

UNKNOTTED PERIODIC ORBITS FOR REEB FLOWS ON THE THREE-SPHERE

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For Louis with admiration and gratitude

It is well known that a Reeb vector field on S^3 has a periodic solution. Sharpening this result we shall show in this note that every Reeb vector field X on S^3 has a periodic orbit which is unknotted and has self-linking number equal to -1 . If the contact form λ is non-degenerate, then there is even a periodic orbit P which, in addition, has an index $\mu(P) \in \{2, 3\}$, and which spans an embedded disc whose interior is transversal to X . The proofs are based on a theory for partial differential equations of Cauchy–Riemann type for maps from punctured Riemann surfaces into $\mathbb{R} \times S^3$, equipped with special almost complex structures related to the contact form λ on S^3 .

1. Introduction and results

We consider the three-sphere (S^3, λ) equipped with a contact form λ . Recall that, by definition, a *contact form* is a one-form having the property that $\lambda \wedge d\lambda$ is a volume form. The two-dimensional kernel, $\xi = \ker \lambda \subset TS^3$, constitutes the contact structure defined by λ . The restriction of $d\lambda$ onto $\xi \oplus \xi$ is non-degenerate so that $(\xi, d\lambda)$ is a symplectic plane field. The contact form λ also determines the so-called *Reeb vector field* $X_\lambda = X$. It is uniquely defined by

$$(1.1) \quad i_X d\lambda = 0 \quad \text{and} \quad i_X \lambda = 1.$$

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The Reeb vector field is transversal to the contact structure ξ , so that the tangent bundle TS^3 naturally splits:

$$(1.2) \quad TS^3 = \mathbb{R}X_\lambda \oplus \xi.$$

If φ^t is the flow of X , then $(\varphi^t)^*\lambda = \lambda$. Consequently, the linearized flow $T\varphi^t : TS^3 \rightarrow TS^3$ leaves the splitting (1.2) invariant:

$$(1.3) \quad T\varphi^t(m) : \xi_m \rightarrow \xi_{\varphi^t(m)},$$

moreover, the map is symplectic with respect to $d\lambda$.

A nowhere vanishing vector field on S^3 need not admit periodic solutions (see [16] and [17]). However, every Reeb vector field on S^3 has a periodic solution (see [10]). The aim of this note is to prove more: every Reeb vector field on S^3 has a periodic orbit which is unknotted and which has some additional properties. In order to formulate the results we first introduce some notations.

Assuming S^3 to be equipped with an orientation we shall consider only contact forms λ satisfying $\lambda \wedge d\lambda > 0$. Let X be the associated Reeb vector field and $\xi = \ker \lambda$ the associated contact structure. We first recall the concept of a self-linking number $sl(x) \in \mathbb{Z}$ of an admissible loop $x : S^1 \rightarrow S^3$. The loop x is called *admissible* if it has the following properties:

$$(1.4) \quad x \text{ is transversal to } \xi \text{ and } x = y \circ \beta,$$

where $y : S^1 \rightarrow S^3$ is an embedded loop and $\beta : S^1 \rightarrow S^1$ is a smooth map. A typical example is a periodic orbit (x, T) of the Reeb vector field X . In this case the loop x_T defined by $x_T(e^{2\pi it}) = x(Tt)$ is admissible. The integer $sl(x)$ for a given admissible loop x is now defined as follows. We choose a smooth map $u : D \rightarrow S^3$, with the disc $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$, satisfying

$$u|_{\partial D} = x, \quad \partial D = S^1.$$

Then we choose a nowhere vanishing section Z of $u^*\xi$. It induces a section along $x^*\xi$ nowhere tangent to x . Now we push x into the direction of Z to obtain a new loop x' disjoint from x . The loops x and x' have natural orientations induced from the orientation of S^1 as the boundary of D . The oriented intersection number of x' with u will be denoted by $sl(x)$. Since we are on S^3 , this integer does not depend on the choices involved and is called the *self-linking number* of the loop x .

If $x : \mathbb{R} \rightarrow S^3$ is a T -periodic solution of $\dot{x} = X(x)$ for some $T > 0$, we define the loop $x_T : S^1 \rightarrow S^3$ by

$$(1.5) \quad x_T(e^{2\pi it}) = x(tT).$$

Since x_T is transversal to ξ we can define the self-linking number $sl(x, T)$ in the obvious way. The smallest positive number T_0 satisfying $x(T_0) = x(0)$ is called

the *minimal period* of x . The positive integer T/T_0 is called the *covering number* of the periodic solution (x, T) and abbreviated in the following by $\text{cov}(x, T)$. The image of a periodic solution (x, T) will be denoted by P_x ; it is an embedded circle in S^3 . P_x has a natural orientation and we define the self-linking number $sl(P_x)$ by

$$(1.6) \quad sl(P_x) = sl(x, T_0).$$

A periodic orbit (x, T) is called *unknotted* if $T = T_0$ and if P_x is the boundary of an embedded disc. Since in this case we can identify P_x and (x, T_0) we will just say that P_x is unknotted. It will also be convenient in the following to denote by P a periodic orbit whose period is minimal. With these notations our first result is the following.

THEOREM 1.1. *If λ is any contact form on S^3 , then the associated Reeb vector field X has a periodic orbit P which is unknotted and has self-linking number $sl(P) = -1$.*

We would like to mention an application to Hamiltonian vector fields X_H in (\mathbb{R}^4, ω_0) , with the standard symplectic form ω_0 . On a regular energy surface $S = \{x \in \mathbb{R}^4 \mid H(x) = \text{const}\}$ the flow lines of X_H are the integral curves of the canonical line bundle $\ker(\omega_0|_S) \subset TS$. Assuming the hypersurface S to be of contact type and diffeomorphic to S^3 we conclude from Theorem 1.1 that S carries an unknotted periodic orbit. We also conclude from Theorem 1.1 that there are spherelike hypersurfaces in \mathbb{R}^4 on which the Hamiltonian flow cannot be conjugate to a Reeb flow. Indeed, K. Cieliebak constructed in [3] spherelike hypersurfaces $S \subset (\mathbb{R}^4, \omega_0)$ whose Hamiltonian flows have only knotted periodic orbits. In his construction, the hypersurface S is even of confoliation type, that is, there exists a one-form λ on S satisfying $\omega_0|_S = d\lambda$ and $\lambda \wedge d\lambda \geq 0$ (in contrast to the contact condition $\lambda \wedge d\lambda > 0$). Since every compact three-manifold admits contact forms, Theorem 1.1 and also Theorem 1.4 below give necessary conditions for a three-manifold to be diffeomorphic to S^3 . For a characterization of the tight S^3 we refer to [11].

We next restrict the class of contact forms under consideration. We call the contact form λ *non-degenerate* if all the periodic orbits (x, T) of the associated Reeb vector fields are non-degenerate. This requires, taking a transversal section of P_x , that the linearizations of the Poincaré section map and all its iterates do not contain 1 in their spectra. There are many non-degenerate contact forms as the following result from [12] shows.

PROPOSITION 1.2. *Fix a contact form λ on S^3 . There exists a dense set $\mathcal{R} \subset C^\infty(S^3, (0, \infty))$ so that for every $f \in \mathcal{R}$ the contact form $f\lambda$ is non-degenerate.*

With a non-degenerate periodic orbit (x, T) of a Reeb vector field X on S^3 we can associate another integer-valued index as follows. Again we take a disc map $u : D \rightarrow S^3$ extending x_T and choose a nowhere vanishing section Z of $u^*\xi$. This section can be used to define a symplectic trivialization of $u^*\xi$. This way we obtain a smooth family of symplectic maps $\Psi(z) : \xi_{x_T(z)} \rightarrow \mathbb{C}$, for $z \in S^1 = \partial D$. If φ^t is the flow of X , the maps

$$(1.7) \quad \widehat{L}(t) := T\varphi^t(x(0))|_{\xi_{x(0)}} : \xi_{x(0)} \rightarrow \xi_{x(t)}$$

are symplectic and we define

$$(1.8) \quad L(t) = \Psi(e^{2\pi it})\widehat{L}(tT)\Psi(1)^{-1}, \quad 0 \leq t \leq 1.$$

This is an arc of linear symplectic maps in \mathbb{C} starting at the identity at time $t = 0$ and ending at time $t = 1$ at a symplectic map which does not have 1 in its spectrum. For such arcs $L(t)$ one can define a Maslov-type index μ as follows.

Denote by Σ the collection of all arcs of linear symplectic maps $\beta : [0, 1] \rightarrow \text{Sp}(1)$ starting at Id and satisfying $1 \notin \sigma(\beta(1))$. Denote by G the set of smooth arcs starting and ending at Id. The homotopy classes in G represent $\pi_1(\text{Sp}(1))$. We observe that G operates on Σ via

$$G \times \Sigma \rightarrow \Sigma, \quad (\alpha, \beta) \mapsto \alpha\beta,$$

where $(\alpha\beta)(t) = \alpha(t)\beta(t)$. Given $\alpha \in \Sigma$ we define the arc $\alpha^{-1} \in \Sigma$ by $\alpha^{-1}(t) = \alpha(t)^{-1}$. We recall that there is a natural isomorphism $\mu_M : \pi_1(\text{Sp}(1)) \rightarrow \mathbb{Z}$, called the *Maslov isomorphism*, mapping the loop $[t \rightarrow e^{2\pi it} \text{Id}]$ onto 1. This induces a homotopy invariant map $\mu_M : G \rightarrow \mathbb{Z}$. The following proposition is a special case of a result in [14].

PROPOSITION 1.3. *There exists a unique map $\mu : \Sigma \rightarrow \mathbb{Z}$ having the following properties:*

- μ is homotopy invariant.
- For $\alpha \in \Sigma$ and $\beta \in G$ we have $\mu(\alpha\beta) = \mu(\alpha) + 2\mu_M(\beta)$.
- $\mu(\alpha) + \mu(\alpha^{-1}) = 0$.
- $\mu(\gamma) = 1$ for the arc $\gamma(t) = e^{\pi it} \text{Id}$, $t \in [0, 1]$.

The *index* $\mu(x, T) \in \mathbb{Z}$ of a non-degenerate periodic orbit (x, T) of the Reeb vector field X associated with λ on S^3 is defined by $\mu(x, T) = \mu(L)$, where $L \in \Sigma$ is the symplectic arc introduced in (1.8) above. Since we are on S^3 , the definition does not depend on the choices involved in the construction. Recall that if $T = T_0$ is the minimal period, we can identify (x, T_0) and P_x . Our second result is as follows.

