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EXISTENCE OF NONNEGATIVE SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS WITH SUBCRITICAL EXPONENTS

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Dedicated to Ky Fan

1. Introduction

Consider the semilinear elliptic boundary value problem

$$\begin{cases} \Delta u = f(x, u, \nabla u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with a smooth boundary $\partial\Omega$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$.

The existence of positive solutions to (1.1) in the case where f depends only on u and grows subcritically has been studied extensively in recent years (see the review article by Lions [3] and the references therein). In this paper, we establish the existence of nonnegative solutions to (1.1) where f has a subcritical growth in u and at most linear growth in ∇u. Aside from the above we do not make any other assumptions on the domain Ω. Our results imply, for instance, the existence of nonnegative solutions to

$$\begin{cases} \Delta u = -\lambda u - \sum_{j=1}^{m} c_j u^{p_j} - b|\nabla u| - h(x), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{cases}$$

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where λ, $c_j \in \mathbb{R}$, $b \geq 0$, $p_j > 1$, provided b, λ and the L^2-norm of h are small.

We also consider one-dimensional cases of (1.1), in particular the equations

$$u'' = f(x, u, u')$$

and

$$u'' + ku' = f(x, u, u'), \qquad k \in \mathbb{R},$$

subject to Dirichlet, Neumann and periodic boundary conditions.

We derive our results using the L^p theory of elliptic partial differential operators as presented in [2] plus some elementary properties of the Leray–Schauder and coincidence degrees (see [4]). Our results were motivated by the studies in [5, 7] and extend the results in these papers in several ways.

We shall denote the norms in $L^p, W^{2,p}$ and C^k by $\|\cdot\|_p, \|\cdot\|_{2,p}$ and $|\cdot|_k$ respectively, and for brevity, we denote the L^2-norm by $\|\cdot\|$.

2. An existence theorem for partial differential equations

In this section we shall establish a general existence theorem solutions of boundary value problems for semilinear elliptic problems subject to zero Dirichlet boundary conditions. In particular, we shall establish the following theorem.

Theorem 2.1. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be continuous and assume:*

(i) *There exist $h \in L^2(\Omega)$ and continuous functions $F, \widetilde{F} : \mathbb{R}^+ \to \mathbb{R}$ with F nondecreasing, $F(0) = 0$ and*

$$F(u) \leq c_1 u^p + c_2, \qquad \widetilde{F}(u) \leq \sum_{j=1}^{m} d_j u^{p_j}$$

for $u \geq 0$, where $0 < p, p_j < p^* - 1$, $p^* = 2N/(N-2)$, c_1, c_2 and d_j are positive constants, such that

$$-\widetilde{F}(u) - b|v| - h(x) \leq f(x, u, v) \leq F(u) + b|v|$$

for a.e. $x \in \Omega$ and all $u, v \in \mathbb{R}$ with $u \geq 0$, where $0 \leq b < \sqrt{\lambda_1}$, λ_1 being the first eigenvalue of $-\Delta$ on H_0^1.

(ii) *There exists $R > 0$ such that*

$$R > \left(1 - \frac{b}{\sqrt{\lambda_1}}\right)^{-1} \left(\sum_{j=1}^{N} d_j |\Omega|^{(p^* - p_j - 1)/p^*}\, \delta^{p_j+1} R^{p_j} + \frac{\|h\|}{\sqrt{\lambda_1}}\right),$$

where $|\Omega|$ denotes the Lebesgue measure of Ω and $\delta > 0$ is such that $\|u\|_{p^*} \leq \delta \|\nabla u\|$ for all $u \in H_0^1$.

Under these assumptions the problem (1.1) *has a nonnegative solution.*

Even though only the values of $u \geq 0$ are of interest here, we shall find it convenient to have F defined for $u < 0$. We hence fix F to be defined by $F(u) = -F(-u)$ for $u < 0$.

Before proving the theorem, we establish an auxiliary result.

Lemma 2.1. *Let $q = p^*/p$. Then for each $v \in L^q$, the problem*

$$\begin{cases} \Delta u = F(u) + b|\nabla u| + v, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

has a unique solution $u = Bv \in H_0^1 \cap W^{2,q}$, and $B : L^q \to H_0^1$ is completely continuous.

