[w] \geq 0$, and this is the desired conclusion. □

Lastly, will need the following results about supersolutions, whose proofs are straightforward.

Proposition 2.8. *Let h' and h'' be in $L^2(\Omega)$; if u' and u'' are supersolutions for the operators $I+\Delta^{-1}(\lambda I-h')$ and $I+\Delta^{-1}(\lambda I-h'')$, respectively, then $u'+u''$ is a supersolution for the operator $I+\Delta^{-1}[\lambda I-(h'+h'')]$. In particular, the assertion is true if u' and u'' are solutions for problems $P_{\psi'}(h')$ and $P_{\psi''}(h'')$, respectively, for some obstacles ψ' and ψ'' in $H_0^{1,2}(\Omega)$.*

Proposition 2.9. *If u is a supersolution for the operator $I+\Delta^{-1}(\lambda I-h)$ (in particular, if u is a solution for some problem $P_\psi(h)$), then u is a supersolution for the operator $I+\Delta^{-1}(\lambda I-h')$ for every h' in $L^2(\Omega)$ such that $h' \geq h$.*

3. Some properties of the set of solutions

Let us define

$$R = \{(\psi,h) \mid \psi \in H_0^{1,2}(\Omega),\ h \in L^2(\Omega),\ P_\psi(h) \text{ has solution}\}.$$

In this section we use supersolutions to study some properties of the set of solutions for problem $P_\psi(h)$ and to describe the set R.

Theorem 3.1. *Let $h \in L^2(\Omega)$ and $\psi \in H_0^{1,2}(\Omega)$; if there exists a solution for $P_\psi(h)$, then there is a solution for problem $P_{\psi'}(h')$ for every pair (ψ',h') with $h' \in L^2(\Omega)$ and $\psi' \in H_0^{1,2}(\Omega)$ such that $\Delta\psi' \geq \Delta\psi$ in weak sense and $h' \geq h$.*

This follows easily from Propositions 2.4 and 2.9.

Theorem 3.2. *The set R is a convex cone whose vertex is the origin.*

Proof. It is clear that if u is a solution for $P_\psi(h)$ then αu solves $P_{\alpha\psi}(\alpha h)$ for every $\alpha \geq 0$. Moreover, if $P_{\psi'}(h')$ and $P_{\psi''}(h'')$ have a solution, say u' and u'' respectively, it follows from Propositions 2.8 and 2.4 that $P_{\psi'+\psi''}(h'+h'')$

also has a solution; in fact, $u' + u'' \in K_{\psi'+\psi''}$ and it is a supersolution for the operator $I + \Delta^{-1}[\lambda I - (h' + h'')]$. This completes the proof. $\square$

Before enunciating some closure properties of R, let us state the following results.

Lemma 3.3. *Let u be in K_ψ; then*

$$\int_\Omega [(\lambda_1 - \lambda)u + (h - \alpha)]e_1 \, dx \geq 0 \quad \forall \alpha \in \partial^- f_{h,\psi}(u). \tag{13}$$

Proof. We have (see Remark 1.5)

$$\int_\Omega \alpha(v - u) \, dx \leq f'_h(u)[v - u] \quad \forall v \in K_\psi; \tag{14}$$

so, if we put $v = u + e_1$ in (14) (notice that $v = u + e_1$ is in K_ψ), we obtain inequality (13). $\square$

Lemma 3.4. *Let $\lambda \neq \lambda_1$ and assume that, for every $m \in \mathbb{N}$, $\psi_m, \psi \in H_0^{1,2}(\Omega)$, $h_m, h \in L^2(\Omega)$. Suppose also that $\Delta\psi_m \geq \Delta\psi$ in weak sense and that $\psi_m \to \psi$ in $H_0^{1,2}(\Omega)$ and $h_m \to h$ in $L^2(\Omega)$ as $m \to \infty$; furthermore, suppose that problem $P_{\psi_m}(h_m)$ has a solution u_m. Then:*

(a) *the sequence $(u_m)_m$ is bounded in $H_0^{1,2}(\Omega)$;*
(b) *if $(u_m)_m$ (or a subsequence) converges to u in $L^2(\Omega)$ and weakly in $H_0^{1,2}(\Omega)$, then u solves problem $P_\psi(h)$;*
(c) *there exists a solution to problem $P_\psi(h)$.*

Proof. In this proof we are using the notations introduced in the first section.

If $\lambda < \lambda_1$, the assertion follows from $f_{h_m,\psi_m}(u_n) \leq f_{h_m,\psi_m}(\overline{u}) \leq \text{const}$, for $\overline{u} \in K_\psi$ fixed, because the solution u_m is the minimum point for the functional f_{h_m} on K_{ψ_m} and $K_\psi \subseteq K_{\psi_m}$ for all m.

