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THE DEGREE OF PROPER C^2 FREDHOLM MAPPINGS: COVARIANT THEORY

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Dedicated to Jean Leray

1. Introduction

In a recent article [10] we introduced a degree for proper C^2 Fredholm mappings with index 0 in general Banach spaces. As subsequently shown in [11], where the case of Banach manifolds is also treated, this degree essentially coincides with the degree of Elworthy and Tromba [8] on completely orientable manifolds. Nevertheless, the conceptually much simpler approach taken in [10] not only facilitates the use of the degree in concrete applications, but also it completely clarifies its behavior under homotopy. Indeed, while it was already known that homotopy need not preserve the sign of the degree, it remained impossible to predict whether a sign change should occur until the new concept of parity was made a crucial part of the definition.

On the other hand, there is a fairly large body of literature devoted to the calculation of the degrees of Brouwer or of Leray-Schauder in the hypothesis that the mapping of interest is covariant under the action of a compact Lie group G. Naturally, the motivation for such studies can be found in Borsuk's theorem, dealing with the simplest case when $G = \mathbb{Z}_2$ acts through $\{I, -I\}$. Elegant theories about covariant properties of fixed point indices in topological spaces or

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about nonlinear periodic actions notwithstanding (see e.g. [14], [16]) the most important applications involve linear actions in Banach spaces. For this case and Leray–Schauder's degree, the following result holds:

THEOREM 1.1. *Let X be a Banach space equipped with a continuous action of a compact Lie group G, and let $f : X \to X$ be a G-covariant compact perturbation of the identity. Denote by X^G the fixed point space of G and by $\mathcal{I}^G$ the ideal of $\mathbb{Z}$ generated by the integers $\chi(G/G_x)$, $x \in X \backslash X^G$, $\operatorname{rank} G_x = \operatorname{rank} G$, where G_x denotes the isotropy subgroup of x and $\chi(G/G_x)$ the Euler-Poincaré characteristic of G/G_x. Finally, let $\Omega \subset X$ be a G-invariant bounded open subset of X such that $0 \notin f(\partial\Omega)$. Then*

$$\deg_{L.S.}(f, \Omega, 0) = \deg_{L.S.}(f_{|X^G}, \Omega \cap X^G, 0) \bmod \mathcal{I}^G, \tag{1.1}$$

where $\deg_{L.S.}$ is Leray-Schauder's degree.

Theorem 1.1 relates the degrees of f and of its restriction to the fixed point space X^G in which the action of G has no effect, and was first proved by Wang [24] when X is a Hilbert space and the action of G is smooth. The above generalization can be found in Rabier [20], where it is also shown that in all the cases when $\mathcal{I}^G \neq \mathbb{Z}$ and hence (1.1) is not trivial, $\mathcal{I}^G$ depends only upon the action of the finite group $N(T)/T$ in the fixed point space X^T of a maximal torus T of $G(N(T)$ = normalizer of T in $G)$. Forerunners of Theorem 1.1 appear in the works of Rubinsztein [22] and Dancer [5], [6], who proves (1.1) for the index when $0 \in f^{-1}(0)$ is an isolated solution. Theorem 1.1 also covers a number of particular cases corresponding to specific and often simple choices of G, e.g. $G = \mathbb{Z}_\wp$ or $SO(2)$, that can be found in the literature (see [24] for a brief comparison).

It is the aim of this article to extend Theorem 1.1 to covariant C^2 Fredholm mappings $f : X \to X$ where X is a real Banach space and the degree is that of [10]. The first difficulty is that Theorem 1.1 does *not* extend to this more general setting without further assumptions. Indeed, if f is Fredholm with index 0 and G-covariant, then $f_{|X^G}$ is Fredholm, but not necessarily of index 0 (see Example 3.1 in Section 3) and the degree of [10] need not be defined. In fact, even if $f_{|X^G}$ is Fredholm with index 0, the degrees of f and $f_{|X^G}$ may still be unrelated: when $G = \mathbb{Z}_\wp$, $\wp$ a prime number, and $X^G = \{0\}$, this is shown in Borisovich et al. ([2, Theorem 3.4]) for the Elworthy-Tromba degree equivalent to ours. They prove that (1.1) holds only if the "$\mathbb{Z}_\wp$-index" of f vanishes. We need not recall the definition of the $\mathbb{Z}_\wp$-index here, but this concept will surface again later in our exposition, in a different form.

When G is a compact Lie group, the generalization of the "vanishing of the $\mathbb{Z}_\wp$-index" condition turns out to be intimately related to the following question: If X is a Banach space equipped with a continuous action of G and $\Phi_{0G}(X)$, $GL_G(X)$ denote the sets of G-covariant (linear) Fredholm operators with index 0 and linear isomorphisms, respectively, what is the closure of $GL_G(X)$ in $\Phi_{0G}(X)$ for the topology of $\mathcal{L}(X)$? When $G = \{1\}$, i.e. no action is involved and hence $\Phi_{0G}(X) = \Phi_0(X), GL_G(X) = GL(X)$, the well-known answer is that $GL_G(X)$ is dense in $\Phi_{0G}(X)$, but this need not be true in general.

As it is central to our purpose, we present a thorough study of the closure of $GL_G(X)$ in $\Phi_{0G}(X)$, denoted by $\Phi_{0G}^{\mathrm{reg}}(X)$ and called the set of *G-regular* covariant elements of $\Phi_0(X)$ (a terminology motivated by the Russian literature). Of necessity, this study includes several equivalent definitions, the proof that $\Phi_{0G}^{\mathrm{reg}}(X)$ is both open and closed in $\Phi_{0G}(X)$, and sufficient conditions for membership in $\Phi_{0G}^{\mathrm{reg}}(X)$ of practical importance. Although the question is strongly related to the G-index in equivariant K-theory (see Remark 3.1 or [1], [21]) we were unable to find a formulation of the corresponding results suitable to our needs.

To define G-regular covariant nonlinear Fredholm mappings $f : X \to X$ with index 0, it suffices to require that f is G-covariant and $Df(x) \in \Phi_{0G}^{\mathrm{reg}}(X)$ for every $x \in X^G$, but we also prove that it suffices that f is G-covariant and $Df(x) \in \Phi_{0G}^{\mathrm{reg}}(X)$ for only *one* $x \in X^G$. These mappings are those to which we may expect that Theorem 1.1 can be extended. Openness (and nonemptiness) of $\Phi_{0G}^{\mathrm{reg}}(X)$ in $\Phi_{0G}(X)$ shows that they form a substantial subset of the set of G-covariant Fredholm mappings with index 0. On the other hand, we shall only be able to prove that (1.1) holds with ($\deg_{L.S.}$ replaced by the degree of [10] and) the *radical* $\sqrt{\mathcal{I}^G}$ replacing $\mathcal{I}^G$. We do not know whether this weaker conclusion is due to an artifact of our method of proof or has deeper reasons. Fortunately, this issue is of secondary importance for most practical purposes since $\mathcal{I}^G$ and $\sqrt{\mathcal{I}^G}$ are distinct from $\mathbb{Z}$ simultaneously.

Several factors contribute to the length of this article. The need for a close investigation of $\Phi_{0G}^{\mathrm{reg}}(X)$ mentioned earlier (and independent of degree considerations) is one of them. A second factor is the partial lack of available results about representations of compact Lie groups in Banach spaces: while the Hilbert space case is completely covered, some useful properties in general Banach spaces seem to have been left out, and must be proved here. A third and last factor is the relative novelty of several of the concepts involved in this paper, which accordingly must be discussed at some length: aside from the degree of [10] whose basic features are briefly reviewed, an important role is played by the notion of intrinsic isotropy subgroup defined in [17], [19], via loose representations (introduced in [20] for the proof of Theorem 1.1 when $\dim X < \infty$).

The properties of group representations in Banach spaces that are needed in the remainder of this paper are listed in Section 2. Section 3 is devoted to the definition of G-regularity of linear and nonlinear Fredholm mappings with index 0. The proofs involving arguments from Lie group representation theory are given in an Appendix at the end of the article.

Parities and parametrices are the major ingredients in the definition of the degree of [10]. Section 4 discusses the covariant aspects of these two concepts. The main result in Section 4 is Theorem 4.2 showing how covariance may rule out the value -1 for the parity.

Section 5 contains the first results about the degree of covariant mappings. The goal here is to show that the general problem can be reduced to a somewhat simpler situation (Proposition 5.1), whose special features are highlighted in Theorem 5.1. Most of Section 6 is devoted to the proof of the generalization of Theorem 1.1 when G is finite (Theorem 6.1), starting with the case when G is a $\wp$-group (Lemma 6.4) and using the reduction of Section 5. Passing from $\wp$-groups to arbitrary finite groups is done via Sylow subgroups. Although the steps here are similar to those in [20], the $\wp$-group case must be given a completely different treatment because the degree of [10] is not defined by finite-dimensional approximation and because we cannot make use of partitions of unity (since C^2 ones need not exist). Theorem 6.2 gives three cases when $\mathcal{I}^G$ can be substituted for $\sqrt{\mathcal{I}^G}$ in Theorem 6.1, and Borsuk's theorem is spelled out explicitly as a trivial corollary (Corollary 6.1). In Section 7, Theorem 6.1 is extended to compact Lie groups.

Remark 1.1. Throughout this paper it is assumed that the Fredholm mapping f of interest is defined in the entire space X. This is *not* merely intended for convenience: although the theory goes through if f is defined over a convex or even more general domain (e.g. star-shaped with respect to a point of X^G) it is not valid for completely arbitrary open domains of definition. This is due to the fact that the degree of [10] is defined only for mappings which are Fredholm of index 0 in a *simply connected* domain $\mathcal{O}$ containing the open subset Ω of interest (no restriction is put on Ω). Since we have to consider here not only the degree of f but also that of its restrictions to some fixed point spaces, hypotheses are needed to ensure that these restrictions are defined in simply connected subsets.

On the other hand, for some *subfamilies* of Fredholm mappings with index 0, limitations on the domain of definition can be dropped (see [11]). □

Some permanent notation and terminology will be used, which are as follows: G denotes a compact Lie group with unity 1 and X is a real Banach space. We use the notation $\mathcal{L}(X)$ (resp. $GL(X)$, $\Phi_\nu(X)$, $\mathcal{K}(X)$) for the set of linear continuous (resp. invertible, Fredholm with index $\nu \in \mathbb{Z}$, compact) operators of X into itself. The representation of G in $GL(X)$ (see Section 2) is denoted by R and, as is customary, we set $R_g = R(g)$ for $g \in G$. A linear operator $A \in \mathcal{L}(X)$ is said to be *G-covariant* if $R_g A = AR_g$, for all $g \in G$, and we use $\mathcal{L}_G(X)$ (resp. $GL_G(X)$, $\Phi_{\nu G}(X)$, $\mathcal{K}_G(X)$) to designate the subset of $\mathcal{L}(X)$ (resp. $GL(X)$, $\Phi_\nu(X)$, $\mathcal{K}(X)$) of those G-covariant elements. No confusion should arise from the fact that the representation R is not explicitly mentioned in this notation. Indeed, we shall rarely use two different representations simultaneously, and when we do so the above notation is not needed.

If H is a closed subgroup of G we write $H \leq G$, and $H < G$ to emphasize that, necessarily, H is a proper subgroup. We denote by $X^H \subset X$ the fixed point space of H relative to R, i.e. the space $\{x \in X : R_g x = x, \ \forall g \in H\}$. If $D \subset X$ is any subset, we set $D^H = D \cap X^H$, and given a G-covariant (hence H-covariant) mapping $f : X \to X$ we let $f^H : X^H \to X^H$ be the restriction $f_{|X^H}$. That f^H maps into X^H is both trivial and well-known. When $H \leq G$, G/H is the set of left cosets gH, $g \in G$. For $x \in X$, G_x is the isotropy subgroup of x relative to R, i.e. $G_x = \{g \in G : R_g x = x\}$, and G/G_x is referred to as the orbit of x (because, as is well-known, G/G_x is homeomorphic to the subset $\{R_g x : g \in G\} \subset X$). If G is finite and $H \leq G$, $|H|$ is the order of H (number of elements) and $[G : H]$ the index of H in G, defined by $[G : H] = |G|/|H|$. Other notation and terminology will be introduced when needed in the paper.

2. General background material

Let G be a compact Lie group and X, Y real Banach spaces. We refer to Lang [15] for the definition of the integral of continuous (and other) Banach space valued functions on G relative to the Haar measure dg of G. Suppose now that $L : G \ni g \to L_g \in \mathcal{L}(X,Y)$ is a mapping such that $G \ni g \mapsto L_g x \in Y$ is continuous for every $x \in X$. If so, the mapping $X \ni x \mapsto \int_G L_g x \, dg \in Y$ is well-defined and linear and denoted by $\int_G L_g \, dg$. Thus, by definition,

$$\left(\int_G L_g \, dg\right) x \equiv \int_G L_g x \, dg, \qquad \forall x \in G. \tag{2.1}$$

The following proposition, in which all integrals must be understood as defined in (2.1), will be useful in several places.