Proof. Without loss of generality, we may assume $p \geq p^*/2$. So $q \leq 2$. Since H^1 is continuously embedded in L^{p^*} the growth conditions on F imply that for each $w \in H_0^1$, we have

$$F(w) + b|\nabla w| + v \in L^q.$$

We now use results about the solvability of boundary value problems for nonhomogeneous linear elliptic equations presented in [2] and let $u = Kw \in W^{2,q} \cap H_0^1$ be the unique solution of

$$\begin{cases} \Delta u = F(w) + b|\nabla w| + v, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2.2}$$

Since the embedding $H_0^1 \hookrightarrow L^{p^*}$ is continuous, we get

$$\begin{aligned} \|u\|_{2,q} &\leq C_1[\|F(w)\|_q + \|\nabla w\|_q + \|v\|_q] \\ &\leq C_2[\|\nabla w\|^p + \|\nabla w\| + \|v\|_q + 1], \end{aligned} \tag{2.3}$$

where C_i are constants. Since $q > p^*/(p^* - 1)$, the embedding $W^{2,q} \hookrightarrow H^1$ is compact. Hence K takes bounded subsets in H_0^1 into relatively compact subsets in H_0^1. We next verify that K is continuous. Let $\{w_n\}_n \subset H_0^1$ be such that $w_n \to w$ in H_0^1 and let $u_n = Kw_n$, $u = Kw$. Then

$$\Delta(u_n - u) = F(w_n) - F(w) - b(|\nabla w_n| - |\nabla w|). \tag{2.4}$$

Multiplying (2.4) by $u_n - u$, integrating and using Poincaré's and Hölder's inequalities we obtain

$$\begin{aligned} \|\nabla(u_n - u)\|^2 \leq{}& C\|F(w_n) - F(w)\|_q \|u_n - u\|_{p^*} \\ &+ \frac{b}{\sqrt{\lambda_1}} \|\nabla(u_n - u)\| \, \|\nabla(w_n - w\|. \end{aligned} \tag{2.5}$$

Next choose any subsequence of $\{w_n\}$, which we again denote by $\{w_n\}$. Then since w_n converges to w in L^{p^*}, there exists a subsequence $\{w_{n_k}\}$ such that $w_{n_k} \to w$ a.e. and $|w_{n_k}| \le w^*$ for every k, and some $w^* \in L^{p^*}$ (see [1]). Hence

$$F(w_{n_k}) \to F(w) \quad \text{a.e.}, \qquad |F(w_{n_k})| \le C[1 + |w^*|^p] \in L^q,$$

from which it follows that $\|F(w_{n_k}) - F(w)\|_q \to 0$ and thus (we use (2.5)) $\|\nabla(u_n - u)\| \to 0$, proving the continuity of K.

We next apply the Leray–Schauder continuation theorem to prove that K has a fixed point. To this end, let $u \in H_0^1$ and $\lambda \in (0,1)$ be such that $u = \lambda K u$. Then

$$\Delta u = \lambda F(u) + \lambda b|\nabla u| + \lambda v. \tag{2.6}$$

Multiplying (2.6) by u and integrating, we obtain

$$\|\nabla u\|^2 \le \frac{b}{\sqrt{\lambda_1}}\|\nabla u\|^2 + C\|v\|_q\|\nabla u\|,$$

which implies (recall that $b < \sqrt{\lambda_1}$)

$$\|\nabla u\| \le C,$$

where C is a constant independent of u and λ.

Hence K has a fixed point u, which is a solution to (2.1). To show uniqueness, let u_1 and u_2 be two solutions of (2.1) and let $u = u_1 - u_2$. Then

$$\Delta u = F(u_1) - F(u_2) + b(|\nabla u_1| - |\nabla u_2|). \tag{2.7}$$

Multiplying (2.7) by u and integrating, we obtain

$$\|\nabla u\|^2 \le \frac{b}{\sqrt{\lambda_1}}\|\nabla u\|^2,$$

and hence $u = 0$.

We next verify that $B : L^q \to H_0^1$ is completely continuous. Let $\{v_n\}_n \subset L^q$ be such that $v_n \to v$ in L^q and let $u_n = Bv_n$, $u = Bv$. Then we have

$$\Delta(u_n - u) = F(u_n) - F(u) + b(|\nabla u_n| - |\nabla u|) + v_n - v. \tag{2.8}$$

Multiplying (2.8) by $u_n - u$ and integrating gives

$$\|\nabla(u_n - u)\|^2 \le \frac{b}{\sqrt{\lambda_1}}\|\nabla(u_n - u)\|^2 + C\|v_n - v\|_q\|\nabla(u_n - u)\|,$$

and hence

$$\|\nabla(u_n - u)\| \to 0,$$

proving the continuity of B.