THEOREM 1.4. *Assume the contact form λ on S^3 is non-degenerate. Then the associated Reeb vector field X has an unknotted periodic orbit P with the following properties. The orbit P spans an embedded disc whose interior is transversal to X . Moreover, P has self-linking number $sl(P) = -1$ and index $\mu(P) \in \{2, 3\}$. If the contact form is overtwisted we find an unknotted periodic orbit P with self-linking number -1 and index $\mu(P) = 2$.*

It is an open question whether for a non-degenerate tight contact form there always exists an unknotted periodic orbit P having self-linking number -1 and index $\mu(P) = 3$. We point out that if $\mu(P) = 3$ in Theorem 1.4 and if, moreover, the period T_0 is minimal among all periodic solutions of X then the periodic orbit P is the binding orbit of an open book decomposition of S^3 into special embedded planes. Every plane of this decomposition is a global surface of section for the Reeb flow on $S^3 \setminus P$. It follows that the contact structure λ considered is tight. This is a special case of results in [11] and [12].

In order to prove Theorems 1.1 and 1.4 we need some results on pseudoholomorphic curves described next. We consider a compact three-manifold equipped with the contact form λ , associated contact structure ξ and Reeb vector field X . We now choose an almost complex structure $J : \xi \rightarrow \xi$ compatible with the symplectic structure $d\lambda$ on ξ in the sense that $(h, k) \mapsto d\lambda(h, Jk)$ defines an inner product on ξ . On the four-manifold $\mathbb{R} \times S^3$ we next define a special, \mathbb{R} -invariant almost complex structure associated with λ and J as follows:

$$(1.9) \quad \tilde{J}(a, m)(h, k) = (-\lambda(m)(k), J(m)\pi k + hX(m)),$$

where $\pi : TM = \mathbb{R}X \oplus \xi \rightarrow \xi$ is the projection along X and $(h, k) \in T_{(a,m)}(\mathbb{R} \times S^3)$. A *finite energy plane* is a smooth map $\tilde{u} = (a, u) : \mathbb{C} \rightarrow \mathbb{R} \times M$ satisfying, in the coordinates $z = s + it$, the following partial differential equation of Cauchy–Riemann type:

$$(1.10) \quad \tilde{u}_s + \tilde{J}(\tilde{u})\tilde{u}_t = 0,$$

and the energy requirement $0 < E(\tilde{u}) < \infty$. The *energy* is defined by

$$(1.11) \quad E(\tilde{u}) = \sup_{\mathbb{C}} \int_{\mathbb{C}} \tilde{u}^* d\lambda_{\varphi}.$$

The one-form λ_{φ} on $\mathbb{R} \times M$ is defined by

$$(1.12) \quad \lambda_{\varphi}(a, m)(h, k) = \varphi(a)\lambda(m)(k),$$

where $\varphi : \mathbb{R} \rightarrow [0, 1]$ is smooth and satisfies $\varphi'(s) \geq 0$ for all $s \in \mathbb{R}$. The supremum in (1.11) is taken over all such φ 's. The asymptotic behaviour of a finite energy plane $\tilde{u}(z)$ as $|z| \rightarrow \infty$ is determined by periodic solutions of the Reeb vector field X and we recall the following crucial result from [10] and [13].

PROPOSITION 1.5. *Assume $\tilde{u} = (a, u) : \mathbb{C} \rightarrow \mathbb{R} \times M$ is a finite energy plane. Then there exists a sequence $R_k \rightarrow \infty$ such that $a(R_k e^{2\pi i t}) \rightarrow \infty$ and $\lim u(R_k e^{2\pi i t}) = x(Tt)$ as $k \rightarrow \infty$ in $C^\infty(S^1)$, for a T -periodic solution x of the Reeb vector field X on M . The period is equal to $T = E(\tilde{u})$. Moreover, if this solution is non-degenerate, then $a(Re^{2\pi i t}) \rightarrow \infty$ and*

$$\lim_{R \rightarrow \infty} u(Re^{2\pi i t}) = x(Tt)$$

in $C^\infty(S^1)$, and the convergence is of exponential nature.

If the periodic solution x in Proposition 1.5 is non-degenerate, we call it the asymptotic limit of the finite energy plane $\tilde{u} = (a, u)$. In this case we can extend $u : \mathbb{C} \rightarrow M$ over a circle at infinity to obtain a map $\bar{u} : D = \mathbb{C} \cup S^1 \rightarrow M$ satisfying $\bar{u}(+\infty \cdot e^{2\pi i t}) = x(tT)$ over S^1 . The map \bar{u} defines a homology class in $H_2(M, P_x; \mathbb{Z})$ denoted by $[\bar{u}]$. Using a symplectic trivialization of $\bar{u}^*\xi$ we can define the index $\mu(\tilde{u}) \in \mathbb{Z}$ by taking the symplectic arc associated with the periodic solution (x, T) at the boundary via this trivialization. This integer does not depend on the choice of the trivialization. It only depends on the homotopy class of disc maps $u : D \rightarrow M$ satisfying $u(e^{2\pi i t}) = x(Tt)$. Therefore, on $M = S^3$ the integer $\mu(\tilde{u}) = \mu(x, T)$ does not depend on the choice of the disc map. We should recall from [14], Theorem 4.4, that

$$(1.13) \quad \mu(\tilde{u}) \geq 2$$

for every finite energy plane $\tilde{u} : \mathbb{C} \rightarrow \mathbb{R} \times M$ having a non-degenerate asymptotic limit (x, T) .

The next results show how the index $\mu(\tilde{u})$ influences the geometry of the plane $u(\mathbb{C}) \subset S^3$. The results extend Proposition 4.12 of [13] to multicovered asymptotic limits.

THEOREM 1.6. *Assume $\tilde{u} = (a, u) : \mathbb{C} \rightarrow \mathbb{R} \times S^3$ is an embedded finite energy plane with non-degenerate asymptotic limit (x, T) . If $\mu(\tilde{u}) \leq 3$ then $u(\mathbb{C}) \cap P_x = \emptyset$ and $u : \mathbb{C} \rightarrow M \setminus P_x$ is an embedding transversal to the Reeb vector field X .*

Theorem 1.6 describes a situation in which the next theorem is applicable. It does not require the asymptotic limits to be non-degenerate nor simply covered.

THEOREM 1.7. *Let $\tilde{u} = (a, u) : \mathbb{C} \rightarrow \mathbb{R} \times S^3$ be a non-constant finite energy plane. Assume $u : \mathbb{C} \rightarrow S^3$ is an injective immersion. Then every asymptotic limit (x, T) of \tilde{u} is simply covered, $\text{cov}(x, T) = 1$. The self-linking number of (x, T) is $sl(x, T) = -1$. Moreover, $P_x = x(\mathbb{R})$ is unknotted and $P_x \cap u(\mathbb{C}) = \emptyset$.*

Our main result (Theorem 1.1) will follow immediately from Theorem 1.7 and Proposition 1.5 in view of the following existence result.

THEOREM 1.8. *Let λ be a contact form on S^3 and J a compatible almost complex structure on the associated contact structure ξ . Then there exists a non-constant finite energy plane $\tilde{u} := (a, u) : \mathbb{C} \rightarrow \mathbb{R} \times S^3$ such that u is an injective immersion.*

The proofs of the theorems will be based on the techniques of pseudoholomorphic curves in symplectizations of contact manifolds developed in [10]–[15].

2. Dealing with the covering number

The aim of this section is to prove Theorems 1.6 and 1.7 of the introduction.

THEOREM 2.1. *Assume $\tilde{u} := (a, u) : \mathbb{C} \rightarrow \mathbb{R} \times S^3$ is an embedded finite energy plane with non-degenerate asymptotic limit (x, T) . If $\mu(\tilde{u}) \leq 3$ then $u(\mathbb{C}) \cap P_x = \emptyset$ and $u : \mathbb{C} \rightarrow M \setminus P_x$ is an embedding transversal to the Reeb vector field X .*

We begin the proof with

LEMMA 2.2. *Assume the finite energy plane $\tilde{u} := (a, u) : \mathbb{C} \rightarrow \mathbb{R} \times S^3$ is non-degenerate and satisfies $\mu(\tilde{u}) \leq 3$. If $R > 0$ is sufficiently large, then*

$$u : \mathbb{C} \setminus B_R \rightarrow S^3 \setminus P_x$$

and it is an embedding.

PROOF. Setting $z = e^{2\pi(s+it)}$ we know from Proposition 1.5 that $u(s, t) := u(e^{2\pi(s+it)}) \rightarrow x(Tt)$ as $s \rightarrow \infty$ and we can study the map u in the convenient local coordinates $S^1 \times \mathbb{R}^2$ of a tubular neighbourhood of $P = P_x \subset S^3$ introduced in [13]. In these coordinates $S^1 \times \{0\}$ corresponds to P and $\{0\} \times \mathbb{R}^2 \subset \mathbb{R} \times \mathbb{R}^2$ corresponds to the contact planes ξ along P . For the period we have $T = kT_0$ where $k = \text{cov}(x, T)$. We set the minimal period $T_0 = 1$ for notational convenience. Working in the covering space \mathbb{R} of $S^1 = \mathbb{R}/\mathbb{Z}$ the map $\tilde{u} : \mathbb{R}^2 \rightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2$ has the following presentation in these local coordinates, where $s \geq s_0$ is large:

$$(2.1) \quad \tilde{u}(s, t) = (a(s, t), \vartheta(s, t), z(s, t))$$

and

$$(2.2) \quad \begin{aligned} a(s, t) &= s + a_0 + \alpha(s, t), & \vartheta(s, t) &= kt + \beta(s, t), \\ z(s, t) &= e^{\int_{s_0}^s \mu(\tau) d\tau} [e(t) + r(s, t)]. \end{aligned}$$

The functions α, β and r are periodic in t of period 1. Moreover, as $s \rightarrow \infty$, $r(s, t) \rightarrow 0$ with all its derivatives, uniformly in t . The functions α and β converge exponentially fast to zero together with all their derivatives. In addition, $\mu(s) \rightarrow \lambda$, where λ is a negative eigenvalue with normalized eigenfunction $e(t) = e(t+1) \in \mathbb{R}^2$ of a linear selfadjoint operator. It is of the form $h \mapsto -J(t)\dot{h} - A(t)h$ in

$L^2(S^1, \mathbb{R}^2)$ on the domain $H^{1,2}(S^1, \mathbb{R}^2)$. Clearly $e(t) \neq 0$. The matrix functions $A(t)$ and $J(t)$ are periodic in t with period $1/k$. In abstract terms one may consider e as a distinguished section of $x_T^* \xi$ so that $e(t) \in \xi_{x(kt)}$. For details we refer to [13].