PROPOSITION 2.1. *Let G be a compact Lie group, X and Y real Banach spaces and $L : G \to \mathcal{L}(X,Y)$ a mapping such that $G \ni g \mapsto L_g x \in Y$ is continuous for every $x \in X$. Then:*

(i) *There is a constant $c \geq 0$ such that $\|L_g\| \leq c$, for all $g \in G$.*

(ii) $\int_G L_g \, dg \in \mathcal{L}(X,Y)$.

In addition, if $Y = X$ and $M : G \to \mathcal{L}(X)$ is another mapping such that $G \ni g \mapsto M_g x \in X$ is continuous for every $x \in X$, then

(iii) *The mapping $\mathcal{L}(X) \ni \Lambda \mapsto \int_G L_g \Lambda M_g \, dg \in \mathcal{L}(X)$ is continuous.*

(iv) *If $K \in \mathcal{K}(X)$, then $\int_G L_g K M_g \, dg \in \mathcal{K}(X)$.*

PROOF. (i) Since G is compact, the set $\{L_g x : g \in G\}$ is a compact subset of Y for each $x \in X$ and hence there is a constant $c_x \geq 0$ such that $\|L_g x\| \leq c_x$ for all $g \in G$. The conclusion follows from the uniform boundedness principle.

(ii) From (2.1) and the properties of the integral we have $\|(\int_G L_g dg)x\| \leq \int_G \|L_g x\| \, dg$ for all $x \in X$, whereas $\int_G \|L_g x\| \, dg \leq c\|x\|$ from part (i).

(iii) That $\int_G L_g \Lambda M_g dg \in \mathcal{L}(X)$ follows from (ii) with $L_g \Lambda M_g$ replacing L_g. Indeed, using (i) it is easily seen that the mapping $G \ni g \mapsto L_g \Lambda M_g x \in X$ is continuous for every $x \in X$. Next, by the same arguments as in (ii), $\| \int_G L_g \Lambda M_g \, dg\| \leq c^2 \|\Lambda\|$ where $c \geq 0$ is such that $\|L_g\| \leq c$, $\|M_g\| \leq c$ for all $g \in G$.

(iv) We must show that the set $(\int_G L_g K M_g \, dg)(B)$ is relatively compact in X for every bounded subset $B \subset X$. To see this, it suffices to prove that the set $C = \bigcup_{g \in G} L_g K M_g(B)$ is relatively compact in X. Indeed, if so, the closed convex hull $\widetilde{C}$ of $\overline{C}$ is compact. As $(\int_G L_g K M_g \, dg)x = \int_G L_g K M_g x \, dg$ (see (2.1)) is a limit of convex combinations of points of C for $x \in B$ (since every continuous function $f : G \to X$ can be approximated by step functions of the form $\sum_{i=1}^m f(g_i)\chi_{E_i}$ with E_i dg-measurable, $\bigcup_{i=1}^m E_i = G$, $E_i \cap E_j = \emptyset$ for $i \neq j$ and $g_i \in E_i$), it follows that $(\int_G L_g K M_g \, dg)x \in \widetilde{C}$ for all $x \in B$, whence $(\int_G L_g K M_g \, dg)(B)$ is relatively compact.

Now, set $B' = \bigcup_{g \in G} M_g(B)$, $B'' = K(B')$ and $B''' = \bigcup_{g \in G} L_g(B'')$.

Obviously, $C \subset B'''$. We claim that B''', hence C, is relatively compact. First, part (i) implies that B' is bounded, so that B'' is relatively compact since K is compact. A sequence $x_n \in B'''$ has the form $x_n = L_{g_n} y_n$ with $g_n \in G$ and $y_n \in B''$. As G is compact and B'' is relatively compact, we may assume that $\lim_{n\to\infty} g_n = g \in G$ and $\lim_{n\to\infty} y_n = y \in X$ with no loss of generality. If so, $x_n - L_g y = L_{g_n}(y_n - y) + (L_{g_n} - L_g)y$, and from part (i) we find $\|x_n - L_g y\| \leq c\|y_n - y\| + \|L_{g_n} y - L_g y\|$. This shows that $\lim_{n\to\infty} x_n = L_g y$ since $G \ni h \mapsto L_h y$ is continuous by hypothesis. $\square$

The other results proven in this section may be called standard in Hilbert space, but they are apparently hard to find in the literature in the more general Banach space setting.

To begin with, recall that a representation of G in $GL(X)$ is a group homomorphism $R : G \to GL(X)$ such that the mapping $G \ni g \mapsto R_g x \in X$ is continuous for every $x \in X$. When $\dim X = \infty$, this requirement is weaker than continuity of R for the topology of $\mathcal{L}(X)$. It can be shown (Dieudonné [7, p. 9], Warner [25, p. 237]) that the definition given above amounts to assuming that R is a group homomorphism and that the mapping $G \times X \ni (g, x) \mapsto R_g x \in X$ is continuous. We will need the following version of Maschke's theorem ([23]).

THEOREM 2.1 (Maschke's theorem for Banach spaces). *Let $Y \subset X$ be a closed split G-invariant subspace. Then Y possesses a (possibly nonunique) G-invariant closed complement Z. Moreover, the (continuous) projections onto Y and Z associated with the splitting $X = Y \oplus Z$ are G-covariant.*

The proof of Theorem 2.1 is based upon the observation that any projector whose image is a closed invariant subspace can be made covariant by integrating over G.

An important case when the invariant complement Z in Theorem 2.1 is unique is $Y = X^G$. In addition, existence of Z as well is always true in this case, as shown in Theorem 2.2 below. The proof uses the concept of "irreducible G-module", which is simply a finite-dimensional vector space equipped with an irreducible representation of G. In this respect, recall that every irreducible representation of G in a Banach space is finite-dimensional; see [3] or [4]. Given two irreducible G-modules U and V, U is said to have type V if there is a G-covariant linear isomorphism $U \to V$. By an irreducible G-module contained in X we mean a finite-dimensional subspace U of X invariant under the action of G in X and such that the subrepresentation of G in $GL(U)$ is irreducible. If V is an arbitrary irreducible G-module, the algebraic sum X_V of all the irreducible G-modules of type V contained in X is called the *isotypical component* of X of type V. In particular, the fixed point space X^G is the isotypical component of X of type V_0, where V_0 is the one-dimensional irreducible G-module in which G acts trivially (trivial irreducible G-module). We agree that $X_V = \{0\}$ if X contains no irreducible G-module of type V.

THEOREM 2.2. *The fixed point space X^G of G has a unique G-invariant closed complement $\widetilde{X}^G$. Furthermore, relative to the splitting $X = X^G \oplus \widetilde{X}^G$, every operator $A \in \mathcal{L}_G(X)$ has the block-diagonal decomposition*

$$A = \begin{pmatrix} A^G & 0 \\ 0 & \widetilde{A}^G \end{pmatrix},$$

with $A^G \in \mathcal{L}(X^G)$ and $\widetilde{A}^G \in \mathcal{L}_G(\widetilde{X}^G)$.

PROOF. See the Appendix. □

We complete this section with a result about isotypical components.

THEOREM 2.3. *Let V be an arbitrary irreducible G-module.*

(i) *If $W \neq \{0\}$ is an irreducible G-module and $U \subset \overline{X}_V$ is an irreducible G-module of type W, then V and W are isomorphic (hence $X_V = X_W$).*
(ii) *The space $\overline{X}_V$ is G-invariant, and for every $A \in \mathcal{L}_G(X)$, we have $A(\overline{X}_V) \subset \overline{X}_V$, whence $A_{|\overline{X}_V} \in \mathcal{L}_G(\overline{X}_V)$. Furthermore, if $\overline{X}_V$ is split and Z is a closed G-invariant complement of $\overline{X}_V$ (see Theorem 2.1), we have $A(Z) \subset Z$, whence $A_{|Z} \in \mathcal{L}_G(Z)$.*

PROOF. See the Appendix. □

REMARK 2.1. Whenever convenient, it is not restrictive to assume that the norm of X is G-invariant, for the mapping

$$X \ni x \mapsto \int_G \|R_g x\|\, dg$$

is a G-invariant norm equivalent to the norm $\|\cdot\|$ (use Proposition 2.1(i)). □

3. Regular covariant Fredholm mappings with index 0

If $A \in \Phi_0(X)$, it is well-known and essentially trivial to prove that there is $\eta \in GL(X)$ such that $\eta A = I - K$ with $K \in \mathcal{K}(X)$. It is equally clear that this property implies that $GL(X)$ is dense in $\Phi_0(X)$. The concept of regular covariant Fredholm operator of index 0 arises in a natural way when A above is in $\Phi_{0G}(X)$, and the question is asked whether there is $\eta \in GL_G(X)$ such that $\eta A = I - K, K \in \mathcal{K}(X)$ (hence $K \in \mathcal{K}_G(X)$), or whether A can be approximated by elements of $GL_G(X)$. Indeed, the answer to these questions is in general negative, as shown in

EXAMPLE 3.1. Let $X = E \times E$ where E is a separable Hilbert space and let $(e_n)_{n=1,\infty}$ be an othonormal basis of E. For $(u,v) \in E \times E$, $u = \sum_{n=1}^{\infty} u_n e_n$, $v = \sum_{n=1}^{\infty} v_n e_n$, set $A(u,v) = (\sum_{n=1}^{\infty} u_n e_{n+1}, \sum_{n=1}^{\infty} v_{n+1} e_n)$. Obviously, $\dim \ker A = \operatorname{codim} \operatorname{rge} A = 1$, whence $A \in \Phi_0(X)$. Also, A is G-covariant with $G = \mathbb{Z}_2 = \{1, -1\}$ represented by $R_{-1} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. If so, $X^G = E \times \{0\}$ and $\widetilde{X}^G = \{0\} \times E$, and $A(u,v) = (A^G u, \widetilde{A}^G v)$ where A^G (resp. $\widetilde{A}^G$) is the right (resp. left) shift operator. Given $\eta \in GL_G(X)$, we have (e.g. by Theorem 2.2) $\eta = \begin{pmatrix} \eta^G & 0 \\ 0 & \widetilde{\eta}^G \end{pmatrix}$. Thus, if $\eta A = I - K$, $K \in \mathcal{K}(X)$, then $\eta^G A^G = I - K^G$, $K^G \in \mathcal{K}(E)$, which is absurd since $\eta^G A^G$ has the same index as A^G, that is, -1. Similar considerations show that A cannot be approximated by a sequence from $GL_G(X)$.

This example justifies the following definition:

DEFINITION 3.1. Let $A \in \Phi_{0G}(X)$. We shall say that A is *G-regular* if there is $\eta \in GL_G(X)$ such that $\eta A = I - K$, $K \in \mathcal{K}(X)$ (hence $K \in \mathcal{K}_G(X)$). The subset of all G-regular elements of $\Phi_{0G}(X)$ will be denoted by $\Phi_{0G}^{\text{reg}}(X)$.

The terminology "G-regular" is motivated by the relevance of this concept regarding parametrices (see Sections 4 and 5) and the use of "regularizor" for "parametrix" in the Russian literature. The remainder of this section is devoted to a detailed study of $\Phi_{0G}^{\text{reg}}(X)$. The most technical proofs are relegated to the Appendix. We shall begin with two elementary examples.

EXAMPLE 3.2. If $K \in \mathcal{K}_G(X)$, then $I - K \in \Phi_{0G}^{\text{reg}}(X)$. More generally, if $A \in \Phi_{0G}^{\text{reg}}(X)$, then $A - K \in \Phi_{0G}^{\text{reg}}(X)$ since, given $\eta \in GL_G(X)$ such that $\eta A = I - L$, $L \in \mathcal{K}(X)$, we have $\eta(A - K) = \eta A - \eta K = I - (L + \eta K)$, and $L + \eta K \in \mathcal{K}(X)$.

EXAMPLE 3.3. If $G = \mathbb{Z}_2 = \{1, -1\}$ is represented by $R_{-1} = -I$, then $\Phi_{0G}^{\text{reg}}(X) = \Phi_{0G}(X)$ since $\Phi_{0G}(X) = \Phi_0(X)$ and $GL_G(X) = GL(X)$.

Our first proposition shows that $\Phi_{0G}^{\text{reg}}(X)$ is always a sizable subset of $\Phi_{0G}(X)$.

PROPOSITION 3.1. *Let $A \in \Phi_{0G}^{\text{reg}}(X)$. Then there is an open neighborhood $\mathcal{N}$ of A in $\Phi_{0G}(X)$ and a continuous mapping $\eta : \mathcal{N} \to GL_G(X)$ such that $\eta(B)B = I - K(B)$, where $K(B) \in \mathcal{K}_G(X)$. In particular, $\Phi_{0G}^{\text{reg}}(X)$ is a nonempty open subset of $\Phi_{0G}(X)$.*

PROOF. Let $X_0 = \ker A$, $Y_1 = \operatorname{rge} A$, and denote by X_1 and Y_0 arbitrary closed complements of X_0 and Y_1, respectively. Also, denote by Q_0 and Q_1 the (continuous) projections associated with $X = Y_0 \oplus Y_1$.