Now let $\mathbb{K}$ be a bounded set in L^q and $v \in \mathbb{K}$. Then using equation (2.1), we deduce $\|\nabla u\| \le C$.

Since

$$\begin{aligned}\|u\|_{2,q} &\leq C[\|F(u)\|_q + b\|\nabla u\|_q + \|v\|_q] \\ &\leq C[\|\nabla u\|^p + b\|\nabla u\| + \|v\|_q + 1],\end{aligned}$$

it follows that $B(\mathbb{K})$ is bounded in $W^{2,q}$ and therefore is relatively compact in H_0^1, completing the proof of lemma 2.1.

PROOF OF THEOREM 2.1. Let $E = \{v \in H_0^1 : v \geq 0\}$, where we use $\|\nabla u\|$ as a norm in H_0^1. For each $v \in E$, let

$$Nv = f(x, v, \nabla v) - F(v) - b|\nabla v|.$$

Then $Nv \in L^q$, and N maps bounded sets in E into bounded sets in L^q. It follows from Lemma 2.1 that for each $v \in E$, there exists a unique solution $u = Av$ of

$$\begin{cases} \Delta u - F(u) - b|\nabla u| = Nv, & x \in \Omega, \\ u = 0, \quad x \in \partial\Omega. \end{cases}$$

Since $Nv \leq 0$, it follows from the maximum principle (recall the convention made about the definition of F for negative values of u) [2] that $u \geq 0$. So $A : E \to E$ and since $A = BN$, it follows that A is completely continuous.

Now let $u \in E$ and $\lambda \in (0, 1)$ be such that

$$u = \lambda A u.$$

Then we have

$$\begin{aligned}\Delta u &= \lambda\left(F\left(\frac{u}{\lambda}\right) - F(u)\right) + (1-\lambda)b|\nabla u| + \lambda f(x, u, \nabla u) \\ &\geq \lambda f(x, u, \nabla u).\end{aligned} \tag{2.9}$$

Multiplying (2.9) by u and integrating, we obtain

$$\begin{aligned}\|\nabla u\|^2 &\leq -\lambda \int f(x, u, \nabla u) u \leq \int \widetilde{F}(u) u + b \int |\nabla u| u + \int |h| u \\ &\leq \sum_{j=1}^m d_j \int u^{p_j+1} + \frac{b}{\sqrt{\lambda_1}}\|\nabla u\|^2 + \frac{\|h\|}{\sqrt{\lambda_1}}\|\nabla u\| \\ &\leq \sum_{j=1}^m d_j |\Omega|^{(p^*-p_j-1)/p^*} \delta^{p_j+1} \|\nabla u\|^{p_j+1} \\ &\quad + \frac{b}{\sqrt{\lambda_1}}\|\nabla u\|^2 + \frac{\|h\|}{\sqrt{\lambda_1}}\|\nabla u\|,\end{aligned}$$

which implies

$$\left(1 - \frac{b}{\sqrt{\lambda_1}}\right)\|\nabla u\|^2 \leq \sum_{j=1}^m d_j |\Omega|^{(p^*-p_j-1)/p^*} \delta^{p_j+1} \|\nabla u\|^{p_j+1} + \frac{\|h\|}{\sqrt{\lambda_1}}\|\nabla u\|$$

and hence $\|\nabla u\| \neq R$, by (ii).

Thus A has a fixed point u, which is a nonnegative solution to (1.1), completing the proof of Theorem 2.1.

Remark 2.1. Condition (ii) is satisfied if either $p_j > 1$ or $p_j < 1$ for all j, and $\|h\|$ is small.

3. Existence theorems for ordinary differential equations

Now we turn to the one-dimensional case of (1.1). We first have an existence result for the Dirchlet boundary value problem.

Theorem 3.1. *Let $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and assume:*

(i) *There exist continuous, nondecreasing functions $F, \widetilde{F} : \mathbb{R}^+ \to \mathbb{R}$ with $F(0) = 0$, and there exist $h \in L^1(0,1)$ and $0 \le b < 4$ such that*

$$-\widetilde{F}(u) - b|v| - h(x) \le f(x,u,v) \le F(u) + b|v|$$

for a.e. $x \in [0,1]$ and all $u, v \in \mathbb{R}$ with $u \ge 0$.