Considering the eigenfunction $e(t)$ we assume that $1 \leq j \leq k$ is the smallest integer satisfying

$$(2.3) \quad e(0) = \tau e(j/k) \quad \text{for some } \tau > 0$$

and show that

$$(2.4) \quad j/k = 1/l$$

for an integer l and

$$(2.5) \quad e(t) = e(t + 1/l), \quad t \in \mathbb{R}.$$

Since $e(t)$ is an eigenfunction of a first order differential equation whose coefficients are $1/k$ -periodic we conclude from (2.3) that $e(t) = \tau e(j/k + t)$ so that $e(t) = \tau^m e(mj/k + t)$ for every integer m and $t \in \mathbb{R}$. Setting $m = k$ and recalling $e(t + 1) = e(t)$ we conclude from $e(t) = \tau^k e(t)$ that $\tau = 1$. It remains to verify that k/j is an integer. Arguing by contradiction we assume $k/j = l + r/j$ and $1 \leq r \leq j - 1$. Then $l \cdot j/k = 1 - r/k$ and

$$e(t) = e(lj/k + t) = e(1 - r/k + t) = e(-r/k + t).$$

Setting $t = r/k$ we obtain $e(0) = e(r/k)$ contradicting the choice of the integer j , which is the smallest such integer. Hence (2.4) and (2.5) are proved.

We next show that the assumption $\mu(\tilde{u}) \leq 3$ implies that $l = 1$. We recall the definition of the winding number $\text{wind}_\infty(\tilde{u}) \in \mathbb{Z}$ associated with a non-degenerate finite energy plane \tilde{u} , introduced in [14]. Recalling that $TS^3 = \mathbb{R}X \oplus \xi$ we take the tubular neighbourhood $V = \Phi(U)$ of the periodic orbit P , where Φ is a diffeomorphism

$$(2.6) \quad \Phi : \xi|P \supset U \rightarrow V = \Phi(U) \subset S^3$$

of a neighbourhood U of the zero section of $\xi|P$ satisfying $\Phi(0_p) = p$ for every $p \in P$. Moreover, the fibrewise derivative of Φ at 0_p is the inclusion of ξ_p into $T_p S^3$. For large s we can represent $u(s, t)$ uniquely as

$$(2.7) \quad u(s, t) = \Phi(x(k\beta(s, t)), w(s, t)),$$

with $w(s, t) \in \xi_{x(k\beta(s, t))}$. The function β satisfies $\beta(s, t + 1) = \beta(s, t) + 1$ and $\beta(s, t) \rightarrow t$ as $s \rightarrow \infty$. We obtain the distinguished nowhere vanishing section v of $x_T^* \xi$ by

$$(2.8) \quad v(t) = \lim_{s \rightarrow \infty} \frac{w(s, t)}{|w(s, t)|} \in \xi_{x(kt)}.$$

Let $D = \{z \mid |z| \leq 1\}$ and choose a disc map $\varphi : D \rightarrow S^3$ satisfying $\varphi(e^{2\pi it}) = x(kt)$. Then choose a nowhere vanishing section Z of $\varphi^*\xi$ so that $0 \neq Z(z) \in \xi_{\varphi(z)}$. The *winding number* of \tilde{u} is defined as

$$(2.9) \quad \text{wind}_\infty(\tilde{u}) = \text{wind}(v|[0, 1], Z|[0, 1]) = \text{wind}(f).$$

Here, $\text{wind}(f) \in \mathbb{Z}$ is the winding number of the function $f : S^1 \rightarrow \mathbb{C} \setminus \{0\}$ defined by $v(t) = f(t) \cdot Z(t) \in \xi_{x(kt)}$, where $0 \leq t \leq 1$. The complex multiplication in ξ is defined by the almost complex structure J . In view of its homotopy invariance, the integer $\text{wind}_\infty(\tilde{u})$ does not depend on the choice of the non-vanishing section Z of $\varphi^*\xi$, and since we are on S^3 it is also independent of the choice of the disc map φ satisfying $\varphi(e^{2\pi it}) = x(kt)$. In particular, we can homotope φ to a special disc map ψ given by $\psi(z) = \psi_0(z^k)$, where $\psi_0 : D \rightarrow S^3$ satisfies $\psi_0(e^{2\pi it}) = x(t)$. This way we find a section Z of $\psi^*\xi$ satisfying, in addition, $Z(t) = Z(t + 1/k)$. Observe that, in the local coordinates (2.2), we have $v(t) = \lambda(t)e(t)$ with $0 \neq \lambda(t) \in \mathbb{R}$. Since $\text{wind}(f) = \text{wind}(\lambda f)$ we conclude from $e(t + 1/l) = e(t)$ and $1/l = j \cdot 1/k$ that

$$(2.10) \quad \text{wind}(v|[0, 1], Z|[0, 1]) = l \text{wind}(v|[0, 1/l], Z|[0, 1/l]).$$

The invariants $\text{wind}_\infty(\tilde{u})$ and $\mu(\tilde{u})$ can be compared with each other in a symplectic trivialization of $\varphi^*\xi$. One finds $0 \leq \text{wind}_\infty(\tilde{u}) - 1 \leq \frac{1}{2}\mu(\tilde{u}) - 1$ (see [14]). Consequently, the assumption $\mu(\tilde{u}) \leq 3$ implies $\text{wind}_\infty(\tilde{u}) = 1$ and we conclude that both integers on the right hand side of (2.10) are equal to 1. In particular, $l = 1$ and we have proved that, if $\mu(\tilde{u}) \leq 3$, then

$$(2.11) \quad e(t + j/k) \neq \tau e(t)$$

for all $t \in \mathbb{R}$, $\tau > 0$ and $1 \leq j \leq k - 1$.

From the asymptotic formula (2.2) one easily deduces for large R that $u(\mathbb{C} \setminus B_R) \cap P = \emptyset$ and $u : \mathbb{C} \setminus B_R \rightarrow S^3 \setminus P$ is an immersion (we refer to [13] for a proof). We shall show that $u|_{\mathbb{C} \setminus B_R}$ is injective. Arguing by contradiction we assume

$$(2.12) \quad u(s_j, t_j) = u(s'_j, t'_j)$$

for sequences satisfying

$$(2.13) \quad (s_j, t_j) \neq (s'_j, t'_j) \quad \text{and} \quad s_j \rightarrow \infty, s'_j \rightarrow \infty.$$

In view of the periodicity in t we may assume $t_j \rightarrow t_* \in [0, 1)$ and $t'_j \rightarrow t'_* \in [0, 1)$. Since $\vartheta(s_j, t_j) = \vartheta(s'_j, t'_j) \pmod{1}$ we deduce from (2.2), as $j \rightarrow \infty$, that $k(t_* - t'_*) = 0 \pmod{1}$ so that

$$t_* - t'_* = j/k \quad \text{and} \quad j \in \{0, 1, \dots, k - 1\}.$$

Since $z(s_j, t_j) = z(s'_j, t'_j)$ we conclude from (2.2), as $j \rightarrow \infty$, that $e(t_*) = \tau e(t_* + j/k)$, for some $\tau > 0$. Consequently, by (2.11), we find $j = 0$ and $\tau = 1$ so that $t_* = t'_*$. Taking norms, $\|z(s_j, t_j)\| = \|z(s'_j, t'_j)\|$, we deduce from (2.2) that $s_j - s'_j \rightarrow 0$. Summarizing,

$$(2.14) \quad s_j - s'_j \rightarrow 0 \quad \text{and} \quad t_j - t'_j \rightarrow 0.$$

Since u is an immersion one concludes from (2.2), (2.14) and $u(s_j, t_j) = u(s'_j, t'_j)$ that $(s_j, t_j) = (s'_j, t'_j)$ if j is large, contradicting (2.13). We have proved that $u : \mathbb{C} \setminus B_R \rightarrow S^3 \setminus P$ is an injective immersion for R sufficiently large. In view of the asymptotic behaviour (2.2) it must be an embedding. The proof of Lemma 2.2 is complete. \square

LEMMA 2.3. *Assume $\tilde{u} := (a, u) : \mathbb{C} \rightarrow \mathbb{R} \times S^3$ is an injective, non-degenerate finite energy plane satisfying $\mu(\tilde{u}) \leq 3$. Then*

$$\tilde{u}_c(\mathbb{C}) \cap \tilde{u}(\mathbb{C}) = \emptyset, \quad c \neq 0,$$

where $\tilde{u}_c(z) = (a(z) + c, u(z))$.

PROOF. Given $0 < r_0 < r_1$ there exists an $R_1 > 0$ such that for all $R \geq R_1$ the following holds true. If $z, z' \in \mathbb{C}$ satisfy $|z| = R, |z'| \leq R$, then $\tilde{u}_c(z) \neq \tilde{u}(z')$ for every $c \in \mathbb{R}$ in $r_0 \leq |c| \leq r_1$. Indeed, arguing indirectly we find a sequence c_j in $r_0 \leq |c_j| \leq r_1$ and sequences z_j, z'_j with $|z'_j| \leq |z_j|$ and $|z_j| \rightarrow \infty$ satisfying

$$(2.15) \quad \tilde{u}_{c_j}(z_j) = \tilde{u}(z'_j).$$

Outside a large ball B_{R_0} the map u is an embedding, in view of Lemma 2.2. Therefore, we may assume that $|z'_j| \leq R_0$. Recall $a(z) \rightarrow \infty$ as $|z| \rightarrow \infty$. Since $|z_j| \rightarrow \infty$ and c_j is bounded we conclude from (2.15) that $a(z'_j) = a(z_j) + c_j \rightarrow \infty$. This, however, is not possible since the sequence z'_j is bounded, and the claim is proved.