As $A_{|X_1} \in GL(X_1, Y_1)$, we have $Q_1 B_{|X_1} \in GL(X_1, Y_1)$ for B in some open neighborhood $\mathcal{V}$ of A in $\mathcal{L}(X)$. Denoting by T any linear isomorphism of Y_0 onto X_0 (T exists since $\dim X_0 = \dim Y_0 < \infty$), the mapping $\mu(B) = (Q_1 B_{|X_1})^{-1} Q_1 + T Q_0 \in GL(X)$ is well-defined and continuous in $\mathcal{V}$. Furthermore, $\mu(B)B = (Q_1 B_{|X_1})^{-1} Q_1 B + T Q_0 B = I + (Q_1 B_{|X_1})^{-1} Q_1 B_{|X_0} + T Q_0 B$ is a compact perturbation $I - L(B)$ of the identity since $B_{|X_0}$ and $Q_0 B$ have finite rank.

The relation $\mu(B)B = I - L(B)$ first yields $\mu(A)^{-1} = A + \mu(A)^{-1} L(A)$ and next $\mu(A)^{-1}\mu(B)B = A - \mu(A)^{-1}(L(B) - L(A))$. By hypothesis, there is $\eta_0 \in GL(X)$ such that $\eta_0 A = I - K_0$, $K_0 \in \mathcal{K}_G(X)$. Multiplying the previous identity by η_0, we find $\eta_0 \mu(A)^{-1} \mu(B) B = I - K_0 - \eta_0 \mu(A)^{-1}(L(B) - L(A)) = I - \widetilde{K}(B)$, where $\widetilde{K}(B) = K_0 + \eta_0 \mu(A)^{-1}(L(B) - L(A)) \in \mathcal{K}(X)$.

For $B \in \mathcal{N} = \mathcal{V} \cap \Phi_{0G}(X)$, covariance of B and η_0 yields

$$\eta_0 R_{g^{-1}} \mu(A)^{-1} \mu(B) R_g B = I - R_{g^{-1}} \widetilde{K}(B) R_g, \qquad \forall g \in G.$$

Thus, integrating over G, we arrive at $\eta(B)B = I - K(B)$, where (the integrals being understood in the sense of (2.1))

$$\eta(B) = \eta_0 \int_G R_{g^{-1}} \mu(A)^{-1} \mu(B) R_g \, dg, \qquad K(B) = \int_G R_{g^{-1}} \widetilde{K}(B) R_g \, dg.$$

Since $R_{g^{-1}} = R_g^{-1}$, Proposition 2.1(iii) shows that η and K are continuous functions of $B \in \mathcal{N}$ with values in $\mathcal{L}(X)$ and, in fact, in $\mathcal{L}_G(X)$ since G-covariance is clear. Also, $\eta(A) = \eta_0 \in GL_G(X)$, whence $\eta(B) \in GL_G(X)$ after shrinking $\mathcal{N}$ if necessary, and $K(B) \in \mathcal{K}(X)$ (hence $\mathcal{K}_G(X)$) by Proposition 2.1(iv).

That $\Phi_{0G}^{\mathrm{reg}}(X)$ is open in $\Phi_{0G}(X)$ follows at once from the above, and $I \in GL_G(X) \subset \Phi_{0G}^{\mathrm{reg}}(X)$ ensures nonemptiness of $\Phi_{0G}^{\mathrm{reg}}(X)$. □

In our second proposition, we prove a result implying denseness of $GL_G(X)$ in $\Phi_{0G}^{\mathrm{reg}}(X)$. Later (Theorem 3.1), we shall show that, in fact, $\Phi_{0G}^{\mathrm{reg}}(X)$ is the closure of $GL_G(X)$ in $\Phi_{0G}(X)$, and that this property can be taken as an equivalent definition for $\Phi_{0G}^{\mathrm{reg}}(X)$.

PROPOSITION 3.2. *For every sequence (A_i) from $\Phi_{0G}^{\mathrm{reg}}(X)$ and for every $\varepsilon > 0$, there is $K \in \mathcal{K}_G(X)$ such that $\|K\| < \varepsilon$ and $A_i - K \in GL_G(X)$, for all $i \in \mathbb{N}$. In particular, $GL_G(X)$ is dense in $\Phi_{0G}^{\mathrm{reg}}(X)$.*

PROOF. Since $\mathcal{K}_G(X)$ is a closed subspace of $\mathcal{L}(X)$, hence a Baire space, it suffices to prove that for every $A \in \Phi_{0G}^{\mathrm{reg}}(X)$, the set $\{K \in \mathcal{K}_G(X) : A + K \in GL_G(X)\}$ is open and dense in $\mathcal{K}_G(X)$. Openness is obvious. To prove denseness, fix $K_0 \in \mathcal{K}_G(X)$ so that $A + K_0 \in \Phi_{0G}^{\text{reg}}(X)$ (see Example 3.2). Let $\eta \in GL_G(X)$ be such that $\eta(A + K_0) = I - L, L \in \mathcal{K}_G(X)$. For $\delta > 0$ small enough, we have $I-(1-\delta)L \in GL_G(X)$ and hence $A+K_0+\delta\eta^{-1}L = \eta^{-1}(I-(1-\delta)L) \in GL_G(X)$. Also, $K_\delta \equiv K_0 + \delta\eta^{-1}L \in \mathcal{K}_G(X)$ and $\|K_\delta - K_0\| = \delta\|\eta^{-1}L\|$ can be made arbitrarily small by shrinking δ if necessary. $\square$

THEOREM 3.1. *For $A \in \Phi_{0G}(X)$, the following statements are equivalent:*

(i) $GL_G(\ker A, Z) \neq \emptyset$ *for every G-invariant closed complement Z of* $\operatorname{rge} A$.
(ii) $GL_G(\ker A, Z) \neq \emptyset$ *for some G-invariant closed complement Z of* $\operatorname{rge} A$.
(iii) $A \in \Phi_{0G}^{\text{reg}}(X)$.
(iv) $A \in \overline{GL_G(X)}$ *(closure in $\Phi_0(X)$ or, equivalently, $\Phi_{0G}(X)$).*
(v) *For every irreducible G-module V, $A_{|\overline{X}_V} \in \Phi_0(\overline{X}_V)$ (hence $A_{|\overline{X}_V} \in \Phi_{0G}(\overline{X}_V)$; that $A \in \mathcal{L}_G(\overline{X}_V)$ follows from Theorem* 2.3).

PROOF. See the Appendix. Note that a G-invariant closed complement of $\operatorname{rge} A$ does exist by Theorem 2.1. $\square$

REMARK 3.1. Conditions (i) and (ii) of Theorem 2.3 mean that the subrepresentations of G in $GL(\ker A)$ and $GL(Z)$ are equivalent. The set of all isomorphism classes of finite-dimensional representations of G is a semigroup under the direct sum operation. Let $R(G)$ be the associated Grothendieck group: by definition, $R(G)$ is the set of equivalence classes of pairs (M, N) under the relation $(M, N) \sim (M', N')$ if and only if $M \oplus N'$ is G-isomorphic to $M' \oplus N$. Since every finite-dimensional representation splits into a sum of irreducible representations it follows that $R(G)$ is a free group generated by the equivalence classes of irreducible representations (see Segal [21]). To each $A \in \Phi_{0G}(X)$, Atiyah and Singer [1] associate an index $\operatorname{ind}_G(A) = [\ker A] - [\operatorname{coker} A] \in R(G)$ (where "[ ]" stands for "equivalence class") which depends continuously on A. If the action of G is trivial, then ind_G reduces to the usual index, but in general ind_G induces a nontrivial invariant $\operatorname{ind}_G : \pi_0(\Phi_{0G}(X)) \to R(G)$. Clearly, (ii) of Theorem 3.1 holds if and only if $\operatorname{ind}_G(A) = 0$ (for $G = \mathbb{Z}_p$, this was considered by Borisovich et al. [2] in their investigation of the covariant properties of the Elworthy-Tromba degree). $\square$

Theorem 3.1 has several useful corollaries, the simplest one being:

COROLLARY 3.1. *$\Phi_{0G}^{\text{reg}}(X)$ is open and closed in $\Phi_{0G}(X)$. In particular, if $A_0, A_1 \in \Phi_{0G}(X)$ are homotopic in $\Phi_{0G}(X)$ and $A_0 \in \Phi_{0G}^{\text{reg}}(X)$, then $A_1 \in \Phi_{0G}^{\text{reg}}(X)$. Furthermore, $\Phi_{0G}^{\text{reg}}(X)$ is the union of all the connected components of $\Phi_{0G}(X)$ intersecting $GL_G(X)$.*

PROOF. Openness of $\Phi_{0G}^{\mathrm{reg}}(X) \subset \Phi_{0G}(X)$ was proved in Proposition 3.1, and closedness follows from (iv) $\Rightarrow$ (iii) in Theorem 3.1. The second part of the corollary is obvious from the first part and the third part follows from the first part and (iii) $\Rightarrow$ (iv) in Theorem 3.1. □

Next, Theorem 3.1 yields a simple criterion to establish membership in $\Phi_{0G}^{\mathrm{reg}}(X)$ which does not require any knowledge of the way G acts in X. This criterion generalizes Example 3.2.

COROLLARY 3.2. *Let $A \in \mathcal{L}_G(X)$ and suppose that 0 is an isolated eigenvalue of A with finite algebraic multiplicity. Then $A \in \Phi_{0G}^{\mathrm{reg}}(X)$. More generally, if there is $B \in GL_G(X)$ such that 0 is an isolated eigenvalue of BA with finite algebraic multiplicity, then $A \in \Phi_{0G}^{\mathrm{reg}}(X)$.*

NOTE. As usual, "isolated eigenvalue" means "isolated in the spectrum" and not only in the set of eigenvalues.

PROOF. The "more generally" part follows at once from the first part and the definition of $\Phi_{0G}^{\mathrm{reg}}(X)$ (Definition 3.1). Now, if 0 is an isolated eigenvalue of A with finite algebraic multiplicity, and if Y is any closed subspace of X such that $A(Y) \subset Y$, then either $A_{|Y}$ is invertible or 0 is an isolated eigenvalue of $A_{|Y}$ with finite algebraic multiplicity. In particular, this is true for $Y = \overline{X}_V$ and V any irreducible G-module (see Theorem 2.3(ii)). But invertible operators or those having 0 as an isolated eigenvalue with finite multiplicity are Fredholm with index 0, whence $A \in \Phi_{0G}(X)$ and $A_{|\overline{X}_V} \in \Phi_{0G}(\overline{X}_V)$ for every irreducible G-module V, and the conclusion follows from (v) $\Rightarrow$ (iii) in Theorem 3.1. □

Also, we obtain a sufficient condition for the relation $\Phi_{0G}^{\mathrm{reg}}(X) = \Phi_{0G}(X)$ to hold, generalizing Example 3.3.

COROLLARY 3.3. *Suppose that there is an irreducible G-module V such that $\overline{X}_V$ is split, and that $\dim X_W < \infty$ for every irreducible G-module $W \not\approx V$. Then $\Phi_{0G}^{\mathrm{reg}}(X) = \Phi_{0G}(X)$. (In particular, this holds if $\dim X_W < \infty$ for every irreducible G-module W.)*

PROOF. See the Appendix. □

REMARK 3.2. (i) Example 3.3 corresponds to the case $V = \mathbb{R}$ and $G = \mathbb{Z}_2 = \{1, -1\}$ represented by $R_{-1} = -I$, so that $X = \overline{X}_V$ where V is the one-dimensional irreducible G-module in which $-1 \in G$ acts by multiplication, and $X_W = \{0\}$ for $W \not\approx V$.

(ii) Except for the case just mentioned in (i), Corollary 3.3 seems to be of little interest when G is finite (and $\dim X = \infty$) because there are only finitely many nonisomorphic irreducible G-modules in this case. If $\dim G \geq 1$, the framework of Corollary 3.3 is frequently encountered. For instance, if $X = C^0_{2\pi}(\mathbb{R};\mathbb{R})$, the circle group $G = SO(2) \sim \mathbb{R}/2\pi\mathbb{Z}$ acts through translation of the independent variable (i.e. $R_\theta f(x) = f(x+\theta)$ for all $x \in \mathbb{R}$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$). Now, if V is an irreducible G-module, then either $V \sim \mathbb{R}$ and G acts trivially in V, or $V \sim \mathbb{R}^2$ and $\mathbb{R}_\theta$ acts through rotation of angle $n\theta$, $n \geq 1$ an integer, relative to some basis $\{f_1, f_2\}$ of V. In the first case, if $f \in V$ we have $f(x+\theta) = f(x)$, whence $f(\theta) = f(0)$, i.e. V is the space of constant functions, hence unique, and $V = X_V$ is one-dimensional. In the second, writing $f_1(x+\theta) = \cos n\theta f_1(x) + \sin n\theta f_2(x)$, $f_2(x+\theta) = -\sin n\theta f_1(x) + \cos n\theta f_2(x)$, and letting $x = 0$, we find that $f_1(\theta) = a\cos n\theta + b\sin n\theta$, $f_2(\theta) = -a\sin n\theta + b\cos n\theta$ for $a, b \in \mathbb{R}$ with $a^2 + b^2 > 0$. This shows that $V = \operatorname{span}\{f_1(x), f_2(x)\} = \operatorname{span}\{\cos nx, \sin nx\}$. Once again, V is unique and $V = X_V$ is two-dimensional. $\square$

Another useful by-product of Theorem 3.1 is the following variant of Theorem 2.2 when $A \in \Phi^{\mathrm{reg}}_{0G}(X)$.