If $\lambda > \lambda_1$, we have

$$f_{h_m,\psi_m}(v) \geq f_{h_m,\psi_m}(u) + \langle \alpha, v - u \rangle + \frac{1}{2}\|v - u\|^2_{H_0^{1,2}} - \frac{\lambda}{2}\|v - u\|^2_{L^2} \qquad \forall u, v \in K_{\psi_m},\ \forall \alpha \in \partial^- f_{h_m,\psi_m};$$

in particular, for $u = u_m$ and $v = \psi$ (notice that $\psi \in K_\psi \subseteq K_{\psi_m}$), we get

$$\begin{aligned} f_{h_m}(\psi) &\geq f_{h_m}(u_m) - \frac{\lambda}{2}\|\psi - u_m\|^2_{L^2} \\ &= \frac{1}{2}\int_\Omega |Du_m|^2 \, dx - \frac{\lambda}{2}\int_\Omega u_m^2 \, dx \\ &\quad + \int_\Omega h_m u_m \, dx - \frac{\lambda}{2}\|\psi - u_m\|^2_{L^2}. \end{aligned} \tag{15}$$

Let us prove that the sequence $(u_m)_m$ is bounded in $L^2(\Omega)$. If we suppose, by contradiction, that this is not the case, then up to taking a subsequence, we have $\lim_{m\to\infty}\|u_m\|_{L^2}=\infty$. If we put $z_m=u_m/\|u_m\|_{L^2}$, from (15) we deduce that $(z_m)_m$ is bounded in $H_0^{1,2}(\Omega)$; so a subsequence of it converges in $L^2(\Omega)$ and a.e. in Ω, to a function $z\in H_0^{1,2}(\Omega)$; then it follows that $\|z\|_{L^2}=1$; moreover, $z\geq 0$ in Ω, because $u_m\geq\psi_m$ a.e. in Ω, and $\psi_m\to\psi$ in $H_0^{1,2}(\Omega)$.

By Lemma 3.3 we have

$$\frac{1}{\|u_m\|_{L^2}}\int_\Omega[(\lambda_1-\lambda)u_m+h]e_1\,dx\geq 0,$$

from which, as $m\to\infty$, we obtain $(\lambda_1-\lambda)\int_\Omega ze_1\,dx\geq 0$, which is impossible because $\lambda>\lambda_1$, $z\geq 0$ and $\|z\|_{L^2}=1$. So the sequence $(u_m)_m$ has to be bounded in $L^2(\Omega)$ and then, from (15), it follows that it is also bounded in $H_0^{1,2}(\Omega)$. So (a) is proved.

Let us prove (b): since $K_\psi\subseteq K_{\psi_m}$ for all $m\in\mathbb{N}$, we have

$$f_{h_m}(v)\geq f_{h_m}(u_m)-\frac{\lambda}{2}\|v-u_m\|_{L^2}^2\quad\forall v\in K_\psi\ \forall m\in\mathbb{N};$$

so, letting $m\to\infty$, we get

$$f_h(v)\geq f_h(u)-\frac{\lambda}{2}\|v-u\|_{L^2}^2\quad\forall v\in K_\psi,$$

which gives (b).

The third conclusion follows from (a) and (b). □

Lemma 3.4 allows us to prove the following closure property of the set R.

Theorem 3.5. *Let $\lambda\neq\lambda_1$ and assume that, for every $m\in\mathbb{N}$, $\psi_m\in H_0^{1,2}(\Omega)\cap H^{2,2}(\Omega)$, $h_m\in L^2(\Omega)$, $\psi_m\to\psi$ in $H^{2,2}(\Omega)$, $h_m\to h$ in $L^2(\Omega)$ and $(\psi_m,h_m)\in R$ (i.e. $P_{\psi_m}(h_m)$ has solution). Then problem $P_\psi(h)$ has solution (i.e. $(\psi,h)\in R$).*

Proof. If $\lambda<\lambda_1$, then the assertion is trivial because of Remark 1.7. If $\lambda>\lambda_1$, let $(\psi_m)_m$ and $(h_m)_m$ be two sequences converging to ψ and to h in $H^{2,2}(\Omega)$ and in $L^2(\Omega)$ respectively, such that, for every $m\in\mathbb{N}$, $P_{\psi_m}(h_m)$ has a solution. If we define $\varphi_n=\Delta^{-1}(\Delta\psi\vee\Delta\psi_m)$, also problem $P_{\varphi_m}(h_m)$ has a solution: indeed, if $\overline{u}_m$ is a solution for $P_{\psi_m}(h_m)$, then $\overline{u}_m$ is a supersolution for $I+\Delta^{-1}(\lambda I-h_m)$. Furthermore, $\Delta\overline{u}_m\leq\Delta\psi_m$ implies that $\Delta\overline{u}_m\leq\Delta\varphi_m$ (in weak sense), that is, $\overline{u}_m\in K_{\varphi_m}$. So we can apply Proposition 2.4 in order to get a solution for $P_{\varphi_m}(h_m)$. If we observe that $\varphi_m\to\psi$ in $H_0^{1,2}(\Omega)$, applying the previous lemma to the sequence $(\varphi_m,h_m)_m$, we have the desired conclusion. □

Remark. Notice that Theorem 3.5 does not hold for $\lambda=\lambda_1$. In fact, if, for example, $(\psi_m)_m$, with $\Delta\psi_m\in C_0^\infty(\Omega)$ for all $m\in\mathbb{N}$, is a sequence converging in $H^{2,2}(\Omega)$ to a function $\psi\in H_0^{1,2}(\Omega)$ such that $\sup_\Omega\psi/e_1=\infty$, then problem

$P_{\psi_m}(0)$ has solution for every $m \in \mathbb{N}$, but $P_\psi(0)$ has no solution because, when $\lambda = \lambda_1$ and $h = 0$, the solutions of problem $P_\psi(h)$ solve the equation $\Delta u + \lambda_1 u = 0$, as we shall prove in Section 6 (see Theorem 6.1 and the related example).

Now we recall a proposition from [8] which is useful to prove the next result about the existence of a minimal solution.