As a consequence, there is a well defined intersection number $\text{int}(\tilde{u}_c, \tilde{u})$ if $c \neq 0$. It is defined by restricting the maps to sufficiently large balls. By the excision and the homotopy property of the intersection number, the integer $\text{int}(\tilde{u}_c, \tilde{u})$ is independent of c for $c \neq 0$. By the positivity of the local intersection numbers of pseudoholomorphic curves, $\text{int}(\tilde{u}_c, \tilde{u}) \geq 0$. Moreover, $\tilde{u}_c(\mathbb{C}) \cap \tilde{u}(\mathbb{C}) = \emptyset$ for $c \neq 0$ if and only if $\text{int}(\tilde{u}_c, \tilde{u}) = 0$. We shall prove that $\text{int}(\tilde{u}_c, \tilde{u}) = 0$. Arguing by contradiction we find a sequence $c_j \rightarrow 0, c_j \neq 0$ and sequences z_j, z'_j such that

$$(2.16) \quad \tilde{u}_{c_j}(z_j) = \tilde{u}(z'_j).$$

Since outside a ball u is an embedding (Lemma 2.2) we may assume, taking a suitable subsequence, that $z'_j \rightarrow z'_0$. Since $\mu(\tilde{u}) \leq 3$, the map $u : \mathbb{C} \rightarrow S^3$ is an immersion in view of Corollary 4.2 of [13]. Therefore, $|z'_0 - z_j| \geq \varepsilon$ if j is large, for

a suitable $\varepsilon > 0$. Since $a(z_j) = a(z_j) - c_j$ is bounded, also the sequence z_j must be bounded. Hence, for a subsequence, $z_j \rightarrow z_0$, so that by (2.16), $\tilde{u}(z_0) = \tilde{u}(z'_0)$. Since $z_0 \neq z'_0$ this contradicts our assumption that \tilde{u} is injective. The proof of Lemma 2.3 is complete. \square

In order to complete the proof of Theorem 2.1 we observe that Lemma 2.3 implies that the map $u : \mathbb{C} \rightarrow S^3$ is injective. We have seen that the assumption $\mu(\tilde{u}) \leq 3$ implies $\text{wind}_\infty(\tilde{u}) = 1$ and conclude, using Corollary 4.2 of [14], that $\dim_{\mathbb{R}} \pi Tu(z)(\mathbb{C}) = 2$ for every $z \in \mathbb{C}$. Recalling $TS^3 = \mathbb{R}X \oplus \xi$ with the projection $\pi : TS^3 \rightarrow \xi$ along X , the map u is an immersion transversal to the Reeb vector field X . It follows that $u(\mathbb{C}) \cap P = \emptyset$. Indeed, arguing indirectly we assume $u(z_0) \in P$. Then $u(\mathbb{C})$ intersects the solution P of X transversally in $u(z_0)$. Defining the loops S_R by $S_R(t) = u(Re^{2\pi it})$ we know that $S_R(t) \rightarrow x(Tt)$ as $R \rightarrow \infty$ in $C^\infty(S^1)$. Hence there is an open neighbourhood $B_\varepsilon(z_0)$ satisfying $u(B_\varepsilon(z_0)) \cap S_R \neq \emptyset$ if $R > 0$ is large. This contradicts the injectivity of u . In view of the asymptotic behaviour of u as $|z| \rightarrow \infty$ it follows that the injective immersion $u : \mathbb{C} \rightarrow S^3 \setminus P$ is an embedding. This finishes the proof of Theorem 2.1. \square

THEOREM 2.4. *Let $\tilde{u} = (a, u) : \mathbb{C} \rightarrow \mathbb{R} \times S^3$ be a non-constant finite energy plane. Assume $u : \mathbb{C} \rightarrow S^3$ is an injective immersion. Then every asymptotic limit (x, T) of \tilde{u} is simply covered, $\text{cov}(x, T) = 1$. In addition, $P_x = x(\mathbb{R})$ is unknotted and has self-linking number $sl(x, T) = -1$.*

We point out that the theorem does not require the contact form λ to be non-degenerate.

PROOF. Recalling the projection $\pi : TS^3 = \mathbb{R}X \oplus \xi \rightarrow \xi$ we conclude from the differential equation (1.10) for \tilde{u} that

$$(2.17) \quad \pi \circ Tu \circ i = J(u) \circ \pi \circ Tu.$$

Hence $\pi \circ Tu(z) : \mathbb{C} \rightarrow \xi_{u(z)}$ is complex linear and since u is an immersion we have

$$(2.18) \quad \pi \circ Tu(z) \neq 0 \quad \text{for every } z \in \mathbb{C}.$$

We shall use this to show that u does not intersect any of its asymptotic limits. Assume (x, T) is an asymptotic limit of \tilde{u} . Then there exists a sequence $R_k \rightarrow \infty$ so that

$$(2.19) \quad u(R_k e^{2\pi it}) \rightarrow x_T(t) = x(Tt)$$

in $C^\infty(\mathbb{R})$ as $k \rightarrow \infty$. Setting $P_x = x(\mathbb{R})$ we claim that

$$(2.10) \quad u(\mathbb{C}) \cap P_x = \emptyset.$$

Arguing indirectly we assume that $u(z_0) \in P_x$. Then u intersects P_x at $u(z_0)$ transversally in view of (2.18). If k is large we therefore conclude from (2.19) that $S_k(\mathbb{R}) \cap u(B_\varepsilon(z_0)) \neq \emptyset$, with $S_k(t) = u(R_k e^{2\pi it})$. This contradicts the postulated injectivity of u and proves the claim (2.20).

Let $D = \{z \mid |z| \leq 1\}$. Given a disc map $v : D \rightarrow S^3$ satisfying

$$v(e^{2\pi it}) = x_T(t),$$

the self-linking number of (x, T) is defined by

$$(2.21) \quad sl(x, T) = \text{int}(v, y_T),$$

where the loop y_T is obtained by pushing x_T into the direction of a nowhere vanishing section Z of $v^*\xi \rightarrow D$, so that $x_T(\mathbb{R}) \cap y_T(\mathbb{R}) = \emptyset$. Introducing $D_{R_k} = \{z \mid |z| \leq R_k\}$ we define the maps $w_k : D_{R_k} \rightarrow S^3$ by $w_k(z) = v(z/R_k)$. Clearly $\text{int}(w_k, y_T) = \text{int}(v, y_T)$ so that $sl(x, T) = \text{int}(w_k, y_T)$ for every k . For large k the maps w_k and $u_k := u|_{D_{R_k}} : D_{R_k} \rightarrow S^3$ are homotopic through maps not intersecting y_T on their boundaries. Consequently, by the homotopy property of the intersection number we obtain for k large enough

$$(2.22) \quad sl(x, T) = \text{int}(u_k, y_T), \quad u_k = u|_{D_{R_k}}.$$

Consider now the nowhere vanishing section Z_k of $u_k^*\xi \rightarrow D_{R_k}$ defined by

$$Z_k(z) = \pi \circ Tu(z)(\partial/\partial s), \quad z = s + it,$$

and the section W_k along ∂D_{R_k} defined by

$$W_k(z) = \pi \circ Tu(z)(\partial/\partial r), \quad |z| = R_k,$$

where r is the radial coordinate $z = re^{i\varphi} \in \mathbb{C}$. For k large enough W_k is homotopic, as a vector field not tangent to $u_k(\partial D_{R_k})$, to the outward pointing tangent vector to the embedded disc $u_k(D_{R_k})$ at its boundary. Along ∂D_{R_k} we can write, using the complex multiplication in ξ defined by the almost complex structure J ,

$$(2.23) \quad W_k(z) = f_k(z) \cdot Z_k(z), \quad |z| = R_k,$$

with a function $f_k : \partial D_{R_k} \rightarrow \mathbb{C} \setminus \{0\}$ having winding number

$$(2.24) \quad \text{wind}(f_k) = 1.$$

We claim that for k large enough there exists a C^∞ -small deformation of $u_k = u|_{D_{R_k}} : D_{R_k} \rightarrow S^3$ through immersions into a deformed map $v_k : D_{R_k} \rightarrow S^3$

with the following properties:

$$\begin{aligned}
 (2.25) \quad & v_k(\partial D_{R_k}) = P_x, \\
 & v_k : \partial D_{R_k} \rightarrow P_x \text{ has degree } \text{cov}(x, T), \\
 & v_k \text{ is an immersion,} \\
 & v_k(\dot{D}_{R_k}) \cap P_x = \emptyset.
 \end{aligned}$$

To prove this claim we identify a closed tubular neighbourhood U of $P_x = P$ with $P \times D$ and define

$$(2.26) \quad \tau : S^1 \times D \rightarrow P \times D, \quad (t, z) \mapsto (x(Tt), z),$$

a $\text{cov}(x, T)$ -fold covering map. Defining $A_k = \{z \in \mathbb{C} \mid |z| \leq R_k \text{ and } u_k(z) \in U\}$ we can lift the map $u_k|_{A_k}$ to the map $\hat{u}_k : A_k \rightarrow S^1 \times D$ so that $u_k = \tau \circ \hat{u}_k$. In view of (2.20),

$$(2.27) \quad \hat{u}_k(A_k) \cap (S^1 \times \{0\}) = \emptyset.$$

For k large enough, the loops $\alpha_k : t \mapsto \hat{u}_k(R_k e^{2\pi i t})$ and $\alpha : t \mapsto (t, 0)$, $t \in [0, 1]$, are C^∞ -close. Hence we find a family Φ_λ , $\lambda \in [0, 1]$, of diffeomorphisms of $S^1 \times D$ which are C^∞ -close to the identity map, compactly supported in the interior of $S^1 \times D$ and satisfy $\Phi_1(\alpha_k) = \alpha$. Note that α corresponds to x_T in our tubular neighbourhood. Since Φ_1 is a diffeomorphism we conclude

$$(S^1 \times \{0\}) \cap \Phi_1 \circ \hat{u}_k(A_k \setminus \partial D_{R_k}) = \emptyset.$$

We now define a C^∞ -small isotopy of immersions $\tilde{\Phi}_\lambda : D_{R_k} \rightarrow S^3$, $\lambda \in [0, 1]$, by

$$(2.28) \quad \tilde{\Phi}_\lambda(z) = \begin{cases} u_k(z), & z \in D_{R_k} \setminus A_k, \\ \tau \circ \Phi_\lambda \circ \hat{u}_k(z), & z \in A_k. \end{cases}$$

Then $v_k(z) = \tilde{\Phi}_1(z)$, $z \in D_{R_k}$, has the desired properties.