COROLLARY 3.4. *Let $H \leq G$ be a closed subgroup, and suppose that X^H is G-invariant (e.g. if H is normal in G). Then:*

(i) *$\widetilde{X}^H$ is G-invariant.*

(ii) *For every $A \in \Phi^{\mathrm{reg}}_{0G}(X)$, we have the block-diagonal decomposition*

$$A = \begin{pmatrix} A^H & 0 \\ 0 & \widetilde{A}^H \end{pmatrix},$$

with $A^H \in \Phi^{\mathrm{reg}}_{0G}(X^H)$ and $\widetilde{A}^H \in \Phi^{\mathrm{reg}}_{0G}(\widetilde{X}^H)$.

PROOF. (i) It follows from Theorem 2.2 that X^H is split, and hence it has a G-invariant closed complement Z by Theorem 2.1. Clearly Z is also H-invariant, whence $Z = \widetilde{X}^H$ by uniqueness of $\widetilde{X}^H$ (Theorem 2.2), i.e. $\widetilde{X}^H$ is G-invariant.

(ii) From (iii) $\Rightarrow$ (iv) in Theorem 3.1 and from Theorem 2.2, there is a sequence $A_n \in GL_G(X)$ such that $\lim_{n\to\infty} \|A_n - A\| = 0$, and A_n, A have the block-diagonal form

$$A_n = \begin{pmatrix} A_n^H & 0 \\ 0 & \widetilde{A}_n^H \end{pmatrix}, \qquad A = \begin{pmatrix} A^H & 0 \\ 0 & \widetilde{A}^H \end{pmatrix},$$

relative to $X = X^H \oplus \widetilde{X}^H$.

Clearly, $A_n^H \in GL_G(X^H)$ and $\widetilde{A}_n^H \in GL_G(\widetilde{X}^H)$ from (i), and $\lim_{n\to\infty} \|A_n^H - A^H\| = \lim_{n\to\infty} \|\widetilde{A}_n^H - \widetilde{A}^H\| = 0$. As $A \in \Phi_0(X)$, we must have $A^H \in \Phi_\nu(X^H)$ and $\widetilde{A}^H \in \Phi_{-\nu}(\widetilde{X}^H)$ for some integer ν. But local constancy of the index and $\lim_{n\to\infty} \|A_n^H - A^H\| = 0$ imply $\nu = 0$ since A_n^H has index 0 for all n, and, further, that $A^H \in \Phi_{0G}^{\mathrm{reg}}(X^H)$ by (iv) $\Rightarrow$ (iii) in Theorem 3.1. Likewise, $\widetilde{A}^H \in \Phi_{0G}^{\mathrm{reg}}(\widetilde{X}^H)$. □

The notion of G-regularity can easily be extended to nonlinear covariant mappings:

DEFINITION 3.2. Let $f : X \to X$ be a G-covariant nonlinear Fredholm mapping with index 0. We shall say that f is *G-regular* if $Df(x) \in \Phi_{0G}^{\mathrm{reg}}(X)$, for all $x \in X^G$.

NOTE. This definition makes sense since $Df(x) \in \Phi_{0G}(X)$ for $x \in X^G$ by G-covariance of f.

When $\Phi_{0G}^{\mathrm{reg}}(X) = \Phi_{0G}(X)$ (see Corollary 3.3) G-regularity is not an additional assumption. Otherwise, the situation may be more complicated, but the following criterion shows that it is actually much simpler than it may look at first sight.

THEOREM 3.2. *Let $f : X \to X$ be a G-covariant Fredholm mapping with index 0. Then f is G-regular if and only if there is $x_0 \in X^G$ such that $Df(x_0) \in \Phi_{0G}^{\mathrm{reg}}(X)$. In particular, if f is G-regular, then f is also H-regular for every closed subgroup $H \leq G$.*

PROOF. Since $Df(tx_0 + (1-t)x)$, $0 \leq t \leq 1$, is a homotopy in $\Phi_{0G}(X)$ for every pair $x, x_0 \in X^G$, the first part follows from Corollary 3.1. For the second part, note only that $X^G \subset X^H$ and $\Phi_{0G}^{\mathrm{reg}}(X) \subset \Phi_{0H}^{\mathrm{reg}}(X)$ when $H \leq G$, and use the first part. □

Theorem 3.3 below follows at once from previous results, but it will be useful for future reference.

THEOREM 3.3. *Let $f : X \to X$ be a G-regular covariant C^k ($k \geq 1$) Fredholm mapping with index 0, and let $H \leq G$ be such that X^H is G-invariant (e.g. if H is normal in G). Then*

(i) *$f^H = f_{|X^H}$ maps X^H into itself.*

(ii) *We have*

$$Df(x) = \begin{pmatrix} Df^H(x) & 0 \\ 0 & B(x) \end{pmatrix}, \qquad \forall x \in X^H,$$

relative to the splitting $X = X^H \oplus \widetilde{X}^H$, and $Df^H(x) \in \Phi_{0G}^{\mathrm{reg}}(\widetilde{X}^H)$ (recall that $\widetilde{X}^H$ is G-invariant; see Corollary 3.4). In particular, f^H is a G-regular covariant C^k Fredholm mapping with index 0.

PROOF. (i) is trivial and well-known, and (ii) follows from Corollary 3.4. □

REMARK 3.3. If H is normal in G in Theorem 3.3, then the action of G in X^H factors through an action of G/H and part (ii) of the theorem implies that f^H is a G/H-regular covariant C^k Fredholm mapping with index 0. □

4. Parametrices and parity: covariant aspects

Given $A \in C^0([a,b];\Phi_0(X))$, a *parametrix* of A is a mapping $\eta \in C^0([a,b]; GL(X))$ such that $\eta(\lambda)A(\lambda) = I - K(\lambda)$, where $K(\lambda) \in \mathcal{K}(X)$, $\forall\lambda \in [a,b]$. Parametrices are essential in our approach to the degree, since they permit us to define parities (a concept reviewed later) which are explicitly involved in the definition of the degree at regular values. If now $A \in C^0([a,b];\Phi_{0G}(X))$, it is natural to ask whether there is a G-covariant parametrix for A, i.e. a mapping $\eta \in C^0([a,b];GL_G(X))$ which is a parametrix of A. The answer is given in

THEOREM 4.1. *Let $A \in C^0([a,b];\Phi_{0G}(X))$. Then there is a G-covariant parametrix for A if and only if $A \in C^0([a,b];\Phi_{0G}^{\mathrm{reg}}(X))$.*

PROOF. If a G-covariant parametrix η of A exists, then $A(\lambda) \in \Phi_{0G}^{\mathrm{reg}}(X)$ for all $\lambda \in [a,b]$, by definition of $\Phi_{0G}^{\mathrm{reg}}(X)$ (see Section 3). Conversely, suppose $A \in C^0([a,b];\Phi_{0G}^{\mathrm{reg}}(X))$. Locally, existence of η is a straightforward consequence of Proposition 3.1 with $B = A(\lambda)$. By compactness of $[a,b]$, this implies that there is a finite subdivision $a = a_0 < a_1 < b_0 < a_2 < b_1 < \ldots < a_k < b_{k-1} < b_k = b$ such that a G-covariant parametrix η_i of A exists in $[a_i,b_i]$, $0 \le i \le k$.

Set $\lambda_0 = a_0 = a$, $\lambda_{k+1} = b_k = b$ and, for $1 \le i \le k$, choose $\lambda_i \in [a_i, b_{i-1}]$. For $\lambda \in [a,b]$, define $\eta(\lambda)$ by

$$\eta(\lambda) = \begin{cases} \eta_0(\lambda), & a = \lambda_0 \le \lambda \le \lambda_1, \\ \eta(\lambda_i)\eta_i(\lambda_i)^{-1}\eta_i(\lambda), & \lambda_i \le \lambda \le \lambda_{i+1},\ 1 \le i \le k. \end{cases}$$

Clearly, $\eta \in C^0([a,b];GL_G(X))$, and it remains to show that $\eta(\lambda)A(\lambda) - I \in \mathcal{K}(X)$ for $\lambda \in [a,b]$. This is clear for $a \le \lambda \le \lambda_1$. Suppose then that the result is true for $a \le \lambda \le \lambda_i$ where $1 \le i \le k$. We have $\eta(\lambda_i)A(\lambda_i) = I - K(\lambda_i)$, $\eta_i(\lambda_i)A(\lambda_i) = I - K_i(\lambda_i)$, with $K(\lambda_i)$, $K_i(\lambda_i) \in \mathcal{K}(X)$ since both η and η_i are parametrices of A in $[a_i, \lambda_i]$. Thus, $A(\lambda_i) = \eta_i(\lambda_i)^{-1}(I - K_i(\lambda_i))$ and $\eta(\lambda_i)\eta_i(\lambda_i)^{-1}(I - K_i(\lambda_i)) = I - K(\lambda_i)$. This shows that $\eta(\lambda_i)\eta_i(\lambda_i)^{-1}$ is a compact perturbation of the identity, and hence $\eta(\lambda)A(\lambda) = \eta(\lambda_i)\eta_i(\lambda_i)^{-1}\eta_i(\lambda)A(\lambda)$ is a compact perturbation of $\eta_i(\lambda)A(\lambda)$ for $\lambda_i \le \lambda \le \lambda_{i+1}$. But $\eta_i(\lambda)A(\lambda) = I - K_i(\lambda)$ with $K_i(\lambda) \in \mathcal{K}(X)$ since η_i is a parametrix of A in $[\lambda_i, \lambda_{i+1}] \subset [a_i, b_i]$. Thus, we have $\eta(\lambda)A(\lambda) - I \in \mathcal{K}(X)$ for $a \le \lambda \le \lambda_{i+1}$, i.e. the conclusion follows by induction. □

Given $A \in C^0([a,b];\Phi_0(X))$ satisfying the condition $A(a), A(b) \in GL(X)$, the *parity* of A on $[a,b]$, denoted by $\sigma(A,[a,b])$, is defined by

$$\sigma(A,[a,b]) = \deg_{L.S.}\eta(a)A(a)\deg_{L.S.}\eta(b)A(b), \tag{4.1}$$

where η is any parametrix of A on $[a,b]$ and $\deg_{L.S.}$ refers to the Leray-Schauder degree[1] of the invertible linear compact perturbations of the identity $\eta(a)A(a)$ and $\eta(b)A(b)$. Naturally, this definition is justified by its independence of the choice of the parametrix η (see [10] and the references therein).

If, above, $A(\lambda) \in \Phi_{0G}(X)$ for $\lambda \in [a,b]$, the parity can always be calculated from a G-covariant parametrix. Indeed:

PROPOSITION 4.1. *Let $A \in C^0([a,b];\Phi_{0G}(X))$ be such that $A(a), A(b) \in GL(X)$ (hence $A(a), A(b) \in GL_G(X)$). Then $A \in C^0([a,b];\Phi_{0G}^{\mathrm{reg}}(X))$ and there is a G-covariant parametrix for A.*

PROOF. Use Corollary 3.1 and the embedding $GL_G(X) \subset \Phi_{0G}^{\mathrm{reg}}(X)$ to get $A(\lambda) \in \Phi_{0G}^{\mathrm{reg}}(X)$ for all $\lambda \in [a,b]$. Next, use Theorem 4.1. □

In general, the parity may take either value ± 1. However, if a group action is involved and $A(\lambda) \in \Phi_{0G}(X)$ for all $\lambda \in [a,b]$, the value -1 may be ruled out. A specific framework when it so happens is described in Theorem 4.2 later, which plays a crucial role in our future results. Before we can state Theorem 4.2 we must recall the concept of *intrinsic isotropy subgroup* (i.i.s. for short) of a compact Lie group, introduced in [17] and further investigated in [19] in the case of finite groups.

A closed subgroup $H \leq G$ is said to be an i.i.s. of G if, for every integer $n \geq 0$, every representation T of G in $GL(\mathbb{R}^n)$ and every T-covariant linear isomorphism $L \in GL(\mathbb{R}^n)$, we have $\operatorname{sgn}\det L^H = \operatorname{sgn}\det L$, where $L^H \in GL(X^H)$ is the restriction of L to the fixed point space X^H of $T_{|H}$.

It is proved in [17] that every compact Lie group different from $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ (k factors, $k \geq 0$) possesses a nontrivial (i.e. $\neq \{1\}$) i.i.s. An i.i.s. is said to be *maximal* (m.i.i.s. for short) if it is contained in no larger i.i.s. By Zorn's lemma, every i.i.s. is contained in a m.i.i.s. ([17]).

Using the concept of m.i.i.s., we may define representations "between" free and fixed point free ones.

[1] That is, $(-1)^m$ where m is the sum of the algebraic multiplicities of the eigenvalues lying in $(1,\infty)$.

DEFINITION 4.1. The representation R of G in $GL(X)$ is said to be *loose* if $X^H = \{0\}$ for at least one m.i.i.s. H of G. Equivalently, R is loose if at least one m.i.i.s. of G does not appear as a subgroup of any isotropy subgroup G_x for $x \in X\backslash\{0\}$. More generally, R is said to be *semi-loose* if R is loose away from X^G, i.e. there is at least one m.i.i.s. of G that does not appear as a subgroup of any isotropy subgroup G_x for $x \in X\backslash X^G$.