Proposition 3.6. *Let H be a Hilbert space and $f : H \to \mathbb{R} \cup \{\infty\}$ a lower semicontinuous function. Suppose there exists $\lambda \in \mathbb{R}$ such that*

$$f(v) \geq f(u) + \langle \alpha, v-u \rangle - \frac{\lambda}{2}\|v-u\|^2 \quad \forall \alpha \in \partial^- f(u)\ \forall u, v \in \mathcal{D}(f).$$

Let $(u_m)_m$ and $(\alpha_m)_m$ be two sequences such that $u_m \in \mathcal{D}(f)$, $\alpha_m \in \partial^- f(u_m)$ for every $m \in \mathbb{N}$, $\lim_{m\to\infty} u_m = u$, and $\alpha_m \rightharpoonup \alpha$ weakly in H. Then $u \in \mathcal{D}(f)$, $\lim_{m\to\infty} f(u_m) = f(u)$ and $\alpha \in \partial^- f(u)$.

Proposition 3.7. *If problem $P_\psi(h)$ has solution, then there exists a minimal solution $\overline{u}$ (that is, $\overline{u} \leq u$ a.e. in Ω for every solution u for $P_\psi(h)$).*

Proof. Let $(u_m)_m$ be a sequence of solutions such that

$$\lim_{m\to\infty} \int_\Omega u_m\,dx = \inf\left\{ \int_\Omega u\,dx \,\middle|\, u \text{ solution of } P_\psi(h) \right\}$$

(notice that this infimum is finite because there exists a solution).

If we fix $v \in K_\psi$, we get

$$\begin{aligned} f_h(v) &\geq f_h(u_m) - \frac{\lambda}{2}\|v-u_m\|^2_{L^2} \\ &= \frac{1}{2}\int_\Omega |Du_m|^2\,dx - \frac{\lambda}{2}\int_\Omega u_m^2\,dx \\ &\quad + \int_\Omega hu_m\,dx - \frac{\lambda}{2}\|v-u_m\|^2_{L^2}. \end{aligned} \tag{16}$$

We say that $\sup_{m\in\mathbb{N}} \|u_m\|_{L^2} < \infty$. If this is not so, then up to taking a subsequence, we can assume $\lim_{m\to\infty} \|u_m\|_{L^2} = \infty$; let us consider $z_m = u_m/\|u_m\|_{L^2}$; from (16) we deduce that $\sup_{m\in\mathbb{N}} \|z_m\|_{H_0^{1,2}} < \infty$ and consequently $(z_m)_m$ (or a subsequence) converges in $L^2(\Omega)$ and a.e. in Ω to a function z with $\|z\|_{L^2} = 1$ and $z \geq 0$, because $u_m \geq \psi$. Hence $\lim_{m\to\infty} \int_\Omega z_m\,dx = \int_\Omega z\,dx > 0$. But this is impossible: in fact, $\lim_{m\to\infty} \int_\Omega z_m\,dx = \lim_{m\to\infty} (1/\|u_m\|_{L^2}) \int_\Omega u_m\,dx \leq 0$ because $\lim_{m\to\infty} \int_\Omega u_m\,dx < \infty$ and $\lim_{m\to\infty} \|u_m\|_{L^2} = \infty$.

So $(u_m)_m$ must be bounded in $L^2(\Omega)$ and, from (16), it follows that it is also bounded in $H_0^{1,2}(\Omega)$; therefore, up to taking a subsequence, it converges in $L^2(\Omega)$ and weakly in $H_0^{1,2}(\Omega)$ to a function $\overline{u}$ that, by Proposition 3.6, is a solution for $P_\psi(h)$. Let us remark that

$$\int_\Omega \overline{u}\,dx = \min\left\{ \int_\Omega u\,dx \,\middle|\, u \text{ solution of } P_\psi(h) \right\}. \tag{17}$$

Hence we deduce that $\overline{u}$ is the minimal solution. In fact, if there exists a solution u such that $\overline{u} \wedge u \neq \overline{u}$, then there exists another solution $w \leq \overline{u} \wedge u$, by Theorem 2.6. Therefore $\int_\Omega w\,dx < \int_\Omega \overline{u}\,dx$, contrary to (17). $\square$

4. Alternative theorems when $\lambda_1 < \lambda < \lambda_2$

In order to study the solvability of problem $P_\psi(h)$ and evaluate the number of solutions (see Theorems 4.7 and 4.8), we use in this section too the functionals f_h and $f_{h,\psi}$ and the other notations introduced in the first section.

Definition 4.1. Let $h \in L^2(\Omega)$, $\psi \in H_0^{1,2}(\Omega)$, $\lambda \in \mathbb{R}$, $\lambda < \lambda_2$; let $S_{h,\psi} : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be the function defined by

$$S_{h,\psi}(t) = \min\{f_{h,\psi} + I_{P_t}\}$$

where $P_t = \{u \in L^2(\Omega) \mid \int_\Omega ue_1\,dx = t\}$ and I_{P_t} is its indicator function.

Let us remark that, if $\lambda < \lambda_2$, such a minimum exists and, if it is finite, it is achieved at a unique point because in this case $f_{h,\psi}$ is strictly convex, lower semicontinuous and coercive on every hyperplane P_t.

The next lemma says that, if $\lambda < \lambda_2$, then searching the solutions of problem $P_\psi(h)$ is equivalent to looking for the lower critical points of $S_{h,\psi}$.