(ii) *There exists $R > 0$ such that*

$$R > \left(1 - \frac{b}{4}\right)^{-1} \left(\frac{1}{\pi}\widetilde{F}\left(\frac{R}{2}\right) + \frac{\|h\|_1}{2}\right).$$

Under these assumptions the problem

$$u'' = f(x,u,u'), \qquad u(0) = u(1) = 0, \tag{3.1}$$

has a nonnegative solution.

Proof. Let $E = \{u \in H_0^1 : u \ge 0\}$. Then using Opial's inequality [6]

$$\int_0^1 |u||u'| \le \frac{1}{4}\int_0^1 |u'|^2, \qquad \forall u \in H_0^1,$$

and the arguments in the proof of Lemma 2.1, it follows that for each $v \in E$, there exists a unique solution $u = Av$ of

$$u'' - F(u) - b|u'| = f(x,v,v') - F(v) - b|v'|, \qquad u(0) = u(1) = 0,$$

and $A : E \to E$ is completely continuous. Let $u \in E$ and $\lambda \in (0,1)$ be such that $u = \lambda A u$. Then

$$u'' \ge \lambda f(x,u,u'). \tag{3.2}$$

Multiplying (3.2) by u and integrating gives

$$\begin{aligned}
\|u'\|^2 &\le \int_0^1 \widetilde{F}(u)u + b\int_0^1 |u||u'| + \int_0^1 |hu| \\
&\le \widetilde{F}\left(\frac{1}{2}\|u'\|\right)\frac{\|u'\|}{\pi} + \frac{b}{4}\|u'\|^2 + \|h\|_1\frac{\|u'\|}{2},
\end{aligned}$$

where we have used that $|u|_0 \le \frac{1}{2}\|u'\|$ and Opial's inequality. From this and (ii), we deduce $\|u'\| \neq R$, completing the proof of Theorem 3.1 (recall that $\|u'\| = \|u\|_{H_0^1}$).

Remark 3.1.

(a) Theorem 2 is valid if continuity assumptions on f are replaced by Carathéodory conditions.

(b) In [5] and [7], the existence of nonnegative solutions of (3.1) was established for f independent of u' and satisfying

$$-c_1 - c_2 u \le f(x,u) \le 0 \tag{$*$}$$

and

$$\beta u \le f(x,u) \le \alpha u \tag{$**$}$$

respectively, where $c_1 > 0$, $0 \le c_2 < 1$, $\beta \in L^1$ and $\alpha > 0$. By applying Theorem 3.1 with $F = 0$, $b = 0$, $\widetilde{F}(u) = c_2 u$ and $h(x) \equiv c_1$, we obtain the condition $0 \le c_2 < 2\pi$ for $(*)$, and by choosing $\widetilde{F} = 0$, $F(u) = \alpha u$, $b = 0$, we obtain the condition $\alpha > 0$ for $(**)$. Thus Theorem 3.1 contains the corresponding results in [5, 7] as special cases.

For the Neumann boundary condition, we have:

Theorem 3.2. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous and assume:*

(i) *There exists $M > 0$ such that $f(x,u) > 0$ for $u > M$.*

(ii) *There exist continuous, increasing functions $F, \widetilde{F} : \mathbb{R}^+ \to \mathbb{R}$ with $F(0) = 0$, $\lim_{t\to\infty} F(t) = \infty$, and $h \in L^1$ such that*

$$-\widetilde{F}(u) - h(x) \le f(x,u) \le F(u)$$

for a.e. $x \in [0,1]$ and all $u \ge 0$.

(iii) *There exists $R > 0$ such that*

$$R^2 > 2[M^2 + \widetilde{F}(R)R + \|h\|_1 R].$$

Then the problem

$$u'' = f(x,u), \qquad u'(0) = u'(1) = 0,$$

has a nonnegative solution.

In order to prove the theorem we first establish a lemma.

Lemma 3.1. *For each* $v \in C^0[0,1]$, *the problem*

$$u'' - F(u) = v, \qquad u'(0) = u'(1) = 0, \tag{3.3}$$

has a unique solution $u = Bv$, *and* $B : C^0 \to C^0$ *is a completely continuous mapping.*

Proof. Let $X = C^0$, $Z = C^0$.