Loose and semi-loose representations were first introduced in [20] (in the case $X = \mathbb{R}^n$) for the proof of Theorem 1.1.

REMARK 4.1. (i) It is easily seen that R is semi-loose if and only if R is loose in $\widetilde{X}^G$.

(ii) Every loose representation is fixed point free (i.e. $X^G = \{0\}$), but the converse need not be true, unless G is its own, unique, m.i.i.s. This happens only in special cases, e.g. $|G|$ finite and odd or $G =$ a torus; see [17] and [19]. Note that if G is a m.i.i.s. of itself, then every representation of G is semi-loose.

(iii) If $G \neq \{1\}$ and $G \neq \mathbb{Z}_2$, every free representation R of G is loose. Indeed, if R is free, then $G_x = \{1\}$, for all $x \in X\backslash\{0\}$, so that $X^H = \{0\}$ for every m.i.i.s. H of G if $G \neq \mathbb{Z}_2^k$. On the other hand, if $G = \mathbb{Z}_2^k$, then G contains $\mathbb{Z}_2 \times \mathbb{Z}_2$ since $G \neq \{1\}$, $G \neq \mathbb{Z}_2$ by hypothesis. But no representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ can act freely in any nontrivial Banach space (see e.g. [3]), and hence this case cannot occur. Thus, $\mathbb{Z}_2$ is the only nontrivial compact Lie group having a free but not loose representation (namely $\{I, -I\}$). □

THEOREM 4.2. *Suppose that the representation R of G in $GL(X)$ is loose. Let $A \in C^0([a,b]; \Phi_{0G}(X))$ be such that $A(a), A(b) \in GL(X)$ (hence $A(a), A(b) \in GL_G(X)$). Then*

$$\sigma(A, [a,b]) = 1. \tag{4.2}$$

PROOF. It is an easy consequence of the following two lemmas.

LEMMA 4.1. *Suppose* $\dim X < \infty$ *and the representation R of G in $GL(X)$ is loose. Let $M \in GL_G(X)$. Then* $\det M > 0$.

PROOF. As R is loose, we have $X^H = \{0\}$ for some m.i.i.s. H of G. By definition of an i.i.s., sgn $\det M =$ sgn $\det M_{|X^H}$ since $X \sim \mathbb{R}^n$, and the right-hand side is 1 since $X^H = \{0\}$. □

LEMMA 4.2. *Suppose that the representation R of G in $GL(X)$ is loose. Let $K \in \mathcal{K}_G(X)$ be such that $I - K \in GL(X)$ (hence $I - K \in GL_G(X)$). Then* $\deg_{L.S.}(I-K) = 1$.

PROOF. By definition, $\deg_{L.S.}(I-K)=(-1)^m$, where m is the sum of the algebraic multiplicities of the eigenvalues of K in $(1,\infty)$. For each such eigenvalue λ, the generalized null-space $\ker(\lambda I-K)^{m_\lambda}$, where m_λ is the multiplicity of λ, is G-invariant since $(\lambda I-K)^{m_\lambda}$ is G-covariant. Thus, the finite-dimensional space $N=\bigoplus_{\lambda\in(1,\infty)}\ker(\lambda I-K)^{m_\lambda}$ is G-invariant. Of course, the subrepresentation of R in $GL(N)$ is loose, and $(I-K)_{|N}\in GL_G(N)$. From Lemma 4.1, $\det(I-K)_{|N}>0$. On the other hand, $(-1)^m=\operatorname{sgn}\det(I-K)_{|N}$ as is well-known. $\square$

From Proposition 4.1, there is a G-covariant parametrix η of A, and hence $\eta(a)A(a)$ and $\eta(b)A(b)$ are G-covariant compact perturbations of the identity. Thus, $\deg_{L.S.}\eta(a)A(a)=\deg_{L.S.}\eta(b)A(b)=1$ by Lemma 4.2, whence $\sigma(A,[a,b])=1$ as claimed in Theorem 4.2.

For a fairly large class of finite groups which is also of special importance in concrete applications, there is a very simple criterion to decide whether a given representation is semi-loose. Recall that a 2-*nilpotent* (finite) group is one in which the elements of odd order form a subgroup (see e.g. [13]). Thus, every group of odd order is 2-nilpotent, and the same is true of supersolvable (e.g. nilpotent, abelian or Dedekind) groups of any finite order.

THEOREM 4.3. *Suppose that G is finite and 2-nilpotent. Then the representation R is semi-loose if and only if $X^H=X^G$ for every subgroup $H<G$ with $[G:H]=2$.*

PROOF. If G is 2-nilpotent, the subgroup G^2 generated by the elements of the form g^2, $g\in G$, is the unique m.i.i.s. of G (see [19]). Hence, R is not semi-loose if and only if $X^G\subsetneq X^{G^2}$. Suppose so and let $H<G$ be maximal with the property that $G^2\le H$ and $X^G\subsetneq X^H$. As is well-known, G^2 is a normal subgroup of G, and $G/G^2\sim\mathbb{Z}_2^k$, $k\ge 0$. Since $\mathbb{Z}_2^k$ is abelian, H/G^2 is normal in G/G^2 and hence H is normal in G. It follows that X^H is invariant under G and that the action of G in X^H factors through an action of G/H. Now, this action is semi-free by maximality of H. As $\widetilde{X}^G\cap X^H$ is a closed G/H-invariant complement of X^G in X^H, the action of G/H in $\widetilde{X}^G\cap X^H$ is free. But since $G/H\sim(G/G^2)/(H/G^2)$, we have $G/H\sim\mathbb{Z}_2^\ell$, $\ell\ge 0$. Necessarily, $\ell\ge 1$ since $H<G$, and $\ell\le 1$ since G/H acts freely in $\widetilde{X}^G\cap X^H\ne\{0\}$ (recall that $\mathbb{Z}_2^\ell$, $\ell\ge 2$, acts freely in no nontrivial Banach space). Thus, $\ell=1$, i.e. $|G/H|=2$, and hence $[G:H]=2$ and $X^G\subsetneq X^H$.

Conversely, let $H<G$ be such that $X^G\subsetneq X^H$ and $[G:H]=2$. Let G_{odd} be the (normal) subgroup of G of elements of odd order, so that $H_{\mathrm{odd}}=H\cap G_{\mathrm{odd}}$ is a (normal) subgroup of H consisting of all the elements of odd order in H. As $|G_{\mathrm{odd}}|$ and $|H_{\mathrm{odd}}|$ are odd, and H_{odd} is a subgroup of G_{odd}, we have $|G_{\mathrm{odd}}|=$

$k|H_{\text{odd}}|$ with k odd. Since $|G| = |G/G_{\text{odd}}||G_{\text{odd}}| = 2|H| = 2|H_{\text{odd}}||H/H_{\text{odd}}|$, we find that k divides $2|H/H_{\text{odd}}|$. This implies $k = 1$ because H/H_{odd} is a 2-group. As a result, $H_{\text{odd}} = G_{\text{odd}}$, i.e. $G_{\text{odd}} \subset H$. But then H/G_{odd} is a subgroup of G/G_{odd} of index 2, hence a normal subgroup since G/G_{odd} is a 2-group. This implies that H is normal in G and that $G/H \sim \mathbb{Z}_2$. Hence, $G^2 \leq H$ and since $X^G \subsetneq X^H$, it follows that $X^G \subsetneq X^{G^2}$, i.e. R is not semi-loose. $\square$

5. The degree of covariant mappings: first results

Throughout this section, we shall assume that $f : X \to X$ is a G-regular covariant C^2 Fredholm mapping with index 0. We shall also consider a G-invariant open subset $\Omega \subset X$ and assume that $f_{|\overline{\Omega}}$ is proper. If $Df(p) \in GL(X)$ for some point $p \in X$ and if $y \in X \backslash f(\partial\Omega)$ is a regular value of $f_{|\Omega}$, so that $f^{-1}(y) = \{x_1, \dots, x_k\}$, $k \geq 0$ an integer, the *base point degree* $d_p(f, \Omega, y)$ introduced in [10] is defined by

$$d_p(f, \Omega, y) = \sum_{i=1}^{k} \sigma_i,$$

where for $1 \leq i \leq k$, $\sigma_i = \sigma(Df \circ \gamma_i, [a_i, b_i])$ and $\gamma_i \in C^0([a_i, b_i]; X)$ is any curve joining p to x_i. Of course, this definition is independent of the choice of γ_i. If $y \in X \backslash f(\partial\Omega)$ is a singular value of $f_{|\Omega}$, $d_p(f, \Omega, y)$ is defined by regular value approximation. On the other hand, if $q \in X$ is another base point for f (i.e. $Df(q) \in GL(X)$), we have $d_q(f, \Omega, y) = \varepsilon d_p(f, \Omega, y)$, where $\varepsilon = \sigma(Df \circ \gamma, [a, b])$ and $\gamma \in C^0([a, b]; X)$ is any curve joining p to q. Thus, $|d|(f, \Omega, y) \equiv |d_p(f, \Omega, y)|$ is independent of the base point p, and $|d|(f, \Omega, y)$ continues to make sense (and equals 0) when no base point exists, i.e. when $Df(x) \notin GL(X)$ for all $x \in X$.

An *Ω-admissible homotopy* is a mapping $h \in C^2([0,1] \times X; X)$ such that $h(t, \cdot)$ is Fredholm with index 0 for $t \in [0,1]$ and $h_{|[0,1]\times\overline{\Omega}}$ is proper. In this case, given $y \notin h([0,1] \times \partial\Omega)$ and base points p, q for $h(0, \cdot)$ and $h(1, \cdot)$, respectively, we have $d_p(h(0, \cdot), \Omega, y) = \varepsilon d_q(h(1, \cdot), \Omega, y)$, where $\varepsilon = \sigma(D_x h \circ \Gamma, [a, b])$ and $\Gamma \in C^0([a, b]; [0,1] \times X)$ is any curve joining $(0, p)$ to $(1, q)$. In particular, $|d|(h(0, \cdot), \Omega, y) = |d|(h(1, \cdot), \Omega, y)$ and equality continues to hold when no base point exists for $h(0, \cdot)$ or for $h(1, \cdot)$ (and hence both sides are 0). This property expresses *homotopy invariance of the absolute degree* $|d|$.

REMARK 5.1. If $p \in X$ is a base point of $h(t, \cdot)$ for $0 \leq t \leq 1$, then $d_p(h(0, \cdot), \Omega, y) = d_p(h(1, \cdot), \Omega, y)$. Indeed, we may choose $\Gamma(t) = (t, p)$, $t \in [0, 1]$, so that $D_x h \circ \Gamma = D_x h(\cdot, p)$ is a curve of isomorphisms and hence has parity 1 (see [10]). $\square$

The Ω-admissible homotopy h will be called *G-covariant* (resp. *G-regular covariant*) if $h(t,\cdot)$ is G-covariant (resp. G-regular covariant) for each $t \in [0,1]$. From Corollary 3.1, h is G-regular covariant if and only if it is G-covariant and $h(t,\cdot)$ is G-regular for some $t \in [0,1]$. When h is G-covariant, we denote by h^G the homotopy $h_{|[0,1]\times X^G} : [0,1] \times X^G \to X^G$.

Since the C^2 mapping f of interest is G-regular covariant, the mapping $f^G \equiv f_{|X^G} : X^G \to X^G$ is C^2 Fredholm with index 0 by Theorem 3.3(ii). With $\Omega^G = \Omega \cap X^G$, it is clear that properness of $f_{|\overline{\Omega}}$ implies properness of $f^G_{|\overline{\Omega}^G}$. As $f^G(\partial\Omega^G) = f(\partial\Omega^G) \subset f(\partial\Omega)$, the base point degrees $d_p(f,\Omega,0)$ and $d_p(f^G,\Omega^G,0)$ are defined provided that $0 \notin f(\partial\Omega)$ and $p \in X^G$ is a base point of f, for then p is also a base point of f^G from the block-diagonal decomposition of $Df(p)$ in Theorem 3.3(ii).

In the next two sections, we investigate the relationship between $d_p(f,\Omega,0)$ and $d_p(f^G,\Omega^G,0)$ and, more generally, between $|d|(f,\Omega,0)$ and $|d|(f^G,\Omega^G,0)$ when Ω is a G-invariant open subset of X. First, we consider the case when Ω is a small enough open neighborhood of Ω^G in Theorem 5.1.