Lemma 4.2. *Let $\lambda < \lambda_2$ and u be a minimum point for $f_{h,\psi} + I_{P_{\overline{t}}}$, where $\overline{t} = \int_\Omega ue_1\,dx$. Then $k \in \partial^- S_{h,\psi}(\overline{t})$ if and only if $ke_1 \in \partial^- f_{h,\psi}(u)$. In particular, if $k = 0$, then $\overline{t}$ is a lower critical point for $S_{h,\psi}$ if and only if the minimum point for $f_{h,\psi} + I_{P_{\overline{t}}}$ is a lower critical point for $f_{h,\psi}$.*

Proof. Assume that $k \in \partial^- S_{h,\psi}(\overline{t})$. We have to estimate

$$L = \liminf_{v \to u} \frac{f_h(v) - f_h(u) - (ke_1, v - u)}{\|v - u\|_{L^2}}.$$

Observe that, if $\int_\Omega ve_1\,dx = \int_\Omega ue_1\,dx$, then $f_h(v) \geq f_h(u)$, by definition of u. Hence, if

$$M = \liminf_{\substack{v \to u \\ \int_\Omega ve_1\,dx \neq \overline{t}}} \frac{f_h(v) - S_{h,\psi}(\overline{t}) - k(\int_\Omega ve_1\,dx - \overline{t})}{|\int ve_1\,dx - \overline{t}|} \geq 0,$$

then $L \geq 0$, because

$$0 < \frac{|\int ve_1\,dx - \overline{t}|}{\|v - u\|_{L^2}} \leq 1.$$

By definition $f_h(v) \geq S_{h,\psi}(\int_\Omega ve_1\,dx)$; furthermore, $v \to u$ in $L^2(\Omega)$ implies $\int_\Omega ve_1\,dx \to \int_\Omega ue_1\,dx$, so

$$M \geq \liminf_{t \to \overline{t}} \frac{S_{h,\psi}(t) - S_{h,\psi}(\overline{t}) - k(t - \overline{t})}{|t - \overline{t}|} \geq 0$$

by assumption.

Conversely, let us remark, first of all, that $f_h = f_h \circ \Pi_1 + f_h \circ \Pi_2 + f_h \circ \Pi_3$, where $f_h \circ \Pi_2$ and $f_h \circ \Pi_3$ are convex. Therefore we have

$$f_{h,\psi}(v) \geq f_{h,\psi}(u) + f_h'(u)[v-u] + \frac{1}{2}(\lambda_1 - \lambda)\left(\int_\Omega (v-u)e_1\,dx\right)^2.$$

By assumption $ke_1 \in \partial^- f_{h,\psi}(u)$ and so

$$f_{h,\psi}(v) \geq f_{h,\psi}(u) + \int_\Omega ke_1(v-u)\,dx + \frac{1}{2}(\lambda_1 - \lambda)\left(\int_\Omega (v-u)e_1\,dx\right)^2;$$

if we set $t = \int_\Omega ve_1\,dx$ and $\bar{t} = \int_\Omega ue_1\,dx$, by simple arguments it follows that

$$S_{h,\psi}(t) \geq S_{h,\psi}(\bar{t}) + k(t-\bar{t}) + \frac{1}{2}(\lambda_1 - \lambda)(t-\bar{t})^2,$$

which implies $k \in \partial^- S_{h,\psi}(\bar{t})$. □

DEFINITION 4.3. If $\lambda < \lambda_2$, let $\sigma_{h,\psi} : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be defined by

$$\sigma_{h,\psi}(t) = \min\{f_h \circ (\Pi_2 + \Pi_3) + I_{K_\psi} + I_{P_t}\}$$

with the usual notations.

The following lemma states a simple property of $\sigma_{h,\psi}$, which can be easily proved.

LEMMA 4.4. *Let $S_{h,\psi}$ and $\sigma_{h,\psi}$ be the functions defined above. Then*

$$S_{h,\psi}(t) = \sigma_{h,\psi}(t) - (\lambda - \lambda_1)\frac{t^2}{2} + t\int_\Omega he_1\,dx.$$

LEMMA 4.5. *The projection $(\Pi_2+\Pi_3)(K_\psi)$ is dense in $(X_2 \oplus X_3) \cap H_0^{1,2}(\Omega)$ with respect to the $H_0^{1,2}(\Omega)$ norm.*

PROOF. Consider $u \in K_\psi$; we have

$$(\Pi_2 + \Pi_3)(K_\psi) \supseteq (\Pi_2 + \Pi_3)(K_u) = (\Pi_2 + \Pi_3)(u) + (\Pi_2 + \Pi_3)(K_0).$$

The projection $(\Pi_2+\Pi_3)(K_0)$ includes $(\Pi_2+\Pi_3)(C_0^\infty(\Omega))$. In fact, if $w \in C_0^\infty(\Omega)$, then there exists $\tau \in \mathbb{R}$ such that $\Delta(w + \tau e_1) = \Delta w + \tau\Delta e_1 \leq 0$, because $\Delta e_1 = -\lambda_1 e_1 < 0$ in Ω. So $w+\tau e_1 \in K_0$ and $(\Pi_2+\Pi_3)(w+\tau e_1) = (\Pi_2+\Pi_3)(w) \in (\Pi_2+\Pi_3)(K_0)$. The projection $(\Pi_2+\Pi_3)(C_0^\infty(\Omega))$ is dense in $(X_2 \oplus X_3) \cap H_0^{1,2}(\Omega)$, and so the lemma follows. □

LEMMA 4.6. *Let $\lambda < \lambda_2$. Then the function $\sigma_{h,\psi}$ has the following properties:*

- (a) $\mathcal{D}\sigma_{h,\psi} = [\int_\Omega \psi e_1\,dx, \infty)$;
- (b) $\sigma_{h,\psi}$ *is bounded from below, nonincreasing, convex and lower semicontinuous;*
- (c) $\lim_{t\to\infty} \sigma_{h,\psi}(t) = \min f_h \circ (\Pi_2 + \Pi_3)$;
- (d) $\sigma_{h,\psi+\mu e_1}(t+\mu) = \sigma_{h,\psi}(t)$ *for all* $\mu \in \mathbb{R}$.