Define $L : X \supset \operatorname{dom} L \to Z$ by $Lu = u''$ where $\operatorname{dom} L = \{u \in C^2 : u'(0) = u'(1) = 0\}$ and set

$$\begin{aligned} \widetilde{A} : X \to Z, &\qquad \widetilde{A}u = F(u), \\ N : X \to Z, &\qquad Nu = F(u) + v. \end{aligned}$$

Then (3.3) is equivalent to

$$Lu = Nu.$$

Note that L is a linear Fredholm operator of index 0 and $\widetilde{A}$ and N are L-completely continuous [4], with $(Lu - \widetilde{A}(u)) = 0$ if and only if $u = 0$.

Let now $u \in \operatorname{dom} L$ and $\lambda \in (0,1)$ be such that

$$Lu - (1-\lambda)\widetilde{A}u - \lambda Nu = 0$$

or

$$u'' - F(u) = \lambda v. \tag{3.4}$$

Integrating (3.4) gives

$$\int_0^1 F(u) = -\lambda \int_0^1 v,$$

which, by the mean value theorem, implies that there exists $\tau \in [0,1]$ such that $F(|u(\tau)|) = |F(u(\tau))| \le |v|_0$. Since $\lim_{t\to\infty} F(t) = \infty$, it follows that $|u(\tau)| \le C$, where C depends only on F and v.

Hence

$$|u|_0 \le C + \|u'\|. \tag{3.5}$$

Multiplying (3.4) by u and integrating gives

$$\|u'\|^2 + \int_0^1 F(u)u \le \int_0^1 |v||u| \le C \int_0^1 |v| + \|u'\| \int_0^1 |v|$$

and so $\|u'\| \le C$, $|u|_0 \le C$ where C is independent of u and λ. We now use Theorem IV.3 of [4] with L and N as above and $H = L - \widetilde{A}$ and conclude that the first condition of that theorem is satisfied on choosing Ω a large ball, and that the second condition holds via Proposition II.18 of [4] as H is odd. Hence there is a solution u to $Lu = Nu$, and hence to (3.3). Uniqueness is proved in a standard way using the monotonicity of F.

We now verify that $B : C^0 \to C^0$ is completely continuous. Let $\{v_n\}_n \subset C^0$ be such that $v_n \to v$ in C^0 and let $u_n = Bv_n$, $u = Bv$. Since $\{v_n\}_n$ is bounded in C^0, it follows from the above argument that $\{u_n\}_n$ is bounded in C^0. Using the equation

$$u_n'' - F(u_n) = v_n$$

we deduce that $\{u_n\}_n$ is bounded in C^2.

Now choose any subsequence of $\{u_n\}_n$ which we again denote by $\{u_n\}_n$. It, in turn, has a subsequence $\{u_{n_k}\}_k$ such that $u_{n_k} \to \widetilde{u}$ in C^1. Since

$$u_{n_k}'' - u'' = F(u_{n_k}) - F(u) + v_{n_k} - v,$$

it follows that

$$\int_0^1 |u_{n_k}' - u'|^2 + \int_0^1 (F(u_{n_k}) - F(u))(u_{n_k} - u) \leq \int_0^1 |v_{n_k} - v||u_{n_k} - u|. \tag{3.6}$$

Passing to the limit in (3.6), we obtain

$$\int_0^1 |\widetilde{u}' - u'|^2 + \int_0^1 (F(\widetilde{u}) - F(u))(\widetilde{u} - u) \leq 0,$$

which implies that $\widetilde{u} = u$. Hence $u_n \to u$ in C^0 and B is continuous. B is completely continuous since B maps bounded sets in C^0 into bounded sets in C^2. This completes the proof of the lemma.

PROOF OF THEOREM 3.2. Let $E = \{u \in C^0 : u \geq 0\}$. It follows from Lemma 3.1 that for each $v \in E$ the problem

$$\begin{aligned} &u'' - F(u) = f(x, v) - F(v) \equiv Nv, \\ &u'(0) = u'(1) = 0, \end{aligned}$$

has a unique solution $u = Av$. Since $Nv \leq 0$, $u \geq 0$ so $A : E \to E$. Since N transforms bounded sets in C^0 into bounded sets in C^0 and $A = BN$, A is completely continuous. Let $u \in E$ and $\lambda \in (0, 1)$ be such that $u = \lambda Au$. Then

$$u'' = \lambda f(x, u) + \lambda\left(F\left(\frac{u}{\lambda}\right) - F(u)\right) \geq \lambda f(x, u). \tag{3.7}$$