REMARK 5.2. To speak of $d_p(f,\Omega,0)$, we must assume that $p \in X$ is a base point of f, i.e. $Df(p) \in GL(X)$. Likewise, $d_p(f^G,\Omega^G,0)$ makes sense only if $p \in X^G$ (and $Df^G(p) \in GL(X^G)$). Thus, to compare $d_p(f,\Omega,y)$ and $d_p(f^G,\Omega^G,0)$, we must assume that p is a base point of f that lies in X^G. By Theorem 3.2, this assumption alone implies that f is G-regular. This shows that the hypothesis of G-regularity certainly cannot be weakened for the comparison of the two degrees to be possible at all. □

THEOREM 5.1. *Let $f : X \to X$ be a G-regular covariant C^2 Fredholm mapping with index* 0, *and suppose that $p \in X^G$ is a base point of f. Let $\Omega \subset X$ be a G-invariant open subset of X such that $f_{|\overline{\Omega}}$ is proper and $0 \notin f(\partial\Omega)$. Suppose also that* 0 *is a regular value of $f^G_{|\Omega^G}$ $(\Omega^G = \Omega \cap X^G)$ and that $Df(x) \in GL(X)$ for all $x \in (f^G)^{-1}(0) \cap \Omega^G$.*

Then there is a G-invariant open neighborhood ω of Ω^G in Ω such that $0 \notin f(\partial\omega)$ and $f^{-1}(0) \cap \omega = (f^G)^{-1}(0) \cap \Omega^G$. Moreover:

(i) *If the representation R is semi-loose* (*see Definition* 4.1), *we have*

$$d_p(f,\omega,0) = d_p(f^G,\Omega^G,0). \tag{5.1}$$

(ii) *If the representation R is not semi-loose, then we still have*

$$d_p(f,\omega,0) = d_p(f^G,\Omega^G,0) \bmod 2\mathbb{Z}. \tag{5.2}$$

PROOF. The fact that all the degrees involved in (5.1) and (5.2) are defined without ambiguity follows from general remarks earlier in this section.

As usual, denote by $\widetilde{X}^G$ the unique G-invariant closed complement of X^G in X (Theorem 2.2). Equip X with a G-invariant equivalent norm (Remark 2.1), and for $\rho > 0$, let $B(0,\rho)$ be the open ball with center 0 and radius ρ in $\widetilde{X}^G$. The set $\Omega_\rho \equiv (\Omega^G \oplus B(0,\rho)) \cap \Omega$ is a G-invariant open neighborhood of Ω^G in Ω. We claim that for $\rho > 0$ small enough, we have $f^{-1}(0) \cap \overline{\Omega}_\rho = (f^G)^{-1}(0) \cap \Omega^G$.

Otherwise, there is a sequence $x_\ell \in f^{-1}(0) \cap \overline{\Omega}_{1/\ell}$, $x_\ell \notin \Omega^G$, for all $\ell \in \mathbb{N}$. As $f_{|\overline{\Omega}}$ is proper, we may assume that $\lim_{\ell\to\infty} x_\ell = x$ exists. Obviously, $x \in f^{-1}(0) \cap \overline{\Omega}^G = (f^G)^{-1}(0) \cap \overline{\Omega}^G$, and hence $x \in f^{-1}(0) \cap \Omega^G$ since $0 \notin f(\partial\Omega^G) \subset f(\partial\Omega)$. But then the hypothesis $Df(x) \in GL(X)$ implies that $f(y) \neq 0$ for y in some neighborhood of x in X, which in turn requires $x_\ell = x$ for ℓ large enough, in contradiction with the hypothesis $x_\ell \notin \Omega^G$.

Let $\rho > 0$ be as above and set $\omega = \Omega_\rho$, so that

$$\left\{\begin{array}{l} 0 \notin f(\partial\omega), \\ f^{-1}(0) \cap \omega = (f^G)^{-1}(0) \cap \Omega^G. \end{array}\right. \tag{5.3}$$

From (5.3), both (5.1) and (5.2) are trivial if $(f^G)^{-1}(0) \cap \Omega^G = \emptyset$, and we shall henceforth assume that $(f^G)^{-1}(0) \cap \Omega^G = \{x_1, \dots, x_m\}$, where $m \geq 1$ is an integer.

By Theorem 3.3(ii) with $H = G$, we find

$$Df(y) = \begin{pmatrix} Df^G(y) & 0 \\ 0 & B(y) \end{pmatrix} \qquad \text{for } y \in X^G, \tag{5.4}$$

relative to the splitting $X = X^G \oplus \widetilde{X}^G$. Since $Df(p), Df(x_i) \in GL(X)$ by hypothesis, it follows from (5.4) with $y = p$ and next $y = x_i$ that $Df^G(p), Df^G(x_i) \in GL(X^G)$ and $B(x_p), B(x_i) \in GL_G(\widetilde{X}^G)$, $1 \leq i \leq m$.

Let $\gamma_i \in C^0([a_i, b_i]; X^G)$ be a curve joining p to x_i, $1 \leq i \leq k$. By definition of the degree at regular values,

$$d_p(f^G, \Omega^G, 0) = \sum_{i=1}^m \sigma_i^G, \tag{5.5}$$

where $\sigma_i^G \equiv \sigma(Df^G \circ \gamma_i, [a_i, b_i])$. On the other hand, since γ_i is also a curve joining p to x_i in X, and by (5.3), we have once again by definition of the degree at regular values

$$d_p(f, \Omega, 0) = \sum_{i=1}^m \sigma_i, \qquad \text{where } \sigma_i \equiv \sigma(Df \circ \gamma_i, [a_i, b_i]). \tag{5.6}$$

As $\gamma_i(\lambda) \in X^G$ for all $\lambda \in [a_i, b_i]$, (5.4) holds with $y = \gamma_i(\lambda)$, and the multiplicative property of the parity with respect to block-triangular decompositions (see [9] or [10]) yields

$$\sigma_i = \sigma(Df^G \circ \gamma_i, [a_i, b_i])\sigma(B \circ \gamma_i, [a_i, b_i]) = \sigma_i^G \sigma(B \circ \gamma_i, [a_i, b_i]). \tag{5.7}$$

Note that $B \circ \gamma_i(\lambda) \in \Phi_{0G}(\widetilde{X}^G)$ by Theorem 3.3(ii) with $H = G$.

Suppose now that R is semi-loose. From Remark 4.1(i), the subrepresentation of R in $GL(\widetilde{X}^G)$ is loose, and since $B \circ \gamma_i \in C^0([a_i, b_i]; \Phi_{0G}(\widetilde{X}^G))$, Theorem 4.2 implies that $\sigma(B \circ \gamma_i, [a_i, b_i]) = 1$. Thus, $\sigma_i^G = \sigma_i$, $1 \le i \le m$ (see (5.7)), and (5.2) follows from (5.5) and (5.6). In any case, it is clear that both (5.5) and (5.6) provide a mod 2 count of the points $x_1, \ldots, x_m$, and hence (5.2) holds even when (5.1) does not. □

Part (i) of Theorem 5.1 is new even in the case when $\dim X < \infty$ (so that the degree is equivalent to Brouwer's). Perhaps more surprisingly, it also seems to be new when $\dim X < \infty$ and G is a finite group of odd order. All the related results we have found in the literature give instead of (5.1) a weaker equality modulo some ideal of $\mathbb{Z}$ and under more stringent assumptions about the representation.

In order to make Theorem 5.1 available as a technical tool in further comparisons of the degrees, we now show that even if f does not meet all the requirements of Theorem 5.1, at least some homotopic mapping does. This will enable us (in Sections 6 and 7) to reduce the problem to the case when Theorem 5.1 can be used.

Proposition 5.1. *Let $f : X \to X$ be a G-regular covariant C^2 Fredholm mapping with index 0, and let $\Omega \subset X$ be a bounded G-invariant open subset of X such that $f_{|\overline{\Omega}}$ is proper and $0 \notin f(\partial\Omega)$. Then there is a G-regular covariant Ω-admissible homotopy $h : [0,1] \times X \to X$ such that $h(0, \cdot) = f$ and*

(i) *$0 \notin h([0,1] \times \partial\Omega)$.*
(ii) *h^G is Ω^G-admissible and $0 \notin h^G([0,1] \times \partial\Omega^G)$.*
(iii) *$f_1 \equiv h(1, 0)$ has a base point $p \in X^G$. Furthermore, if $p \in X^G$ is a base point of f in the first place, h may be chosen so that p is a base point of $h(t, \cdot)$ for all $0 \le t \le 1$.*
(iv) *0 is a regular value of $f^G_{1|\Omega^G}$, and $Df_1(x) \in GL_G(X)$ for every $x \in (f_1^G)^{-1}(0) \cap \Omega^G$.*

Proof. Let $p \in X^G$ be a base point of f^G if such a point exists (e.g. a base point of f, if there is any in X^G), an arbitrary point otherwise. If $y \in X^G$ is a regular value of $f^G_{|\Omega^G}$, we have $(f^G)^{-1}(y) \cap \Omega^G = \{x_1, \ldots, x_m\}$ for some integer

$m \geq 0$. Relative to the splitting $X = X^G \oplus \widetilde{X}^G$, $Df(p)$ and $Df(x_i)$, $1 \leq i \leq m$, have the form

$$Df(p) = \begin{pmatrix} Df^G(p) & 0 \\ 0 & B(p) \end{pmatrix}, \qquad Df(x_i) = \begin{pmatrix} Df^G(x_i) & 0 \\ 0 & B(x_i) \end{pmatrix},$$

with $B(p), B(x_i) \in \Phi_{0G}^{\mathrm{reg}}(\widetilde{X}^G)$, $Df^G(p) \in \Phi_{0G}^{\mathrm{reg}}(X^G) = \Phi_0(X^G)$ (see Theorem 3.3 with $H = G$) and $Df^G(x_i) \in GL_G(X^G) = GL(X^G)$.

Given $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, it follows from Proposition 3.2 that there are $K \in \mathcal{K}_G(\widetilde{X}^G)$ and $L \in \mathcal{K}(X^G)$ $(= \mathcal{K}_G(X^G))$ such that $\|K\| \leq \varepsilon_1$, $\|L\| \leq \varepsilon_2$ and $B(p) + K, B(x_i) + K \in GL_G(\widetilde{X}^G)$, $1 \leq i \leq m$, and $Df^G(p) + L \in GL(X^G)$ $(= GL_G(X^G))$. Fix ε_1 and K, and for $x \in X$, set $g(x) = f(x) + K\widetilde{Q}x - y$, where $\widetilde{Q}$ denotes the projection onto $\widetilde{X}_G$ relative to $X = X^G \oplus \widetilde{X}^G$. Clearly, $g^G = f^G - y$, so that 0 is a regular value of $g^G_{|\Omega^G}$. Also, since $\widetilde{Q}$ is G-covariant (Theorem 2.1), we have

$$Dg(x_i) = \begin{pmatrix} Df^G(x_i) & 0 \\ 0 & B(x_i) + K \end{pmatrix}, \qquad 1 \leq i \leq m,$$

so that $Dg(x) \in GL_G(X)$ for all $x \in (g^G)^{-1}(0) = \{x_1, \dots, x_m\}$. Note that G-covariance of $\widetilde{Q}$ implies G-covariance of g since $y \in X^G$. Now, set $f_1(x) = f(x) + K\widetilde{Q}x + LQx - y = g(x) + LQx$, where $Q = I - \widetilde{Q}$ is the projection onto X^G. As Q and g are G-covariant, so is f_1. Next, $f_1^G = g^G + L$. Thus, by taking ε_2 (i.e. $\|L\|$) small enough in the first place, 0 remains a regular value of f_1^G, and $(f_1^G)^{-1}(0) \cap \Omega^G$ consists of exactly m points x_{1i} with $\|x_i - x_{1i}\|$ arbitrarily small, hence small enough for continuity of B to imply $B(x_{1i}) + K \in GL_G(\widetilde{X}^G)$, $1 \leq i \leq m$. This yields

$$Df_1(x_{1i}) = \begin{pmatrix} Df_1^G(x_{1i}) & 0 \\ 0 & B(x_{1i}) + K \end{pmatrix} \in GL_G(X), \qquad 1 \leq i \leq m,$$

which shows that $Df_1(x) \in GL_G(X)$ for all $x \in (f_1^G)^{-1}(0) \cap \Omega^G$. In addition, from the choice of K and L we have

$$Df_1(p) = \begin{pmatrix} Df^G(p) + L & 0 \\ 0 & B(p) + K \end{pmatrix} \in GL_G(X),$$

hence p is a base point of f_1.

It remains to show that f_1 can be obtained as $h(1, \cdot)$ where h is a homotopy satisfying all the desired properties.

For $(t, x) \in [0,1] \times X$, set $h(t,x) = f(x) + t(K\widetilde{Q}x + LQx - y)$, so that $h(0, \cdot) = f$, $h(1, \cdot) = f_1$, and $0 \notin h([0,1] \times \partial\Omega)$ if $\varepsilon_1, \varepsilon_2$ (i.e. $\|K\|$ and $\|L\|$) have been chosen small enough and if y has been taken close enough to 0 (which is possible due to denseness of regular values). That h is C^2 is obvious, and G-covariance of h follows from G-covariance of f, K, L, Q and $\widetilde{Q}$, and from $y \in X^G$. For $x \in X$, we have $D_x h(t,x) = Df(x) + t(K\widetilde{Q} + LQ) \in \Phi_0(X)$ since $Df(x) \in \Phi_0(X)$ and $t(K\widetilde{Q} + LQ) \in \mathcal{K}_G(X) \subset \mathcal{K}(X)$. In addition, if $x \in X^G$, then $Df(x) \in \Phi_{0G}^{\mathrm{reg}}(X)$ since f is G-regular by hypothesis, and hence $D_x h(t,x) \in \Phi_{0G}^{\mathrm{reg}}(X)$ (see Example 3.2). As a result, $h(t,\cdot)$ is Fredholm with index 0 and G-regular covariant. Furthermore, if p is a base point of f and hence $Df(p) \in GL(X)$, it is clear that $D_x h(t,p) \in GL(X)$ for all $t \in [0,1]$ if ε_1 and ε_2 are small enough, i.e. p is a base point of $h(t,\cdot)$ for $t \in [0,1]$. Properness of $h_{|[0,1]\times\overline{\Omega}}$ follows at once from properness of $f_{|\overline{\Omega}}$, compactness of $K\widetilde{Q} + LQ$ and boundedness of Ω.