Proof. (a) If $u \in K_\psi$, then $u \geq \psi$ and so $\int_\Omega ue_1\,dx \geq \int_\Omega \psi e_1\,dx$ because $e_1 > 0$ on Ω. Furthermore, for every $t \geq 0$, $\psi + te_1 \in K_\psi$.

(b) $\sigma_{h,\psi}$ is bounded from below because $\min f_h \circ (\Pi_2 + \Pi_3) > -\infty$ if $\lambda < \lambda_2$. It is nonincreasing because, since $\Delta e_1 < 0$, we have

$$(\Pi_2 + \Pi_3)(K_\psi \cap P_{t_1}) \subset (\Pi_2 + \Pi_3)(K_\psi \cap P_{t_2}) \quad \text{if } \ t_1 < t_2.$$

$\sigma_{h,\psi}$ is convex because $f_h \circ (\Pi_2 + \Pi_3)$ is convex.

$\sigma_{h,\psi}$ is continuous in the interior of the interval $[\int_\Omega \psi e_1\,dx, \infty)$, which is its domain, because it is convex; in order to prove also in the extremum the lower semicontinuity, it suffices to remark that $f_h \circ (\Pi_2 + \Pi_3)$ is lower semicontinuous in the $L_2(\Omega)$ norm and, furthermore, that its sublevels are bounded in $H_0^{1,2}(\Omega)$ on every set like

$$\left\{u \in L^2(\Omega) \,\middle|\, t_1 \leq \int_\Omega ue_1\,dx \leq t_2\right\}, \quad t_1, t_2 \in \mathbb{R},$$

because

$$f_h \circ (\Pi_2 + \Pi_3)(u) \geq \frac{\lambda_2 - \lambda}{2} \int_\Omega |D(\Pi_2 + \Pi_3)(u)|^2\,dx - \int_\Omega h(\Pi_2 + \Pi_3)(u)\,dx.$$

(c) Let $\overline{u}$ be the minimum point for f_h on $(X_2 \oplus X_3) \cap H_0^{1,2}(\Omega)$. By the previous lemma there exists a sequence $(w_n)_n$ in K_ψ such that $(\Pi_2 + \Pi_3)w_n \to \overline{u}$, which implies

$$\lim_{n\to\infty} f_h \circ (\Pi_2 + \Pi_3)(w_n) = \min f_h \circ (\Pi_2 + \Pi_3).$$

Then we have

$$\begin{aligned}\min f_h \circ (\Pi_2 + \Pi_3) &\leq \liminf_{t\to\infty} \sigma_{h,\psi}(t) \leq \lim_{n\to\infty} \sigma_{h,\psi}\left(\int_\Omega w_n e_1\,dx\right)\\ &\leq \min f_h \circ (\Pi_2 + \Pi_3),\end{aligned}$$

which proves the statement.

(d) This follows because $f_h \circ (\Pi_2 + \Pi_3)$ is invariant under translations along the e_1 axis. □

Theorem 4.7. *Let $\psi \in H_0^{1,2}(\Omega)$ and $h \in L^2(\Omega)$; assume $\lambda_1 < \lambda < \lambda_2$. If we write $\psi = \psi_0 + \tau e_1$, with $\psi_0 \in X_2 \oplus X_3$, then there exists $\overline{\tau} \in \mathbb{R}$ ($\overline{\tau}$ depending on ψ_0 and h) such that:*

(a) *if $\tau > \overline{\tau}$, then problem $P_\psi(h)$ has no solution;*
(b) *if $\tau = \overline{\tau}$, then problem $P_\psi(h)$ has at least one solution;*
(c) *if $\tau < \overline{\tau}$, then problem $P_\psi(h)$ has at least two solutions.*

Proof. By Lemma 4.2 it is sufficient to find $t \in \mathbb{R}$ such that $0 \in \partial^- S_{h,\psi}(t)$, that is, by Lemma 4.4, $t \in \mathbb{R}$ such that

$$(\lambda - \lambda_1)t - \int_\Omega he_1\,dx \in \partial\sigma_{h,\psi}(t). \tag{18}$$

By the properties of $\sigma_{h,\psi}$, $\partial\sigma_{h,\psi}$ is an increasing maximal monotone operator (see [4]); furthermore, according to Lemma 4.6(a),

$$\partial\sigma_{h,\psi}(t) = \emptyset \quad \text{if} \quad t < \int_\Omega \psi e_1\,dx = \tau;$$

moreover, $\lim_{t\to\infty}\partial\sigma_{h,\psi}(t) = 0$ because $\sigma_{h,\psi}$ is convex, nonincreasing and bounded from below.