Along the above deformation $\tilde{\Phi}_\lambda$ of u_k , the sections Z_k resp. W_k can be deformed as nowhere vanishing sections of ξ along v_k resp. its boundary. We shall denote these new sections by the same letters. We may assume that W_k is an outward pointing tangent vector field to the immersed disc v_k at the boundary $v_k(\partial D_{R_k})$. In view of the continuous deformation we still have $W_k = f_k Z_k$ and $\text{wind}(f_k) = 1$. Since Z_k is a nowhere vanishing section of $v_k^* \xi$, the loop y_T is homotopic in the complement of P_x to the loop obtained from x_T by pushing it into the direction of Z_k . Therefore, we may assume that y_T is obtained from x_T by pushing slightly into the direction of Z_k , so that, by definition of the self-linking number,

$$(2.29) \quad \text{sl}(x, T) = \text{int}(v_k, y_T).$$

Since, by (2.25), $v_k(\dot{D}_{R_k}) \cap P_x = \emptyset$, the curve $c_k := v_k|_{\partial D_{R_k - \varepsilon_k}}$ for some small $\varepsilon_k > 0$ is homotopic in $S^3 \setminus P_x$ to a constant map. The curve c_k may be viewed

as obtained from x_T by pushing it into the direction of $-W_k$. For the homology classes $[y_T]$ and $[c_k]$ in $H_1(S^3, P_x; \mathbb{Z})$ we have the relation

$$(2.30) \quad [c_k] = [y_T] + \text{wind}(f_k) \cdot [\partial\Delta_k],$$

where Δ_k is any sufficiently small disc transversal to P_x with orientation induced by ξ . Since $[c_k] = 0$ we deduce from (2.30) that

$$(2.31) \quad \text{int}(v_k, y_T) = -\text{wind}(f_k) \cdot \text{int}(v_k, [\partial\Delta_k]).$$

Recall that $\text{wind}(f_k) = 1$ and observe that $\text{int}(v_k, [\partial\Delta_k]) = \text{cov}(x, T)$ for a sufficiently small disc Δ_k . Since $sl(x, T) = \text{int}(v_k, y_T)$, in view of (2.29), we have proved

$$(2.32) \quad sl(x, T) = -\text{cov}(x, T).$$

If T_0 is the minimal period of (x, T) such that $T = \text{cov}(x, T) \cdot T_0$ we have, since we are on S^3 , the formula

$$(2.33) \quad sl(x, T) = \text{cov}(x, T)^2 \cdot sl(x, T_0).$$

Consequently, in view of (2.32),

$$-\text{cov}(x, T) = \text{cov}(x, T)^2 \cdot sl(x, T_0),$$

implying that $\text{cov}(x, T) = 1$ and $sl(x, T) = -1$. In particular, $(x, T) = (x, T_0)$ is simply covered. Since (x, T_0) is the C^∞ -limit of unknotted loops, it must be unknotted as well. This completes the proof of Theorem 2.4. \square

3. Proof for overtwisted contact structures

We consider S^3 equipped with the contact form λ which determines the contact structure ξ and the Reeb vector field X . We have to distinguish two classes of contact forms according to their induced structures ξ which can be either tight or overtwisted (see [5, 6, 8]). Recall that a contact structure ξ is called *overtwisted* if there exists an embedded disc $\mathcal{D} \subset S^3$ such that $T_m\mathcal{D} \not\subset \xi_m$ for all $m \in \partial\mathcal{D}$ and $T\partial\mathcal{D} \subset \xi|_{\partial\mathcal{D}}$. If no such disc exists, the contact structure is called *tight*. We mention that the contact structures on S^3 have been classified by Ya. Eliashberg in [4, 6]. In this section we shall prove Theorem 1.4 for overtwisted contact structures.

By assumption, λ is a non-degenerate contact form inducing an overtwisted contact structure ξ . Hence, arguing as in [10] we find an overtwisted disc $\mathcal{D} \subset S^3$ whose characteristic foliation $T\mathcal{D} \cap \xi$ contains precisely one singularity, $e \in \text{interior}(\mathcal{D})$, which is positively elliptic. Moreover, the boundary $\partial\mathcal{D}$ is the only limit cycle for the characteristic foliation of \mathcal{D} . In addition, we may assume that \mathcal{D} does not contain a periodic orbit for X . This can be achieved by a C^∞ -small perturbation disjoint from $\partial\mathcal{D}$. We choose an almost complex structure

J_0 on ξ compatible with $d\lambda$ and take the associated \mathbb{R} -invariant almost complex structure \tilde{J}_0 on $\mathbb{R} \times S^3$ defined by (1.9). It is determined by λ and J_0 .

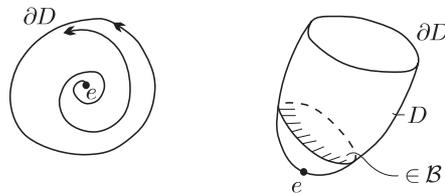
We study disc maps $\tilde{u} = (a, u) : D \rightarrow \mathbb{R} \times S^3$, with the closed disc $D := \{z \in \mathbb{C} \mid |z| \leq 1\}$, satisfying, in the coordinates $z = s + it$, the partial differential equation

$$(3.1) \quad \tilde{u}_s + \tilde{J}_0(\tilde{u})\tilde{u}_t = 0 \quad \text{on } D$$

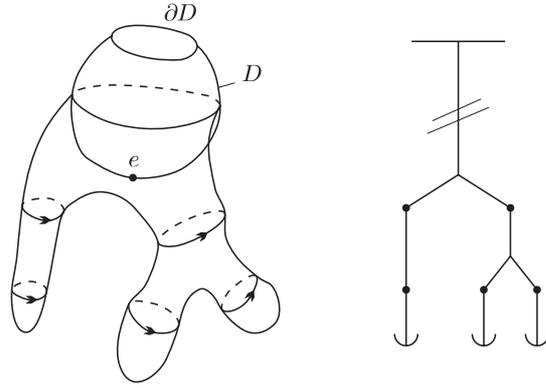
and the boundary conditions

$$(3.2) \quad \begin{aligned} &\bullet \tilde{u}(\partial D) \subset \{0\} \times (\mathcal{D} \setminus \{e\}) \subset \mathbb{R} \times S^3, \\ &\bullet \text{the winding number of } u|\partial D : \partial D \rightarrow \mathcal{D} \setminus \{e\} \text{ is equal to } 1. \end{aligned}$$

The disc \mathcal{D} is oriented in such a way that the unique singularity e of the characteristic foliation is positive. For a suitable choice of J_0 near e we can achieve that the associated structure \tilde{J}_0 is integrable near $(0, e) \in \mathbb{R} \times S^3$, hence a complex structure. Therefore, by a classical result due to Bishop, there is a 1-parameter family of embedded holomorphic discs satisfying (3.1) and (3.2) emerging from the singularity $(0, e)$, which have the property that the discs $\tilde{u}(D)$ are mutually disjoint and the boundaries $\tilde{u}(\partial D) = (0, u(\partial D))$ foliate a punctured neighbourhood of e in \mathcal{D} (see [1, 2, 5, 10]).



Still following [10], this local family can be continued by means of an implicit function theorem to an (essentially unique) maximal 1-parameter family \mathcal{B} of embedded, mutually disjoint disc maps satisfying (3.1) and (3.1). The boundary $u(\partial D) \subset \mathcal{D}$ for every $\tilde{u} = (a, u) \in \mathcal{B}$ is transversal to the singular foliation of \mathcal{D} by the maximum principle. \mathcal{B} , however, is not a compact family, since otherwise there would be a disc in \mathcal{B} touching the boundary $\partial\mathcal{D}$ which is a leaf of the singular foliation on \mathcal{D} . This would contradict the maximum principle. Analysing the failure of compactness one concludes that even a reparametrization of the discs in \mathcal{B} by Möbius transformations does not prevent the gradients $|\nabla\tilde{u}|$ of $\tilde{u} \in \mathcal{B}$ from exploding. In view of the gradient bounds near the boundary ∂D , the gradients blow up at finitely many points $\Gamma \subset \text{interior}(D)$. A bubbling off analysis of these singularities based on rescaling methods, and carried out in detail in [10] and [11], establishes the existence of a complicated structure of various kinds of finite energy surfaces $\tilde{u} = (a, u)$ in $\mathbb{R} \times S^3$ all having $\pi \circ Tu$ not identically vanishing. The geometric realization in S^3 is illustrated by the following picture together with its associated graph.



The top part of the graph represents a map $\tilde{u} := (a, u) : D \setminus \Gamma \rightarrow \mathbb{R} \times S^3$, where $\Gamma \neq \emptyset$ is a finite set of points in $\text{interior}(D)$, which solves the following problem:

$$\begin{aligned}
 (3.3) \quad & \tilde{u}_s + \tilde{J}_0(\tilde{u})\tilde{u}_t = 0 \quad \text{on } D \setminus \Gamma, \\
 & a|_{\partial D} = 0, \\
 & u : \partial D \rightarrow \mathcal{D} \setminus \{e\} \text{ has winding number } 1, \\
 & 0 < E(\tilde{u}) < \infty.
 \end{aligned}$$

The energy $E(\tilde{u})$ is obtained by maximizing the integrals

$$\int_{D \setminus \Gamma} \tilde{u}^* d(\lambda_\varphi)$$

over the set of smooth $\varphi : \mathbb{R} \rightarrow [0, 1]$ satisfying $\varphi'(s) \geq 0$. The points of Γ are negative punctures of \tilde{u} near which u tends asymptotically to periodic solutions of the Reeb vector field X . Recall that, by assumption, the periodic solutions are non-degenerate.

It will be important later on that \tilde{u} is an embedding. Near ∂D this is a consequence of the maximum principle, as shown in [10].

Near the non-degenerate periodic orbits \tilde{u} is either multiple-covered or embedded in view of the arguments in [14] and [13]. The first alternative cannot occur, since by the similarity principle \tilde{u} would have to be multiple-covered also near ∂D where it is an embedding. By the asymptotic behaviour \tilde{u} must be an embedding near the boundary in the sense of D. McDuff. Moreover, by the bubbling off construction \tilde{u} is a limit of embeddings obtained by properly rescaled disc maps. Consequently, \tilde{u} has to be an embedding, in view of the results of D. McDuff in [18, 20].

The bottom parts of the graph represent finite energy planes $\tilde{u} := (a, u) : \mathbb{C} \setminus \{\infty\} \rightarrow \mathbb{R} \times S^3$. At the positive puncture $\{\infty\}$, the plane is asymptotic to a periodic solution of X . Every middle piece of the graph represents a punctured

finite energy sphere $\tilde{u} : \dot{S} \rightarrow \mathbb{R} \times S^3$, where $\dot{S} = S^2 \setminus \Gamma$ is the sphere with a non-empty finite set Γ of points removed, namely one positive puncture and a positive number of negative punctures where the map tends asymptotically to periodic solutions of X .