Finally, $h^G(t,\cdot) = f^G + t(L - y)$ is C^2 and an affine compact perturbation of f^G, hence Fredholm with index 0 (f^G is Fredholm with index 0 since f is G-regular). That $0 \notin h^G([0,1] \times \partial\Omega^G)$ and $h^G_{|[0,1]\times\overline{\Omega}^G}$ is proper is a straightforward consequence of the analogous properties for h and of $\partial\Omega^G \subset \partial\Omega$. □

6. The case of finite groups

As in the previous section, $f : X \to X$ denotes a G-regular covariant C^2 Fredholm mapping with index 0, Ω is a G-invariant open subset of X and $f_{|\overline{\Omega}}$ is proper. In addition, the group G is supposed to be *finite* in all subsequent considerations.

Let $\mathcal{I}^G$ denote the ideal of $\mathbb{Z}$ generated by the integers $[G : G_x]$ where G_x is the isotropy subgroup of $x \in X \backslash X^G$. Equivalently, $\mathcal{I}^G$ is the ideal of $\mathbb{Z}$ generated by the g.c.d. Δ of the number of points in all *nontrivial* orbits. As usual, $\sqrt{\mathcal{I}^G}$ denotes the radical of $\mathcal{I}^G$: if $\Delta > 1$ and $\Delta = \wp_1^{\beta_1} \ldots \wp_k^{\beta_k}$ is the decomposition of Δ as a product of distinct primes $\wp_1, \ldots, \wp_k$, then $\sqrt{\mathcal{I}^G}$ is the ideal of $\mathbb{Z}$ generated by the product $\wp_1 \ldots \wp_k$.

For the proof of our main result (Theorem 6.1), we need a sequence of preliminary lemmas.

LEMMA 6.1. *Suppose that Ω is bounded, $\Omega \cap X^G = \emptyset$, $0 \notin f(\partial\Omega)$ and 0 is a regular value of $f_{|\Omega}$. Then*

$$|d|(f,\Omega,0) = 0 \bmod \mathcal{I}^G. \tag{6.1}$$

PROOF. It is not restrictive to assume that f has a base point $p \in X^G$. Indeed, by a simple contradiction argument, it is easily seen that if $\varepsilon > 0$ is small enough and $K \in \mathcal{K}(X)$ satisfies $\|K\| \le \varepsilon$, then 0 is a regular value of

$(f+K)_{|\Omega}$ and $h(t,x) = f(x) + tKx$ is an Ω-admissible homotopy such that $0 \notin h([0,1] \times \partial\Omega)$ (argue as in the proof of Proposition 5.1). Furthermore, if in addition $K \in \mathcal{K}_G(X)$, then h is G-regular covariant. Thus, choosing $p \in X^G$ arbitrarily and next $K \in \mathcal{K}_G(X)$, $\|K\| \leq \varepsilon$, such that $Df(p) + K \in GL(X)$ (existence of K follows from Proposition 3.2 with $A_i = Df(p)$), we may replace f by $f+K$ in the lemma by homotopy invariance of the absolute degree. This proves the claim.

Let $x \in f^{-1}(0) \cap \Omega$. By G-covariance of f and G-invariance of Ω, the orbit $\mathcal{O}_x = \{R_g x : g \in G\}$ lies in $f^{-1}(0) \cap \Omega$. Our hypotheses ensure that $f^{-1}(0) \cap \Omega$ consists of finitely many points and hence $d_p(f, \Omega, 0)$ is the sum of the contributions of all the disjoint orbits, the orbit $\mathcal{O}_x$ contributing the sum $\sum_{\overline{g} \in G/G_x} \sigma_{\overline{g}}$ where $\sigma_{\overline{g}}$ is the parity of $Df \circ \gamma_{\overline{g}}$ and $\gamma_{\overline{g}}$ is any curve in X joining p to $R_g x$, $g \in \overline{g}$ (note that $R_g x$ is independent of $g \in \overline{g}$).

If $\gamma \in C^0([a,b]; X)$ is any curve in X joining p to x and if $g \in G$, then $R_g\gamma$ is a curve in X joining $R_g p = p$ to $R_g x$. Thus, $\sigma_{\overline{g}}$ is just the parity of $Df \circ R_g \gamma$ for $g \in \overline{g}$. Now, from $f(R_g y) = R_g f(y)$ for all $y \in X$, it follows that $Df(R_g y) = R_g Df(y) R_g^{-1}$. In particular, $Df \circ R_g \gamma = R_g (Df \circ \gamma) R_g^{-1}$. This implies that the parity of $Df \circ R_g \gamma$ equals the parity of $Df \circ \gamma$, e.g. because the constant curves of the isomorphisms R_g and R_g^{-1} have parity 1 (see [10] if details are needed).

The above shows that $\sigma_{\overline{g}} = \sigma(Df \circ \gamma, [a,b])$ for every $\overline{g} \in G/G_x$, and hence the contribution of the orbit $\mathcal{O}_x$ to $d_p(f, \Omega, 0)$ is $[G : G_x]\sigma(Df \circ \gamma, [a,b])$, an integer divisible by the generator Δ of $\mathcal{I}^G$ since $G_x \neq G$ due to the hypothesis $\Omega \cap X^G = \emptyset$. This holds for every orbit, hence $d_p(f, \Omega, 0)$ is divisible by Δ, and the same thing is true of $|d|(f, \Omega, 0) = |d_p(f, \Omega, 0)|$. $\square$

We now work towards removing the hypothesis that 0 is a regular value of $f_{|\Omega}$ in Lemma 6.1. Both the G-covariance requirement and the lack of (C^2) partitions of unity in general contribute to making this task rather delicate. Lemma 6.2 next is only a technical step needed in the proof of Lemma 6.3.

LEMMA 6.2. *Let $N \geq 0$, $M \geq 0$ be integers and let S and T be representations of G in $GL(\mathbb{R}^N)$ and $GL(\mathbb{R}^M)$, respectively. Let $z \in \mathbb{R}^N$ be such that $S_g z \neq z$ for all $g \in G$, $g \neq 1$, and let $y \in \mathbb{R}^M$ be arbitrary. Then there is an (S,T)-covariant polynomial mapping $\xi : \mathbb{R}^N \to \mathbb{R}^M$ (i.e. $\xi(S_g w) = T_g \xi(w)$ for all $w \in \mathbb{R}^N$ and $g \in G$) with $d^0\xi \leq |G| - 1$ such that $\xi(z) = y$.*

PROOF. To find ξ, we construct a polynomial mapping $\pi : \mathbb{R}^N \to \mathbb{R}^M$ with $d^0\pi \leq |G| - 1$ such that $\pi(S_g z) = T_g y$ for all $g \in G$. Once π is available, it suffices to take $\xi(w) = \frac{1}{|G|}\sum_{g \in G} T_g^{-1}\pi(S_g w)$ for all $w \in \mathbb{R}^N$. Indeed, ξ is obviously an (S,T)-covariant polynomial mapping with $d^0\xi \leq d^0\pi \leq |G|-1$, and $\xi(z) = \frac{1}{|G|}\sum_{g\in G} T_g^{-1}T_g y = y$. The polynomial mapping π can be found as follows: for every $g \in G$, let π_g be a real-valued polynomial with $d^0\pi_g \leq |G|-1$ such that $\pi_g(S_g z) = 1$, $\pi_g(S_{g'}z) = 0$ for all $g' \in G$, $g' \neq g$. Because $S_g z \neq S_{g'}z$ for $g \neq g'$, such a polynomial π_g can be obtained via the Hahn-Banach theorem as the product of affine continuous mappings with value 1 at $S_g z$ and 0 at a different one of the $|G|-1$ points $S_{g'}z$, $g' \neq g$. The mapping $\pi(w) = \sum_{g\in G}\pi_g(w)T_g y$ for $w \in \mathbb{R}^N$ possesses the desired properties. □

Lemma 6.3. *Suppose that Ω is bounded, that $R_g x \neq x$ for all $x \in \Omega$, $g \in G$, $g \neq 1$, and that $0 \notin f(\partial\Omega)$. Then there is a G-regular covariant Ω-admissible homotopy $h : [0,1] \times X \to X$ such that $0 \notin h([0,1]\times\partial\Omega)$, $h(0,\cdot) = f$ and 0 is a regular value of $h(1,\cdot)$.*

Proof. We shall not be able to construct h explicitly and will be content with a proof of its existence, which will follow from Sard's theorem and some technicalities described below.

Let $C = f^{-1}(0) \cap \Omega$, a compact subset of Ω since $f_{|\overline{\Omega}}$ is proper and $0 \notin f(\partial\Omega)$. As f is Fredholm, it follows from compactness of C that there is a finite-dimensional subspace $Y \subset X$ such that $\operatorname{rge} Df(x) + Y = X$ for all $x \in C$. If so,

$$\widetilde{Y} = \sum_{g\in G} R_g Y \tag{6.2}$$

is a G-invariant finite-dimensional subspace of X containing Y and hence

$$\operatorname{rge} Df(x) + \widetilde{Y} = X \qquad \text{for } x \in C. \tag{6.3}$$

Fix $x_0 \in C$: As $R_g x_0 \neq x_0$ for all $g \in G - \{1\}$ by hypothesis, the orbit of x_0 consists of $|G|$ distinct points. Hence, there is a linear continuous functional $\lambda_{x_0} \in X^*$ such that $\langle\lambda_{x_0}, R_g x_0\rangle \neq \langle\lambda_{x_0}, R_{g'}x_0\rangle$ for all $g, g' \in G$, $g \neq g'$. By continuity of λ_{x_0}, there is an open neighborhood $\mathcal{N}_{x_0}$ of x_0 in Ω such that $\langle\lambda_{x_0}, R_g x\rangle \neq \langle\lambda_{x_0}, R_{g'}x\rangle$, for all $g, g' \in G$, $g \neq g'$, and all $x \in \mathcal{N}_{x_0}$. Covering C with finitely many open neighborhoods $\mathcal{N}_{x_0}$, we obtain finitely many continuous linear functionals $\lambda_i \in X^*$, $1 \leq i \leq n$, such that

$$\forall x \in C, \exists 1 \leq i \leq n, \quad \langle\lambda_i, R_g x\rangle \neq \langle\lambda_i, R_{g'}x\rangle, \qquad \forall g, g' \in G,\ g \neq g'. \tag{6.4}$$

At this point, it will be convenient to view G as a subgroup of the group of permutations of $\{1, \dots, |G|\}$ (the reader familiar with the concept of regular representation will notice that the representation S below is just the tensor product $\tau \otimes \rho$ where τ is the trivial representation of G in $\mathbb{R}^n$ and ρ the regular representation of G in $\mathbb{R}^{|G|}$). The usual way to do so is to number the elements of G, say $G = \{g_j\}_{1 \le j \le |G|}$ beginning with $g_1 = 1$, and to identify g_k with the permutation τ_k characterized by the condition

$$g_{\tau_k(j)} \equiv g_k g_j, \qquad \text{for } 1 \le j \le |G|. \tag{6.5}$$

Consistent with the above numbering of the elements of G, we shall set $R_j \equiv R_{g_j}$, $1 \le j \le |G|$. By (6.5), we have $R_{\tau_k(j)} = R_k R_j$ for $1 \le j, k \le |G|$. Observe in passing that $R_1 = I$ since $g_1 = 1$, and note also that (6.4) reads

$$\forall x \in C, \exists 1 \le i \le n, \quad \langle R_j^* \lambda_i, x \rangle \ne \langle R_k^* \lambda_i, x \rangle, \qquad \forall 1 \le j, k \le |G|,\ j \ne k, \tag{6.6}$$

where of course $R_j^* \in GL(X^*)$ is the adjoint of R_j.