From Lemma 4.6(d) it follows that

$$\partial\sigma_{h,\psi}(t) = \partial\sigma_{h,\psi_0+\tau e_1}(t) = \partial\sigma_{h,\psi_0}(t-\tau);$$

hence (18) is equivalent to

$$(\lambda-\lambda_1)\tau - \int_\Omega he_1\,dx \in \partial\sigma_{h,\psi_0}(t-\tau) - (\lambda-\lambda_1)(t-\tau),$$

that is, to

$$(\lambda-\lambda_1)\tau \in \partial\sigma_{h,\psi_0}(t-\tau) - (\lambda-\lambda_1)(t-\tau) + \int_\Omega he_1\,dx. \tag{19}$$

Since

$$\lim_{t\to\infty}\left(\partial\sigma_{h,\psi_0}(t-\tau) - (\lambda-\lambda_1)(t-\tau) + \int_\Omega he_1\,dx\right) = -\infty,$$

we have

$$M = \max\left\{\alpha - (\lambda-\lambda_1)t + \int_\Omega he_1\,dx \;\middle|\; t\in\mathbb{R},\ \alpha \in \partial\sigma_{h,\psi_0}(t)\right\} \in \mathbb{R}.$$

Such a maximum depends only on ψ_0 and h, and $M \le \int_\Omega he_1\,dx$ because $\partial\sigma_{h,\psi_0}(t-\tau) \subseteq (-\infty,0]$ if $t \ge \tau$ and $\partial\sigma_{\psi_0}(t-\tau) = \emptyset$ if $t < \tau$.

Now, if we set $\overline{\tau} = M/(\lambda-\lambda_1)$, then the theorem follows from the equation (19) and from the shape of $\partial\sigma_{h,\psi_0}$. □

Theorem 4.8. *Let $\psi \in H_0^{1,2}(\Omega)$ and $h \in L^2(\Omega)$; assume $\lambda_1 < \lambda < \lambda_2$. If we write $h = h_0 + \tau e_1$, with $h_0 \in X_2 \oplus X_3$, then there exists $\overline{\tau} \in \mathbb{R}$ ($\overline{\tau}$ depending on h_0 and ψ) such that:*

(a) *if $\tau < \overline{\tau}$, then problem $P_\psi(h)$ has no solution;*
(b) *if $\tau = \overline{\tau}$, then problem $P_\psi(h)$ has at least one solution;*
(c) *if $\tau > \overline{\tau}$, then problem $P_\psi(h)$ has at least two solutions.*

Proof. To prove this theorem we proceed as for the previous one, starting from (18); in this case

$$-\tau \in \partial\sigma_{h,\psi}(t) - (\lambda - \lambda_1)t. \tag{20}$$

The right hand side term does not depend on τ because σ depends only upon $(\Pi_2 + \Pi_3)(h) = h_0$; so, if we set

$$\overline{\tau} = -\max\{\alpha - (\lambda - \lambda_1)t \mid t \in \mathbb{R},\ \alpha \in \partial\sigma_{h,\psi}(t)\},$$

we obtain the desired conclusion. □

5. The case $\lambda = \lambda_2$

In this section we want to extend to the case $\lambda = \lambda_2$ the results obtained in the preceding sections when $\lambda_1 < \lambda < \lambda_2$.

In particular, we show that Theorems 4.7 and 4.8 still hold for $\lambda = \lambda_2$.

However, let us point out that for $\lambda = \lambda_2$ the functional f_h is not coercive on the hyperplanes P_t (see notations introduced in the previous sections) and so an essential role in order to apply the previous methods is played by the following lemma.

Lemma 5.1. *Let $\lambda = \lambda_2$; if $t \in \mathbb{R}$ and the set $P_t = \{u \in L^2(\Omega) \mid \int_\Omega ue_1\,dx = t\}$ meets K_ψ, then the minimum of the functional f_h on $P_t \cap K_\psi$ exists.*

Proof. It is sufficient to show that the sublevels of $f_h + I_{K_\psi} + I_{P_t}$ are bounded in $H_0^{1,2}(\Omega)$.

If we write

$$f_h = f_h \circ \Pi_1 + f_h \circ \Pi_2 + f_h \circ \Pi_3,$$

we observe that $f_h \circ \Pi_1$ is constant on the hyperplanes P_t and that, if $\lambda = \lambda_2$, there exist positive constants c_1, c_2 and c_3, depending on h, such that for every $u \in H_0^{1,2}(\Omega)$ we have

$$f_h \circ \Pi_3(u) \geq -c_1 + c_2\|\Pi_3 u\|^2_{H_0^{1,2}}, \quad f_h \circ \Pi_2(u) \geq -c_3\|\Pi_2 u\|.$$

It follows that on the sublevels of $f_h + I_{P_t}$ we have

$$\|\Pi_2 u\| \geq -c_4 + c_5\|\Pi_3 u\|^2_{H_0^{1,2}} \tag{21}$$

where c_4 and c_5 are suitable positive constants.

It remains to prove that $\|\Pi_2 u\|$ is bounded in the sublevels of $f_h + I_{K_\psi} + I_{P_t}$. If this is not so, then there exists a subsequence $(u_n)_n$ in a sublevel such that $\lim_{n\to\infty}\|\Pi_2 u\| = \infty$; it follows from (21) that

$$\lim_{n\to\infty} \|\Pi_3 u_n\|_{H_0^{1,2}}/\|\Pi_2 u_n\| = 0.$$

If we fix $u_0 \in K_\psi \cap P_t$ (a convex set) we have

$$u_0 + \frac{k}{\|\Pi_2 u_n\|}(\Pi_1 u_n + \Pi_2 u_n + \Pi_3 u_n - u_0) \in K_\psi \cap P_t \qquad \text{for } 0 \le k \le \|\Pi_2 u_n\|.$$

Hence, letting $n \to \infty$, we get (up to taking a subsequence), for every $k \ge 0$, $u_0 + kv \in K_\psi \cap P_t$ for a function $v \in X_2$ such that $\|v\| = 1$; this is impossible, because v is negative on a set of nonzero measure; this completes the proof. □

The previous lemma allows us to define the functions $S_{h,\psi}$ and $\sigma_{h,\psi}$ (see Section 4) also when $\lambda = \lambda_2$; moreover, Lemma 4.2 can be stated also in the case $\lambda = \lambda_2$ with an analogous proof and so we can again look for lower critical points of the function $S_{h,\psi}$ in order to obtain solutions of problem $P_\psi(h)$.