We now prove that at least one of the bottom finite energy planes \tilde{u} has an asymptotic limit (x, T) with index $\mu(\tilde{u}) = 2$. From (1.13) we recall that $\mu(\tilde{u}) \geq 2$. Arguing indirectly we therefore assume that $\mu(\tilde{u}) \geq 3$ for all bottom finite energy planes. We now cancel in the graph all elements at the bottom representing a finite energy plane and write at the lower ends of the remaining graph the index of the corresponding asymptotic limit which, by assumption is at least 3. We now study the bottom elements of the new graph. These elements are trees with one positive puncture having an index μ^+ and a positive number of negative punctures having corresponding indices ≥ 3 . To this situation we can apply the following special case of Theorem 5.8 of [14] formulated as

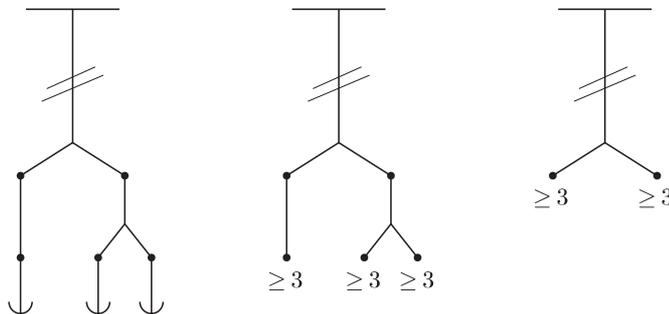
LEMMA 3.1. *Let $\tilde{u} : \dot{S} \rightarrow \mathbb{R} \times S^3$ be a punctured finite energy sphere with $\pi \circ Tu$ not identically vanishing. Then*

$$(3.4) \quad \mu(\tilde{u}) = \mu^+ - \mu^- \geq \text{wind}_\pi(\tilde{u}) + 4 - 2\#\Gamma_0 - \#\Gamma_1.$$

Here μ^+ is the sum of all μ -indices of the positive punctures in Γ while μ^- is the sum over all negative punctures. Furthermore, $\text{wind}_\pi(\tilde{u}) \geq 0$ is a non-negative integer introduced in [14]. Finally, Γ_0 is the set of punctures in Γ having even μ -index, while Γ_1 is the set of those having odd μ -index, $\Gamma = \Gamma_0 \cup \Gamma_1$. Applying Lemma 3.1 to our situation, observing that $\mu^- \geq 3(\#\Gamma - 1)$, we estimate the μ -index of the positive puncture by

$$\mu^+ \geq \mu^- + 4 - 2\#\Gamma \geq 3(\#\Gamma - 1) + 4 - 2\#\Gamma \geq \#\Gamma + 1 \geq 2 + 1 = 3.$$

We now iterate the procedure starting in the next round again with negative punctures having all μ -indices ≥ 3 .



After a finite number of steps only the top piece of the graph remains, namely the piece representing the solution $\tilde{u} : D \setminus \Gamma \rightarrow \mathbb{R} \times S^3$ of the problem (3.3). We recall that it is an embedding. By the inductive construction, its punctures Γ are

negative and have μ -indices at least 3. Note that our construction is independent of the choice of the almost complex structure J_0 on ξ . Choosing now a generic structure J we have arrived at a contradiction with the following result from [15], Theorem 1.15, proved by means of Fredholm theory. Denote by \mathcal{J} the set of all almost complex structures on ξ compatible with $d\lambda$.

LEMMA 3.2. *Given an almost complex structure $J_0 \in \mathcal{J}$ and an embedded disc \mathcal{D} as above, there exists a $J \in \mathcal{J}$ in every C^∞ neighbourhood of J_0 , whose associated almost complex structure \tilde{J} on $\mathbb{R} \times S^3$ defined by (1.9) has the following property. There exists no embedded solution*

$$\tilde{u} = (a, u) : D \setminus \Gamma \rightarrow \mathbb{R} \times S^3$$

of the mixed boundary value problem (3.3) satisfying $\pi \circ Tu(z) \neq 0$ for some z , if $\Gamma \neq \emptyset$ and if all the corresponding asymptotic limits (x_j, T_j) are non-degenerate and have indices

$$\mu(x_j, T_j) \geq 3.$$

With this contradiction we have established the existence of a finite energy plane $\tilde{u} := (a, u) : \mathbb{C} \rightarrow \mathbb{R} \times S^3$ having an asymptotic limit with index $\mu(x, T) = 2$. Now, a finite energy plane \tilde{u} is either somewhere injective or $\tilde{u} = \tilde{v} \circ P$, with a somewhere injective finite energy plane \tilde{v} and a higher degree complex polynomial P (see [14]). If $\mu(\tilde{u}) = 2$, the latter is not possible since the asymptotic limit of \tilde{v} would have an index at least 2 in view of (1.13), and consequently \tilde{u} would have an index at least $2 \cdot \deg(P) \geq 4$. Hence \tilde{u} is somewhere injective and we can apply Lemma 3.1 to the finite energy plane $\tilde{u} : \mathbb{C} = S^2 \setminus \{\infty\} \rightarrow \mathbb{R} \times S^3$. We have $\Gamma_1 = \emptyset$ and $\#\Gamma_0 = 1$ so that

$$(3.5) \quad \mu(\tilde{u}) = 2 \geq \text{wind}_\pi(\tilde{u}) + 2.$$

Hence, in view of $\text{wind}_\pi \geq 0$, we conclude $\text{wind}_\pi(\tilde{u}) = 0$. This implies, in view of the properties of wind_π in [14] that $u : \mathbb{C} \rightarrow S^3$ is an immersion. Consequently, \tilde{u} is an immersion. Near $\{\infty\}$ it is an embedding by Lemma 2.2. Assume that \tilde{u} has selfintersections. Then they are necessarily isolated and moreover have positive intersection indices, since we are dealing with pseudoholomorphic curves (see [18, 20]). By the bubbling off analysis, \tilde{u} can be approximated by a rescaled family of disc maps and we conclude that also the discs in this family have selfintersections if they are sufficiently close to \tilde{u} , by the positivity of the intersection index. This, however, contradicts the fact, that, by our construction based on the Bishop family \mathcal{B} , the discs are all embedded. We have proved that \tilde{u} is an injective immersion. In view of its asymptotic behaviour described in Lemma 2.2 the map \tilde{u} is an embedding. Since $\mu(\tilde{u}) = 2$, Theorem 1.4 is now an immediate consequence of Theorems 1.6 and 1.7. This finishes the proof of Theorem 1.4 in the overtwisted case. \square

For later use we point out an a priori bound for the period T of the distinguished periodic solution of X found above, namely

$$(3.6) \quad T = E(\tilde{u}) = \int_{\mathbb{C}} u^* d\lambda \leq \frac{1}{2} \int_{\mathcal{D}} |d\lambda|.$$

The bound follows from the bubbling off analysis; the proof is given in [10].

4. Proof for tight contact structures

On S^3 the tight contact structures are well understood due to the work of Ya. Eliashberg and E. Giroux (see [5–9]). Consider the round sphere $S^3 = \{z \in \mathbb{C}^2 \mid |z|^2 = 1\}$ equipped with its standard contact form $\lambda_0 = \tilde{\lambda}_0|_{S^3}$, where the Liouville form $\tilde{\lambda}_0$ on \mathbb{C}^2 is, in the coordinates $z_j = q_j + ip_j$, given by

$$(4.1) \quad \tilde{\lambda}_0 = \frac{1}{2} \sum_{j=1}^2 (q_j dp_j - p_j dq_j).$$

If λ is any tight contact form on S^3 such that the volume form $\lambda \wedge d\lambda$ is compatible with the given orientation of S^3 , then there exists an orientation preserving diffeomorphism $\psi : S^3 \rightarrow S^3$ such that $\psi^* \lambda = f \lambda_0$ for a smooth function $f : S^3 \rightarrow (0, \infty)$. This is proved in Eliashberg [6].

In order to prove Theorem 1.4 in the tight case, we can therefore assume that our tight contact form λ on S^3 is given by

$$(4.2) \quad \lambda = f \lambda_0,$$

for a smooth positive function f on S^3 . We assume λ , in addition, to be non-degenerate and denote by X_λ the associated Reeb vector field. Using a modification of a construction in [12] we shall find an embedded finite energy plane \tilde{u} whose asymptotic limit has an index $\mu(\tilde{u}) \in \{2, 3\}$.

We choose a function f_E on S^3 , defined by

$$(4.3) \quad f_E(z) = \left(\sum_{j=1}^2 |z_j|^2 / r_j^2 \right)^{-1/2},$$

with numbers $0 < r_1 < r_2$ such that r_1^2 / r_2^2 is irrational and

$$(4.4) \quad f(z) < f_E(z), \quad z \in S^3,$$

where f is given in (4.2). We denote by X_E the Reeb vector field defined by the contact form

$$(4.5) \quad \lambda_E = f_E \lambda_0.$$

The vector field X_E has precisely 2 non-degenerate, in fact elliptic, periodic orbits P_0 and P_1 having minimal periods $T_0 < T_1$ and indices

$$(4.6) \quad \mu(P_0) = 3 \quad \text{and} \quad \mu(P_1) = 2k + 1 \geq 5,$$

where the integer $k \geq 2$ is determined by $k < 1 + (r_2/r_1)^2 < k + 1$. This is easily verified by a direct computation carried out in [11]. We now interpolate the contact forms $\lambda = f\lambda_0$ and $\lambda_E = f_E\lambda_0$ in a monotonic way by choosing a function $h : \mathbb{R} \times S^3 \rightarrow (0, \infty)$ satisfying $f(z) \leq h(a, z) \leq f_E(z)$, $(a, z) \in \mathbb{R} \times S^3$, and, moreover,

$$(4.7) \quad \begin{aligned} h(a, z) &= \begin{cases} f(z) & \text{if } a \leq -2, \\ f_E(z) & \text{if } a \geq 2, \end{cases} \\ \frac{\partial h}{\partial a}(a, z) &\geq 0, \quad \frac{\partial h}{\partial a}(a, z) \geq \sigma > 0 \quad \text{if } |a| \leq 1. \end{aligned}$$