Integrating (3.7), we obtain

$$\int_0^1 \lambda f(x, u) + \lambda\left(F\left(\frac{u}{\lambda}\right) - F(u)\right) = 0,$$

which implies by (i) that there exists $\tau \in [0, 1]$ such that $u(\tau) \leq M$. By the mean value theorem we get

$$|u|_0 \leq M + \|u'\|. \tag{3.8}$$

Multiplying (3.7) by u and integrating, we obtain

$$\begin{aligned}\|u'\|^2 &\le \lambda \int_0^1 f(x,u)u \le \int_0^1 \widetilde{F}(u)u + \int_0^1 |h|u \\ &\le \widetilde{F}(|u|_0)|u|_0 + \|h\|_1 |u|_0\end{aligned}$$

and hence by (3.8),

$$|u|_0^2 \le 2M^2 + 2\|u'\|^2 \le 2[M^2 + \widetilde{F}(|u|_0)|u|_0 + \|h\|_1|u|_0],$$

which, together with (iii), implies $|u|_0 \neq R$. This completes the proof.

Using similar arguments one immediately obtains the following result for boundary value problems subject to periodic boundary conditions.

Theorem 3.3. *Under the assumption of Theorem* 3.2, *the problem*

$$u'' = f(x,u), \qquad u(0) - u(1) = u'(0) - u'(1) = 0,$$

has a nonnegative solution.

In Theorems 3.1–3.3, the constant b has to be small or equal to zero, so we cannot apply these theorems to the problem

$$u'' + ku' + f(x,u,u') = 0 \tag{3.9}$$

if $|k|$ is large. But, as we shall see in the next theorems, problem (3.9) with Dirichlet, Neumann or periodic boundary conditions always has a nonnegative solution for f satisfying

$$-au - b|v| \le f(t,u,v) \le au + b|v| + c \qquad \forall u,v \in \mathbb{R},\ u \ge 0, \tag{3.10}$$

where a, b, c are positive constants, provided $|k|$ is sufficiently large.

Theorem 3.4. *Let* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *be continuous and satisfy* (3.10) *with*

$$a + 2b < \frac{|k|}{1 - e^{-|k|}}.$$

Then the problem

$$u'' + ku' + f(t,u,u') = 0, \qquad u(0) = u(1) = 0,$$

has a nonnegative solution.

We again need a lemma.

Lemma 3.2. *Let $k, a, b > 0$ satisfy*

$$a + 2b < \frac{k}{1 - e^{-k}}.$$

Then for each $v \in C^0$, the problem

$$u'' + ku' - au - b|u'| = v, \qquad u(0) = u(1) = 0, \tag{3.11}$$

has a unique solution $u = Bv$, and $B : C^0 \to C^1$ is completely continuous.

Proof. For each $w \in C^1$, let $u = Kw$ be the unique solution of

$$u'' + ku' = aw + b|w'| + v, \qquad u(0) = u(1) = 0.$$

Then $K : C^1 \to C^1$ is completely continuous. Let $u \in C^1$ and $\lambda \in (0, 1)$ be such that $u = \lambda K u$. Then

$$u'' + ku' = \lambda(au + b|u'| + v). \tag{3.12}$$

Multiplying (3.12) by $e^{-kt}u'$ and integrating gives

$$u'(t) = e^{kt}\left[u'(0) + \lambda \int_0^t (au + b|u'| + v)e^{ks}\, ds\right]. \tag{3.13}$$

Since $\int_0^1 u' = 0$, this implies

$$\begin{aligned} u'(0) &= \frac{-\lambda \int_0^1 e^{-kt}(\int_0^t (au + b|u'| + v)e^{ks}\, ds)}{\int_0^1 e^{-kt}} \\ &= \frac{-\lambda}{1 - e^{-k}} \int_0^1 (1 - e^{-k(1-s)})(au + b|u'| + v)\, ds \end{aligned}$$

so that

$$|u'(0)| \le a\|u\|_1 + b\|u'\|_1 + \|v\|_1 \le \left(\frac{a}{2} + b\right)\|u'\|_1 + \|v\|_1. \tag{3.14}$$