It is clear that also for $\lambda = \lambda_2$ the functions $S_{h,\psi}$ and $\sigma_{h,\psi}$ have the same properties that we have seen in the previous section. In particular, Lemma 4.6 also holds for $\lambda = \lambda_2$, with the only difference that, since in this case $\inf_{P_t} f_h \circ (\Pi_2 + \Pi_3) = -\infty$ if $\Pi_2 h \neq 0$, we have $\lim_{t\to\infty} \sigma_{h,\psi}(t) = -\infty$. But the properties of $\sigma_{h,\psi}$ allow us to state Theorems 4.7 and 4.8 also in the case $\lambda = \lambda_2$. Their proofs are similar to the case $\lambda_1 < \lambda < \lambda_2$; but, since for $\lambda = \lambda_2$ the functional $f_h \circ (\Pi_2 + \Pi_3)$ is not strictly convex, it could happen that, for a lower critical point t for $S_{h,\psi}$, the functional $f_{h,\psi} + I_{P_t}$ could have more than one minimum point. So the set of solutions of $P_\psi(h)$ has a different structure, which will be described in Proposition 5.3; its proof needs the following lemma.

Lemma 5.2. *Suppose $\lambda = \lambda_2$; if u and v solve problem $P_\psi(h)$ and we have $\int_\Omega (v-u)e_1\,dx = 0$, then $v - u \in X_2$ and we have $\int_\Omega h(v-u)\,dx = 0$.*

Proof. First remark that u and v minimize $f_{h,\psi}$ on P_t, with $t = \int_\Omega ue_1\,dx = \int_\Omega ve_1\,dx$, because they are solutions of problem $P_\psi(h)$ and $f_{h,\psi} + I_{P_t}$ is convex.

Consider the function $D : [0,1] \to \mathbb{R}$ defined by

$$D(s) = f_{h,\psi}(u + s(v-u))$$

(notice that $u + s(v-u) \in P_t \cap K_\psi$ for every $s \in [0,1]$). We have

$$D''(s) = \int_\Omega |D(v-u)|^2\,dx - \lambda_2 \int_\Omega (v-u)^2\,dx \ge 0$$

because $\Pi_1(v-u) = 0$. But, since u and v minimize $f_{h,\psi} + I_{P_t}$, we must have $D''(s) = 0$, that is, $v - u \in X_2$. This implies

$$D'(s) = \int_\Omega h(v-u)\,dx,$$

which must be equal to zero because $D(0) = D(1) = S_{h,\psi}(t)$. □

Proposition 5.3. *Under the same assumptions of Theorem* 4.7 (*or, equivalently, of Theorem* 4.8), *with* $\lambda = \lambda_2$, *the set* S *of solutions of problem* $P_\psi(h)$, *if it is not empty, is the union of a point* u_0 *and of a family* $(S_i)_{i\in I}$ *of pairwise disjoint convex sets:* $S = \{u_0\} \cup \bigcup_{i\in I} S_i$; *furthermore:*

(a) $u_0 \le u_i$ *for all* $u_i \in S_i$, $i \in I$;
(b) $u_i \in S_i \Rightarrow S_i = \{u_i + v \mid v \in X_2,\ u_i + v \in K_\psi,\ \int_\Omega hv\,dx = 0\}$ *for all* $i \in I$.

Proof. Since $P_\psi(h)$ has solution, $S_{h,\psi}$ has lower critical points (by Lemma 4.2). Moreover, since $P_\psi(h)$ has a minimal solution u_0 (see Proposition 3.7), $t_0 = \int_\Omega u_0 e_1\,dx$ is the minimum lower critical point of $S_{h,\psi}$ (because $e_1 > 0$). Let t_0 and $(t_i)_{i\in I}$ be the lower critical points of $S_{h,\psi}$ and set

$$S_i = \{u \in P_{t_i} \cap K_\psi \mid u \text{ is a minimum point for } f_{h,\psi} + I_{P_{t_i}}\}.$$

We claim that $S_0 = \{u_0\}$. Suppose, contrary to our claim, that there exists $\widetilde{u} \ne u_0$, $\widetilde{u} \in S_0$; then, by Theorem 2.6, there exists a solution $u \le u_0 \wedge \widetilde{u}$ and moreover, by Lemma 4.2, $\tau = \int_\Omega ue_1\,dx$ is a lower critical point for $S_{h,\psi}$. By the previous lemma, $\widetilde{u} - u_0 \in X_2$, and so it cannot have constant sign. Therefore

$$\tau = \int_\Omega ue_1 \le \int_\Omega (u_0 \wedge \widetilde{u})e_1 < \int_\Omega u_0 e_1 = t_0,$$

which is a contradiction since t_0 is the minimum lower critical point of $S_{h,\psi}$.

(b) follows easily from the previous lemma. □

Now let us show in a simple example the situation of the previous proposition.

Example. Let $\lambda = \lambda_2$, $h = 0$ and $\psi = -e_1$. Then one can easily verify that the minimal solution is $u_0 = \psi = -e_1$ (because $u_0 + \lambda_2\Delta^{-1}u_0 > 0$ in Ω) and the set S of solutions of problem $P_{-e_1}(0)$ is $S = \{u_0\} \cup S_1$ where

$$S_1 = \left\{ u \in X_2 \,\middle|\, u \ge -\frac{\lambda_1}{\lambda_2} e_1 \right\}.$$

For the proof it suffices to remark that

$$S_{h,\psi}(t) = \begin{cases} (\lambda_1 - \lambda_2)t^2/2 & \text{if } t \ge -1, \\ \infty & \text{if } t < -1, \end{cases}$$

and so $t_0 = -1$ and $t_1 = 0$ are the unique lower critical points of $S_{h,\psi}$.