For every $a \in \mathbb{R}$, the contact form $\lambda_a := h(a, \cdot)\lambda_0$ on S^3 determines the associated Reeb vector field X_a . By construction, $\lambda_a = \lambda$, $X_a = X_\lambda$ if $a \leq -2$ and $\lambda_a = \lambda_E$, $X_a = X_E$ if $a \geq 2$. The contact structure ξ is, of course, independent of a . We next choose a smooth family $a \mapsto J_a$ of almost complex structures of ξ , compatible with $d\lambda_a$ and such that J_a does not depend on a in $|a| \geq 2$. We denote by \widehat{J} the associated almost complex structure on $\mathbb{R} \times S^3$ defined by

$$(4.8) \quad \widehat{J}(a, z)(\beta, k) = (-\lambda_a(z)k, J_a(z)\pi_a k + \beta X_a(z)).$$

Finally, we define a special almost complex structure \widetilde{J} on $\mathbb{R} \times S^3$ as follows. We set $\widetilde{J}(a, z) = \widehat{J}(a, z)$ if $|a| \geq 1$. If $|a| < 1$ we require that \widetilde{J} is compatible with the symplectic form $\Omega = d(h\lambda_0)$ on $[-1, 1] \times S^3$. We denote by $\dot{S}^2 = S^2 \setminus \Gamma$ the punctured sphere, where $\Gamma \subset S^2$ is a finite subset of points to which we refer as *punctures*. We shall study the equation

$$(4.9) \quad \widetilde{u} = (a, u) : \dot{S}^2 \rightarrow \mathbb{R} \times S^3, \quad \widetilde{J} \circ T\widetilde{u} = T\widetilde{u} \circ i,$$

where i is a complex multiplication on S^2 . In complex coordinates, $z = s + it$, the equation (4.9) becomes the familiar Cauchy–Riemann equation $\widetilde{u}_s + \widetilde{J}(\widetilde{u})\widetilde{u}_t = 0$ for the map \widetilde{u} . We are only interested in solutions having finite energy $E(\widetilde{u})$ defined as follows. We denote by Σ the set of smooth functions $\varphi : \mathbb{R} \rightarrow [0, 1]$ satisfying $\varphi'(s) \geq 0$ and $\varphi = 1/2$ on $[-1, 1]$. Given $\varphi \in \Sigma$ we define the one-forms τ_φ and τ on $\mathbb{R} \times S^3$ by

$$\tau_\varphi(a, x)(\beta, k) = \varphi(a)\lambda_a(x)(k), \quad \tau(a, x)(\beta, k) = \lambda_a(x)(k),$$

where $(\beta, k) \in T_{(a,x)}(\mathbb{R} \times S^3)$. Then the energy $E(\widetilde{u})$ for a solution \widetilde{u} of the above equation is now defined as the supremum over all $\varphi \in \Sigma$ of the numbers

$$\int_{\mathbb{C}} \widetilde{u}^* d\tau_\varphi.$$

Any non-constant solution of (4.9) having finite energy will be called a *generalized finite energy sphere*. Next we define positive and negative punctures as follows.

A neighbourhood of a puncture looks like the complement of a compact set in \mathbb{C} and one shows that near the puncture the limit

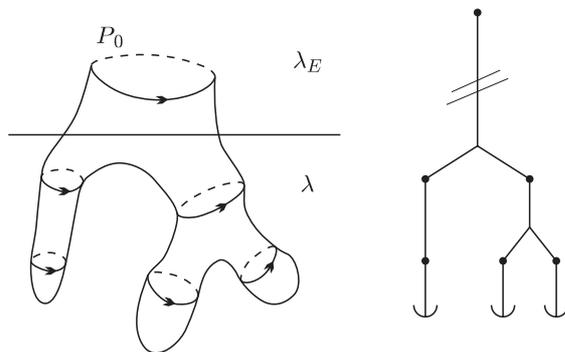
$$T = \lim_{R \rightarrow \infty} \int_{S^1} u(Re^{2\pi it})^* \lambda$$

always exists in \mathbb{R} . If $T > 0$, the puncture is called *positive*, in this case the \mathbb{R} -component of \tilde{u} is unbounded, but bounded from below. If $T < 0$, the puncture is called *negative*, the \mathbb{R} -component of \tilde{u} is unbounded, but bounded from above. If $T = 0$, the solution \tilde{u} can be extended smoothly over the puncture by Gromov’s removable singularity theorem. In case of a non-constant finite energy plane, the puncture is always positive. There are no non-constant finite energy surfaces without positive punctures, in particular no closed surfaces. We refer to [14] for a detailed discussion. By definition of $\tilde{\mathcal{J}}$, a solution $\tilde{u}(z) = (a(z), u(z))$ of (4.9) solves the equation associated with λ_E at points z where $a(z) \geq 2$, while at points where $a(z) \leq -2$ it is a solution of the desired equation associated with our given contact form λ .

Following [11] one establishes an embedded generalized finite energy plane $\tilde{u} = (a, u) : \dot{S} = \mathbb{C} \rightarrow \mathbb{R} \times S^3$ with $\inf(a) \geq 2$ which is asymptotic to the distinguished periodic solution P_0 of X_E having index $\mu(P_0) = 3$. (It is again the result of an explosion of a non-compact Bishop family \mathcal{B} .) As described in [12], this energy plane lies in a 2-parameter family \mathcal{F} of mutually disjoint and embedded generalized finite energy planes which are, as $|z| \rightarrow \infty$, all asymptotic to the same periodic orbit P_0 of X_E . The family \mathcal{F} is non-compact and contains elements $\tilde{u} = (a, u) : \mathbb{C} \rightarrow \mathbb{R} \times S^3$ having arbitrary negative \mathbb{R} -components a :

$$(4.10) \quad \sup_{\tilde{u} \in \mathcal{B}} (\max_z |\nabla \tilde{u}(z)|) = \infty, \quad \inf_{\tilde{u} \in \mathcal{B}} (\min_z a(z)) = -\infty.$$

A bubbling off analysis of the singularities of \mathcal{F} , carried out in [12], gives rise to a tree of generalized energy spheres which is illustrated in the following picture:



The top part of the graph represents an embedded (since P_0 is simply covered) generalized energy sphere

$$(4.11) \quad \tilde{u} = (a, u) : S^2 \setminus \Gamma \rightarrow \mathbb{R} \times S^3$$

with one positive puncture, ∞ , asymptotic to the distinguished periodic solution P_0 of X_E , and a positive number of negative punctures which are all asymptotic to non-degenerate periodic solutions of X_λ . The bottom parts of the graph represent finite energy planes (not necessarily embedded) $\tilde{u} : (a, u) : \mathbb{C} = S^2 \setminus \{\infty\} \rightarrow \mathbb{R} \times S^3$, for the λ -equation, hence asymptotic to periodic solutions of X_λ . All intermediate parts represent finite energy spheres $\tilde{u} : S^2 \setminus \Gamma \rightarrow \mathbb{R} \times S^3$ for the λ -equation, having one positive puncture and a positive number of negative punctures, all asymptotic to periodic solutions of X_λ .

We now prove that there is at least one finite energy plane at the bottom of the graph whose μ -index is either 2 or 3. Arguing indirectly we can assume that all the finite energy planes at the bottom have indices ≥ 4 . Starting at the bottom we work our way up the graph as we already did in the overtwisted case and conclude inductively that all the negative punctures of the energy spheres have indices at least 4. Here we use Lemma 3.1 in order to estimate the positive puncture of the middle parts in the graph as follows:

$$\mu^+ \geq 4(\#\Gamma - 1) + 4 - 2\#\Gamma = 2\#\Gamma \geq 4.$$

After finitely many steps we arrive at the top piece of the graph. It represents the generalized finite energy sphere \tilde{u} in (4.11) having one positive puncture with index $\mu(P_0) = 3$ and (as we have just proved) a positive number of negative punctures, having all an index at least 4. This will lead to a contradiction. We can choose the almost complex structure \tilde{J} to be, in addition, generic (see [15]). Since $\tilde{u} : S^2 \setminus \Gamma \rightarrow \mathbb{R} \times S^3$ is an embedding, the linearized map along \tilde{u} is a Fredholm map having a non-negative Fredholm index $\text{Fred}(\tilde{u}) \geq 0$. Denoting by $\mu^-(\tilde{u})$ the sum over all μ -indices associated with the negative punctures, and by $\mu(P_0)$ the index of the positive puncture, the Fredholm index is given by

$$(4.12) \quad \text{Fred}(\tilde{u}) = \mu(P_0) - \mu^-(\tilde{u}) - \chi(S^2) + \#\Gamma \geq 0.$$

The formula is proved in [15]. Consequently, in view of $\chi(S^2) = 2$, $\mu(P_0) = 3$ and $\mu^-(\tilde{u}) \geq 4(\#\Gamma - 1)$ we arrive at the contradiction

$$\begin{aligned} 3 &\geq \mu^-(\tilde{u}) - (\#\Gamma - 1) + 1 \geq 4(\#\Gamma - 1) - (\#\Gamma - 1) + 1 \\ &= 3(\#\Gamma - 1) + 1 \geq 4. \end{aligned}$$

In view of this contradiction, there exists a finite energy plane \tilde{u} for λ with asymptotic index $\mu(x, T) \in \{2, 3\}$. Arguing now as in the overtwisted case one shows, using Lemma 3.1, that \tilde{u} is an immersion. Again, in view of the

asymptotic behaviour in the non-degenerate case, \tilde{u} has to be an embedding. Indeed, otherwise the isolated selfintersections have positive intersection indices. But, by the bubbling off analysis in [12], \tilde{u} can be approximated by a sequence of rescaled generalized finite energy planes, which would have selfintersections too. This would contradict the fact that, by their construction based on elements of \mathcal{F} , they are embeddings.

To sum up, we have established the existence of an embedded finite energy plane $\tilde{u} : \mathbb{C} \rightarrow \mathbb{R} \times S^3$ for the λ -equation having a non-degenerate asymptotic limit with index $\mu(x, T) \in \{2, 3\}$. Hence Theorem 1.4 is consequence of Theorems 1.6 and 1.7. The proof of Theorem 1.4 in the tight case is complete.

For later use we should mention that there is an a priori bound for the minimal period T of the special asymptotic orbit found above in terms of the parameter r_1 occurring in the definition (4.3) of the function f_E (see [12]).