Combining (3.13) and (3.14), we obtain

$$\begin{aligned} \|u'\|_1 &\le |u'(0)|\frac{1 - e^{-k}}{k} + \int_0^1 \frac{1 - e^{-k(1-s)}}{k}(a|u| + b|u'| + |v|)\, ds \\ &\le 2\frac{1 - e^{-k}}{k}\left[\left(\frac{a}{2} + b\right)\|u'\|_1 + \|v\|_1\right], \end{aligned}$$

which implies, by the assumption on k, that

$$\|u'\|_1 \le C, \tag{3.15}$$

where C is independent of u and λ. Using this in (3.12), we deduce

$$\|u''\|_1 \le C_1, \tag{3.16}$$

where C_1 is independent of u and λ. It now follows that

$$|u|_1 \le C + C_1.$$

We now use the Leray–Schauder continuation theorem to deduce that there exists a solution u to (3.11). If u_1 and u_2 are two solutions of (3.11), let $u = u_1 - u_2$. Then u satisfies

$$u'' + ku' = au + b(|u_1'| - |u_2'|), \qquad u(0) = u(1) = 0.$$

Since $|au + b(|u_1'| - |u_2'|)| \le a|u| + b|u'|$, we deduce as in the existence proof that

$$\|u'\|_1 \le 2\frac{1-e^{-k}}{k}\left(\frac{a}{2} + b\right)\|u'\|_1,$$

and hence $u' = 0$. So $u = 0$.

We now verify that $B : C^0 \to C^1$ is completely continuous. Let $v \in C^0$, $|v|_0 \le M$ and let $u = Bv$. Then we have as above

$$\|u'\|_1 \le M_1,$$

where M_1 depends only on M, a, b and k. Hence, by using the equation in (3.11),

$$\|u''\|_1 \le M_2,$$

where $M_2 = kM_1 + aM_1 + bM_1 + M$ and so

$$|u'|_0 \le \|u''\|_1 \le M_2, \qquad |u''|_0 \le M_3,$$

where $M_3 = kM_2 + aM_1 + bM_2 + M$. Thus B transforms bounded subsets in C^0 into relatively compact subsets in C^1. Now, let $\{v_n\}_n \subset C^0$ be such that $v_n \to v$ in C^0 and let $u_n = Bv_n$, $u = Bv$. Then

$$(u_n - u)'' + k(u_n - u)' = a(u_n - u) + b(|u_n'| - |u'|) + v_n - v, \tag{3.17}$$

which implies

$$\|u_n' - u'\|_1 \le c_1\|v_n - v\|_1, \tag{3.18}$$

where c_1 depends only on a, b and k. Using (3.18) in (3.17), we deduce

$$\|u_n'' - u''\|_1 \le c_2\|v_n - v\|_1, \tag{3.19}$$

where $c_2 = kc_1 + ac_1 + bc_1 + 1$, and so

$$|u_n - u|_1 \le (c_1 + c_2)\|v_n - v\|_1,$$

i.e., B is continuous.

Proof of Theorem 3.4. Note first that u is a solution of

$$u'' + ku' + f(x, u, u') = 0$$

if and only if $v(x) = u(1-x)$ is a solution of

$$v'' - kv' + g(x, v, v') = 0,$$

where $g(x, u, v) = f(1-x, u, -v)$. Therefore we may assume that $k > 0$. Let $E = \{u \in C^1 : u \geq 0\}$. For each $v \in E$, let $u = Av$ be the unique solution of

$$\begin{aligned} u'' + ku' - au - b|u'| &= -f(x, v, v') - av - b|v'| \equiv Nv, \\ u(0) = u(1) &= 0. \end{aligned}$$

Since N transforms bounded subsets in C^1 into bounded subsets in C^0, and $A = BN$, it follows that $A : E \to E$ is completely continuous. Let $u \in E$ and $\lambda \in (0, 1)$ be such that $u = \lambda Au$. Then

$$u'' + ku' = -\lambda f(t, u, u') + (1-\lambda)(au + b|u'|). \tag{3.20}$$

Since

$$|-\lambda f(t, u, u') + (1-\lambda)(au + b|u'|)| \leq au + b|u'| + c, \tag{3.21}$$

by (3.10), it follows as in the proof of Lemma 3.2 that $\|u'\|_1 \leq C$, where C is independent of u and λ. Hence, by (3.20) and (3.21), we deduce $\|u''\|_1 \leq C_1$, where C_1 is independent of u and λ, and so $|u|_1 \leq C_2$, where C_2 is independent of u and λ.

For the Neumann problem, we have the following result.

THEOREM 3.5. *Let $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy* (3.10) *with*

$$a + 2b < \frac{|k|}{1 - e^{|k|}}$$

and suppose that there exists $M > 0$ such that

$$f(x, u, v) < 0 \qquad \text{for } u > M.$$

Then the problem

$$u'' + ku' + f(x, u, u') = 0, \qquad u'(0) = u'(1) = 0,$$

has a nonnegative solution.

The following lemma will be needed.