6. The case $\lambda = \lambda_1$

If $\lambda = \lambda_1$, the solvability of problem $P_\psi(h)$ is described by the following theorem.

THEOREM 6.1. *Assume $\lambda = \lambda_1$; then the solutions of problem $P_\psi(h)$ are the minimum points of the functional $f_{h,\psi}$; furthermore:*

(a) *if $\int_\Omega he_1\,dx < 0$, then there is no solution for $P_\psi(h)$;*
(b) *if $\int_\Omega he_1\,dx = 0$, then u is a solution of $P_\psi(h)$ if and only if $u \in K_\psi$ and $\Delta u + \lambda_1 u = h$, that is, the solution set of problem $P_\psi(h)$ is $K_\psi \cap (\Delta + \lambda_1)^{-1}h$ and so, if it is not empty, it is a half-line parallel to e_1;*
(c) *if $\int_\Omega he_1\,dx > 0$, then there is a unique solution for $P_\psi(h)$.*

REMARK. Let us observe that, while in case (b) the existence of a solution depends upon the obstacle ψ, in cases (a) and (c) it is independent of it.

EXAMPLE. Let $h = 0$ and assume $\sup_\Omega \psi/e_1 = \infty$. Then there is no solution to $P_\psi(h)$: indeed, by Theorem 6.1(b), a solution would be an eigenfunction for the first eigenvalue (which cannot belong to K_ψ under our assumption).

PROOF OF THEOREM 6.1. If $\lambda = \lambda_1$, the functional $f_{h,\psi}$ is convex and then the solutions of $P_\psi(h)$, the lower critical points of $f_{h,\psi}$, are its minimum points.

(a) It is sufficient to take the function $v = u + e_1$ as a test function and to remark that $f'_h(u)[e_1] < 0$ for every $u \in H^{1,2}_0(\Omega)$.

(b) Let u be a solution for $P_\psi(h)$; for every $\gamma \in C^\infty_0(\Omega)$ let $t > 0$ be small enough so that it is $\Delta(e_1 + t\gamma) < 0$ (observe that $\Delta e_1 < 0$). If we set $v = u + t\gamma + e_1$, we obtain $tf'_h(u)[\gamma] \geq 0$ and the assertion follows.

(c) Let us prove that the minimum of $f_{h,\psi}$ exists: we remark that

$$f_h = f_h \circ (\Pi_2 + \Pi_3) + f_h \circ \Pi_1,$$
$$f_h \circ \Pi_1(u) = \left(\int_\Omega he_1\,dx\right)\left(\int_\Omega ue_1\,dx\right) \quad \forall u \in H^{1,2}_0(\Omega);$$

it results, for suitable positive constants c_1 and c_2, that

$$f_h \circ (\Pi_2 + \Pi_3)(u) \geq -c_1 + c_2\|(\Pi_2 + \Pi_3)u\|_{H^{1,2}_0} \quad \forall u \in H^{1,2}_0(\Omega);$$

moreover, we have, obviously, $\int_\Omega ue_1\,dx \geq \int_\Omega \psi e_1\,dx$ for every $u \in K_\psi$; it follows that the sublevels of $f_{h,\psi}$ are bounded in $H^{1,2}_0(\Omega)$ and so there exists at least one minimum point, because $f_{h,\psi}$ is lower semicontinuous in $L^2(\Omega)$.

Let us prove that there exists a unique minimum point (note that $f_{h,\psi}$ is not strictly convex): let u and v be two minimum points, and define

$$N(t) = f_{h,\psi}(u + t(v - u));$$

then $N : [0,1] \to \mathbb{R}$ because $u, v \in K_\psi$ and K_ψ is convex.

We have

$$N''(t) = \int_\Omega |D(v - u)|^2\,dx - \lambda_1 \int_\Omega (v - u)^2\,dx,$$

which implies $(\Pi_2+\Pi_3)(v-u)=0$ because the function N cannot be strictly convex (u and v are minimum points); furthermore, $(\Pi_2+\Pi_3)u=(\Pi_2+\Pi_3)v$ implies

$$N'(t)=\int_\Omega h(v-u)\,dx=\int_\Omega he_1\,dx\int_\Omega (v-u)e_1\,dx,$$

which must be zero because $N(0)=N(1)=\min f_{h,\psi}$. Therefore also $\Pi_1 v=\Pi_1 u$ and so u and v coincide. $\square$

Notice that Theorem 6.1(b) implies that, if $\lambda=\lambda_1$ and $\int_\Omega he_1\,dx=0$, and if the minimum of $f_{h,\psi}$ exists, then $\min f_{h,\psi}=\min f_h$.

Let us point out that, however, in this case, $\inf f_{h,\psi}=\min f_h$ (even if $f_{h,\psi}$ has no minimum).

In fact, one can infer from Lemma 4.5 that

$$\inf[f_h\circ(\Pi_2+\Pi_3)+I_{K_\psi}]=\inf f_h\circ(\Pi_2+\Pi_3)$$

and so, in order to get $\inf f_{h,\psi}=\min f_h$, it suffices to remark that $f_h=f_h\circ(\Pi_2+\Pi_3)$ if $\lambda=\lambda_1$ and $\int_\Omega he_1\,dx=0$.

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