5. The degenerate case

The results in the previous sections for non-degenerate contact forms on S^3 will now be used in order to prove Theorem 1.8 by an approximation argument.

THEOREM 5.1. *Let λ be a contact form on S^3 and J a compatible almost complex structure on the associated contact structure ξ . Then there exists a finite energy plane $\tilde{u} = (a, u) : \mathbb{C} \rightarrow \mathbb{R} \times S^3$ such that u is an injective immersion.*

The main result (Theorem 1.1) follows immediately from Theorem 5.1 in view of Proposition 1.5 and Theorem 1.7.

PROOF OF THEOREM 5.1. Let λ be a contact form on S^3 . By Proposition 1.2 we find a sequence $f_k : S^3 \rightarrow (0, \infty)$ of smooth functions converging in C^∞ to the constant function $f_0 \equiv 1$ so that the contact forms $\lambda_k = f_k \lambda$ are non-degenerate. The associated contact structures $\xi_k = \xi$ coincide. Choose a J on ξ compatible with λ and choose a sequence J_k compatible with λ_k so that for the associated almost complex structure \tilde{J}_k on $\mathbb{R} \times S^3$ the constructions in Sections 3 and 4 can be carried out. Then there exists a sequence of embedded non-degenerate finite energy planes $\tilde{u}_k = (a_k, u_k) : \mathbb{C} \rightarrow \mathbb{R} \times S^3$ with indices $\mu(\tilde{u}_k) \leq 3$. In view of Theorem 1.7, and the a priori estimates mentioned at the end of Sections 3 and 4, we find a constant $c > 0$ such that

$$(5.1) \quad \begin{aligned} \tilde{J}_k \circ T\tilde{u}_k &= T\tilde{u}_k \circ i, \quad c^{-1} \leq E(\tilde{u}_k) \leq c, \\ u_k : \mathbb{C} &\rightarrow S^3 \text{ is an embedding.} \end{aligned}$$

Moreover, if P_k is the non-degenerate asymptotic limit of \tilde{u}_k , we have

$$(5.2) \quad u_k(\mathbb{C}) \cap P_k = \emptyset.$$

Denote by $\gamma_0 > 0$ a constant smaller than the smallest period of the periodic solutions of X_λ and also smaller than c^{-1} . Reparametrizing the solutions \tilde{u}_k we may assume, in addition to (5.1),

$$(5.3) \quad \inf_z a_k(z) = a_k(0) = 0, \quad \int_D u_k^* d\lambda_k = \int_{\mathbb{C}} u_k^* d\lambda_k - \gamma_0,$$

where $D = \{z \mid |z| \leq 1\}$. We observe that

$$(5.4) \quad \int_D u_k^* d\lambda_k \geq \varepsilon \geq c^{-1} - \gamma_0.$$

Assume (at first) that the gradients of \tilde{u}_k remain uniformly bounded on compact subsets of \mathbb{C} . Then, possibly passing to a subsequence, $\tilde{u}_k \rightarrow \tilde{u}$ in $C_{\text{loc}}^\infty(\mathbb{C})$ by the standard compactness results (see e.g. [11]). Clearly, in view of (5.4), the map $\tilde{u} = (a, u)$ is a non-constant finite energy plane for the structure \tilde{J} having $d\lambda$ -energy at least ε and hence at least γ_0 . We shall prove that u is an injective immersion. In order to prove that u is an immersion we consider the section $\pi \circ Tu$ of the bundle

$$(5.5) \quad \text{Hom}_{\mathbb{C}}(T\mathbb{C}, u^*\xi) \rightarrow \mathbb{C}.$$

By the similarity principle, this section either vanishes identically or has isolated zeros having each a positive index. For non-constant finite energy planes the section does not vanish identically (see [14]). Let $R > 0$ and assume that $\pi \circ Tu$ does not vanish on ∂D_R . Since $u_k|_{D_R}$ converges in C^∞ to $u|_{D_R}$ it follows that the winding numbers w_k of the nowhere vanishing sections $(\pi_k \circ Tu_k)|_{\partial D_R}$ coincide with the winding number w of $(\pi \circ Tu)|_{\partial D_R}$. Since $\pi_k \circ Tu_k(z) \neq 0$ for every $z \in \mathbb{C}$ we conclude $w_k = 0$ and hence $w = 0$. Consequently, $\pi \circ Tu(z) \neq 0$ for $z \in D_R$ (see [14]). This argument shows that $\pi \circ Tu(z) \neq 0$ for every $z \in \mathbb{C}$. Consequently, $u : \mathbb{C} \rightarrow S^3$ is an immersion. To prove that u is injective we argue by contradiction and assume $u(z) = u(z')$ for $z \neq z'$. Hence we find a number $c \in \mathbb{R}$ satisfying

$$\tilde{u}_c(z) = \tilde{u}(z'), \quad z \neq z'.$$

In view of the standard results in the intersection theory of pseudoholomorphic curves (see [18–20]), the intersection is isolated and has, moreover, a positive intersection index. Since $\tilde{u}_{k,c}$ converges in C_{loc}^∞ to \tilde{u}_c , the pseudoholomorphic curves $\tilde{u}_{k,c}$ have to intersect \tilde{u}_k if k is sufficiently large. This, however, contradicts the injectivity of the maps u_k . We have proved that $u : \mathbb{C} \rightarrow S^3$ is an injective immersion, under the assumption that the gradients of \tilde{u}_k are bounded.

It remains to consider the case in which the gradients of \tilde{u}_k are not uniformly bounded. In view of the normalization (5.3) it follows from a standard bubbling off analysis (see [11]) that the bubbling off points must be contained in the unit disc D . For a subsequence we find a finite set Γ of points in D so that \tilde{u}_k

converges in $C_{\text{loc}}^\infty(\mathbb{C} \setminus \Gamma)$ to some finite energy curve \tilde{v} . Carrying out a bubbling off analysis near one of the punctures, say $z_0 \in \Gamma$, we find a sequence $z_k \rightarrow z_0$ and a sequence $\varepsilon_k \rightarrow 0$ satisfying

$$(5.6) \quad \begin{aligned} & |(\nabla \tilde{u}_k)(z_k)| \cdot \varepsilon_k \rightarrow \infty, \\ & |(\nabla \tilde{u}_k)(z)| \leq 2|(\nabla \tilde{u}_k)(z_k)| \quad \text{for all } |z - z_k| \leq \varepsilon_k. \end{aligned}$$

Set $r_k = |(\nabla \tilde{u}_k)(z_k)|$. Rescaling we define a new sequence \tilde{w}_k of finite energy planes by

$$(5.7) \quad \tilde{w}_k(z) = (a_k(z_k + z/r_k) - a_k(z_k), u_k(z_k + z/r_k)).$$

The gradients of \tilde{w}_k are uniformly bounded on every compact subset of \mathbb{C} and, moreover,

$$(5.8) \quad |\nabla \tilde{w}_k(0)| = 1.$$

Being a reparametrization of u_k , the map $w_k : \mathbb{C} \rightarrow S^3$ is an injective immersion. Passing now, as in the first part of the proof, to the limit in C_{loc}^∞ , we obtain a non-constant finite energy plane $\tilde{w} = (a, w) : \mathbb{C} \rightarrow \mathbb{R} \times S^3$. Arguing again as in the first part we deduce that the map $w : \mathbb{C} \rightarrow S^3$ is an injective immersion. The proof of Theorem 5.1 is complete. \square

REFERENCES

- [1] C. ABBAS AND H. HOFER, *Holomorphic Curves and Global Questions in Contact Geometry*, Birkhäuser (to appear).
- [2] E. BISHOP, *Differentiable manifolds in complex Euclidean space*, Duke Math. J. **32** (1965), 1–21.
- [3] K. CIELIEBAK, *Symplectic boundaries: Creating and destroying closed characteristics*, preprint, ETH Zürich, 1996.
- [4] Y. ELIASHBERG, *Classification of overtwisted contact structures on 3-manifolds*, Invent. Math. **98** (1989), 623–637.
- [5] ———, *Filling by holomorphic discs and its applications*, London Math. Soc. Lecture Note Ser., vol. 151, 1991, pp. 45–67.
- [6] ———, *Contact 3-manifolds twenty years since J. Martinet’s work*, Ann. Inst. Fourier (Grenoble) **42** (1992), 165–192.
- [7] ———, *Classification of contact structures on \mathbb{R}^3* , Internat. Math. Res. Notices **3** (1993), 87–91.
- [8] ———, *Legendrian and transversal knots in tight contact-manifolds*, Topological Methods in Modern Mathematics, Publish or Perish, 1993, pp. 171–193.
- [9] E. GIROUX, *Convexité en topologie de contact*, Comm. Math. Helv. **66** (1991), 637–677.
- [10] H. HOFER, *Pseudoholomorphic curves in symplectisations with application to the Weinstein conjecture in dimension three*, Invent. Math. **114** (1993), 515–563.
- [11] H. HOFER, K. WYSOCKI AND E. ZEHNDER, *A characterization of the tight three-sphere*, Duke Math. J. **81** (1995), 159–226.
- [12] ———, *The dynamics on a strictly convex energy surface in \mathbb{R}^4* , preprint, ETH Zürich, 1996.

- [13] H. HOFER, K. WYSOCKI AND E. ZEHNDER, *Properties of pseudoholomorphic curves in symplectisations I: Asymptotics*, Ann. Inst. H. Poincaré Anal. Non Linéaire **13** (1996), 337–379.
- [14] ———, *Properties of pseudoholomorphic curves in symplectisations II: Embedding controls and algebraic invariants*, Geom. Funct. Anal **5** (1995), 270–328.
- [15] ———, *Properties of pseudoholomorphic curves in symplectisations III: Fredholm theory*, preprint, ETH Zürich, 1996.
- [16] G. KUPERBERG, *A volume-preserving counterexample to the Seifert conjecture*, preprint, 1994.
- [17] K. KUPERBERG, *A smooth counterexample to the Seifert conjecture*, Ann. of Math. **140** (1994), 723–732.
- [18] D. MCDUFF, *The local behaviour of holomorphic curves in almost complex 4-manifolds*, J. Differential Geom. **34** (1991), 143–164.
- [19] ———, *Singularities of J -holomorphic curves in almost complex 4-manifolds*, J. Geom. Anal. **2** (1992), 249–266.
- [20] M. MICALLEF AND B. WHITE, *The structure of branch points in minimal surfaces and in pseudoholomorphic curves*, Ann. of Math. **114** (1994), 35–85.

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