LEMMA 3.3. *Let $k, a, b > 0$ satisfy*

$$a + 2b < \frac{k}{1 - e^{-k}}.$$

Then for each $v \in C^0$, the problem

$$u'' + ku' - au - b|u'| = v, \qquad u'(0) = u'(1) = 0, \tag{3.22}$$

has a unique solution $u = Bv$, and $B : C^0 \to C^1$ is completely continuous.

Proof. Let $X = C^1, Z = C^0$. Define $L : X \supset \operatorname{dom} L \to Z$ by $Lu = u'' + ku'$, where $\operatorname{dom} L = \{u \in C^2 : u'(0) = u'(1) = 0\}$, and let

$$\begin{aligned} \widetilde{A} &= X \to Z, & \widetilde{A}u &= au, \\ N &= X \to Z, & Nu &= au + b|u'| + v. \end{aligned}$$

Then L is a linear Fredholm mapping of index 0, $\widetilde{A}$ and N are L-completely continuous and (3.22) is equivalent to $Lu = Nu$. Since $\ker(L - \widetilde{A}) = \{0\}$, we need only prove that all possible solutions of the family

$$Lu - (1-\lambda)\widetilde{A}u - \lambda Nu = 0, \qquad \lambda \in (0,1), \tag{3.23}$$

are bounded independently of u and λ. Let $u \in \operatorname{dom} L$ and $\lambda \in (0,1)$ satisfy (3.23). Then

$$u'' + ku' = au + \lambda(b|u'| + v). \tag{3.24}$$

Multiplying (3.24) by e^{kt} and integrating gives

$$u'(t) = e^{-kt} \int_0^t e^{ks}(au + b|u'| + v)\, ds. \tag{3.25}$$

Since $u'(1) = 0$, there exists $\tau \in [0,1]$ such that

$$|u(\tau)| \le \frac{b}{a}|u'|_0 + \frac{|v|_0}{a}$$

and so

$$|u|_0 \le \left(\frac{b}{a} + 1\right)|u'|_0 + \frac{|v|_0}{a}. \tag{3.26}$$

From (3.25) and (3.26), we deduce

$$|u'(t)| \le \frac{1 - e^{-k}}{k}[(a + 2b)|u'|_0 + 2|v|_0], \tag{3.27}$$

which implies $|u'|_0 \le C$, and thus, by using (3.26), $|u|_1 \le C_1$, where C_1 is independent of u and λ. So by Theorem IV.5 of [4], there exists a solution u to (3.22).

Now, let u_1 and u_2 be two solutions to (3.22) and let $u = u_1 - u_2$. Then we have

$$u'' + ku' = au + b(|u_1'| - |u_2'|),$$

which implies

$$|u'|_0 \le \frac{1 - e^{-k}}{k}(a + 2b)|u'|_0,$$

and hence $u' = 0$, $u = 0$. Finally, by using (3.27) it can be proved that $B : C^0 \to C^1$ is completely continuous, and the proof is complete.

Proof of Theorem 3.5. Let $E = \{u \in C^1 : u \geq 0\}$. For each $v \in E$, let $u = Av$ be the unique solution of

$$\begin{aligned} u'' + ku' - b|u'| - au &= -f(x, v, v') - av - b|v'|, \\ u'(0) = u'(1) &= 0. \end{aligned}$$

Then $A : E \to E$ is completely continuous. As in Theorem 3.4, we assume $k > 0$. Let $u \in E$ and $\lambda \in (0, 1)$ be such that $u = \lambda Au$. Then we have

$$u'' + ku' = (1 - \lambda)(au + b|u'|) - \lambda f(x, u, u'). \tag{3.28}$$

Multiplying (3.28) by e^{kt} and integrating, we obtain

$$\int_0^1 e^{kt}[(1 - \lambda)(au + b|u'|) - \lambda f(t, u, u')]\, dt = 0,$$

which implies that there exists $\tau \in [0, 1]$ such that $u(\tau) \leq M$, from which we deduce as in the proof of Lemma 3.3 that $|u|_1 \leq M_1$, where M_1 is independent of u and λ.

Using a similar argument we obtain the following result.

Theorem 3.6. *Let the assumptions of Theorem* 3.5 *hold, with*

$$a + 2b < |k|.$$

Then the problem

$$u'' + ku' + f(t, u, u') = 0, \quad u(0) - u(1) = 0, \quad u'(0) - u'(1) = 0,$$

has a nonnegative solution.

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