For each index $1 \le \ell \le |G|$, let $S_\ell \in GL(\mathbb{R}^n \otimes \mathbb{R}^{|G|})$ be defined by $w = (w_{ij}) \mapsto S_\ell(w) = (w_{i\tau_\ell(j)})$. It is straightforward to check that the S_ℓ's form a representation of G in $GL(\mathbb{R}^n \otimes \mathbb{R}^{|G|})$ and that the mapping $\varphi = (\varphi_{ij}) : X \to \mathbb{R}^n \otimes \mathbb{R}^{|G|}$ defined by

$$\varphi(x) = \frac{1}{|G|} \sum_{\ell=1}^{|G|} S_\ell^{-1}((\langle R_j^* \lambda_i, R_\ell x \rangle)) \tag{6.7}$$

is (R, S)-covariant. Since S_ℓ^{-1} represents g_ℓ^{-1} and g_ℓ^{-1} is identified with τ_ℓ^{-1} (i.e. $g_{\tau_\ell^{-1}(j)} = g_\ell^{-1} g_j$ for all $1 \le j \le |G|$), we have $S_\ell^{-1}((\langle R_j^* \lambda_i, R_\ell x \rangle)) = ((\langle R^*_{\tau_\ell^{-1}(j)} \lambda_i, R_\ell x \rangle)) = ((\langle R_\ell^* R_j^* R_\ell^{-*} \lambda_i, x \rangle))$. Equivalently,

$$\varphi_{ij}(x) = \left\langle \frac{1}{|G|} \sum_{\ell=1}^{|G|} R_\ell^* R_j^* R_\ell^{-*} \lambda_i, x \right\rangle$$

for $1 \le i \le n$, $1 \le j \le |G|$, $x \in X$, and in particular, $\varphi_{i1}(x) = \langle \lambda_i, x \rangle$, $1 \le i \le n$, $x \in X$. As a result,

$$\forall x \in C, \quad S_k \varphi(x) = \varphi(R_k x) \ne \varphi(x), \qquad \text{for } 2 \le k \le |G| \tag{6.8}$$

for otherwise $\varphi_{i1}(R_k x) = \varphi_{i1}(x)$ for some index $2 \le k \le |G|$ and every $1 \le i \le n$, whence $\langle R_k^* \lambda_i, x \rangle = \langle \lambda_i, x \rangle$, for all $1 \le i \le n$, in contradiction with (6.6).

Denote by $\mathcal{P}$ the space of (S, R)-covariant polynomial mappings with degree $\le |G| - 1$ from $\mathbb{R}^n \otimes \mathbb{R}^{|G|}$ to $\widetilde{Y}$. This definition makes sense since $\widetilde{Y}$ is G-invariant. Moreover, as $\dim \widetilde{Y} < \infty$, we have $\dim \mathcal{P} < \infty$. For $(x, \zeta) \in X \times \mathcal{P}$, set

$$\widetilde{f}(x, \zeta) = f(x) + \zeta(\varphi(x)) \in X, \tag{6.9}$$

where φ is defined by (6.7).

For fixed $\zeta \in \mathcal{P}$, $\widetilde{f}(\cdot,\zeta)$ is a C^∞ finite-dimensional perturbation of f and hence $\widetilde{f}(\cdot,\zeta) \in C^2(X;X)$ is Fredholm with index 0. Moreover, $\widetilde{f}(\cdot,\zeta)$ is G-covariant since φ is (R,S)-covariant and ζ is (S,R)-covariant (and f is G-covariant). Also, $\widetilde{f} \in C^2(X \times \mathcal{P};X)$ and for $x \in X$ we have $D\widetilde{f}(x,0)(v,\xi) = Df(x)v + \xi(\varphi(x))$ for all $v \in X$ and $\xi \in \mathcal{P}$, where $D\widetilde{f}$ denotes the total derivative of $\widetilde{f}$. As $Df(x)$ has index 0, it follows that $D\widetilde{f}(x,0)$ is Fredholm with index $\dim \mathcal{P}$.

For $x \in C$, $D\widetilde{f}(x,0)$ is onto X. Indeed, from (6.3), every element of X can be written in the form $Df(x)v + y$ with $v \in X$ and $y \in \widetilde{Y}$, and it suffices to show that there is $\xi \in \mathcal{P}$ such that $\xi(\varphi(x)) = y$. But this follows from (6.8) and Lemma 6.2 with $T = R$ and $z = \varphi(x)$ (and after identifying $\mathbb{R}^n \otimes \mathbb{R}^{|G|}$ with $\mathbb{R}^N$, $N = n|G|$, and $\widetilde{Y}$ with $\mathbb{R}^M$).

At this stage, the implicit function theorem yields that in the vicinity of each point $(x,0) \in C \times \mathcal{P}$, the zero set of $\widetilde{f}$ coincides with a C^2 manifold with dimension $\dim \mathcal{P}$ (= index of $D\widetilde{f}(x,0)$). By compactness of C, there is an open neighborhood of $C \times \{0\}$ in $X \times \mathcal{P}$ of the form $\mathcal{N}_C \times B_\rho$ with $\mathcal{N}_C$ an open neighborhood of C in X and B_ρ the open ball with center 0 and radius $\rho > 0$ in $\mathcal{P}$, such that $(x,\zeta) \in \mathcal{N}_C \times B_\rho$ and $\widetilde{f}(x,\zeta) = 0$ if and only if (x,ζ) lies in some $\dim \mathcal{P}$-dimensional C^2-submanifold $\mathcal{M}$ of $\mathcal{N}_C \times B_\rho$. Furthermore, after shrinking $\rho > 0$ if necessary, every solution $x \in \overline{\Omega}$ of $\widetilde{f}(x,\zeta) = 0$ lies in $\mathcal{N}_C$ if $\zeta \in B_\rho$: otherwise, there are sequences ζ_m tending to 0 in $\mathcal{P}$ and $x_m \in \overline{\Omega}\backslash\mathcal{N}_C$ such that $\widetilde{f}(x_m,\zeta_m) = 0$. As Ω is bounded by hypothesis, the sequence (x_m) is bounded in X, hence $(\varphi(x_m))$ is bounded in $\mathbb{R}^n \otimes \mathbb{R}^{|G|}$ (see (6.7)). Thus we may assume that $\lim_{m\to\infty} \varphi(x_m) = u \in \mathbb{R}^n \otimes \mathbb{R}^{|G|}$ with no loss of generality. But then $\zeta_m(\varphi(x_m))$ tends to 0 in $\widetilde{Y}$ since ζ_m tends to 0 in $\mathcal{P}$. As a result,

$$\lim_{m\to\infty} f(x_m) = -\lim_{m\to\infty} \zeta_m(\varphi(x_m)) = 0, \tag{6.10}$$

which, by properness of $f_{|\overline{\Omega}}$, implies $\lim_{m\to\infty} x_m = x^* \in \overline{\Omega}$ after replacing (x_m) by a subsequence if necessary. From (6.10), $f(x^*) = 0$, i.e. $x^* \in C$ since $0 \notin f(\partial\Omega)$. If so, $x_m \in \mathcal{N}_C$ for m large enough, a contradiction.

From the above and the definition of the submanifold $\mathcal{M}$ of $\mathcal{N}_C \times B_\rho$, we have

$$(x,\zeta) \in \widetilde{f}^{-1}(0) \cap (\overline{\Omega} \times B_\rho) \Leftrightarrow (x,\zeta) \in \mathcal{M}\,^{2} \tag{6.11}$$

if $\rho > 0$ is small enough. Applying Sard's theorem to the projection $\pi : X \times B_\rho \supset \mathcal{M} \to B_\rho \subset \mathcal{P}$, we find a regular value $\zeta_0 \in B_\rho$ of π (note $\dim \mathcal{M} = \dim B_\rho =$

[2] In particular, $x \in \Omega$ since $\mathcal{M} \subset \mathcal{N}_C \times B_\rho \subset \Omega \times B_\rho$.

$\dim \mathcal{P}$). This means that either $\mathcal{M} \cap \pi^{-1}(\zeta_0) = \emptyset$ (i.e. $\widetilde{f}(x, \zeta_0) \neq 0$ for all $x \in \Omega$; see (6.11)) or $\pi \in GL(T_{(x,\zeta_0)}\mathcal{M}, \mathcal{P})$ for every point $(x, \zeta_0) \in \mathcal{M} \cap \pi^{-1}(\zeta_0)$ (i.e. every point $x \in \Omega$ such that $\widetilde{f}(x, \zeta_0) = 0$; see again (6.11)). As $T_{(x,\zeta_0)}\mathcal{M}$ is identified with $\ker D\widetilde{f}(x, \zeta_0)$, this amounts to $(X \times \{0\}) \cap \ker D\widetilde{f}(x, \zeta_0) = \{0\}$ or, equivalently, to $\ker D_x\widetilde{f}(x, \zeta_0) = \{0\}$. Since $\widetilde{f}(\cdot, \zeta_0)$ is Fredholm with index 0, it follows that $D_x\widetilde{f}(x, \zeta_0) \in GL(X)$ for all $x \in \widetilde{f}(\cdot, \zeta_0)^{-1}(0) \cap \Omega$, i.e. 0 is a regular value of $\widetilde{f}(\cdot, \zeta_0)_{|\Omega}$.

A possible choice for the desired homotopy h is now given by

$$h(t, x) = f(x) + t\zeta_0(\varphi(x)), \qquad 0 \leq t \leq 1,\ x \in X.$$

Obviously, $h \in C^2([0,1] \times X; X)$ and since $t\zeta_0 \in B_\rho$, $0 \leq t \leq 1$, we have $0 \notin h([0,1] \times \partial\Omega)$ by (6.11). Also, $h(t, \cdot)$ is Fredholm with index 0, $0 \leq t \leq 1$, and $h_{|[0,1]\times\overline{\Omega}}$ is proper (for the latter point, use boundedness of Ω in the same way as in the proof of (6.11) given above). Next, $h(0, \cdot) = f$, and $h(1, \cdot) = \widetilde{f}(\cdot, \zeta_0)$ has 0 as a regular value. Finally, h is G-covariant, and even G-regular covariant since $h(0, \cdot) = f$ is G-regular covariant by hypothesis (see Corollary 3.1 or the related comments at the beginning of Section 5). This completes the proof. $\square$

In our final lemma, we handle the case when G is a $\wp$-group, $\wp \geq 2$ a prime number, i.e. $|G| = \wp^\alpha$ for some integer $\alpha \geq 1$.

LEMMA 6.4. *Suppose that G is a $\wp$-group, $\wp \geq 2$ a prime number. Suppose also that Ω is bounded, $f_{|\overline{\Omega}}$ is proper and $0 \notin f(\partial\Omega)$. Then*

$$|d|(f, \Omega, 0) = \pm|d|(f^G, \Omega^G, 0) \bmod \sqrt{\mathcal{I}^G}. \tag{6.12}$$

Moreover, if $p \in X^G$ is a base point for f, we have

$$d_p(f, \Omega, 0) = d_p(f^G, \Omega^G, 0) \bmod \sqrt{\mathcal{I}^G}. \tag{6.13}$$

PROOF. If $X^G = X$, the result is trivial (and $\mathcal{I}^G = \{0\}$). From now on, we assume $X^G \neq X$. Since G is a $\wp$-group, every nontrivial orbit contains $\wp^\beta$ points where $\beta \geq 1$ is an integer. This implies that $\mathcal{I}^G = \wp^\gamma\mathbb{Z}$ where $\gamma \geq 1$ is an integer, whence $\sqrt{\mathcal{I}^G} = \wp\mathbb{Z}$.

We prove the desired results first when $G = \mathbb{Z}_\wp$ and next in general.

Step 1: $G = \mathbb{Z}_\wp$. Since f is G-regular by hypothesis, it follows from homotopy invariance of $|d|$ and Proposition 5.1 that we may assume that f has a base point $p \in X^G$ without changing $|d|(f, \Omega, 0)$ or $|d|(f^G, \Omega^G, 0)$. Furthermore, still by Proposition 5.1, we may assume that 0 is a regular value of f^G and that $Df(x) \in GL_G(X)$ for every $x \in (f^G)^{-1}(0) \cap \Omega^G$.

On the other hand, every representation of $\mathbb{Z}_\wp$ is semi-free, hence semi-loose if $\wp \geq 3$ (see Remark 4.1(ii)). Thus, Theorem 5.1 yields that, irrespective of the prime number $\wp \geq 2$, we have

$$d_p(f,\omega,0) = d_p(f^G,\Omega^G,0) \bmod \wp\mathbb{Z}, \tag{6.14}$$

where $\omega \subset \Omega$ is a G-invariant open neighborhood of Ω^G in Ω.

Because $0 \notin f(\partial\omega) \cup f(\partial\Omega)$, we have $f^{-1}(0) \cap \Omega = (f^{-1}(0) \cap \omega) \cup (f^{-1}(0) \cap (\Omega\backslash\overline{\omega}))$ and $0 \notin f(\partial(\Omega\backslash\overline{\omega}))$. By additivity of the degree with respect to the domain, we find

$$d_p(f,\Omega,0) = d_p(f,\omega,0) + d_p(f,\Omega\backslash\overline{\omega},0). \tag{6.15}$$

But $\Omega\backslash\overline{\omega}$ is G-invariant and $(\Omega\backslash\overline{\omega}) \cap X^G = \emptyset$ since ω is an open neighborhood of $\Omega^G = \Omega \cap X^G$. In particular, $R_g x \neq x$ for all $g \in G = \mathbb{Z}_\wp$ and $x \in \Omega\backslash\overline{\omega}$ (using once again the fact that every representation of $\mathbb{Z}_\wp$ is semi-free). From Lemma 6.1 and Lemma 6.3 with $\Omega\backslash\overline{\omega}$ replacing Ω, and from homotopy invariance of the absolute degree, we find $|d|(f,\Omega\backslash\overline{\omega},0) = 0 \bmod \wp\mathbb{Z}$, hence $d_p(f,\Omega\backslash\overline{\omega},0) = 0 \bmod \wp\mathbb{Z}$. Combining this result with (6.14) and (6.15), we get

$$d_p(f,\Omega,0) = d_p(f^G,\Omega^G,0) \bmod \wp\mathbb{Z}, \tag{6.16}$$

and (6.12) follows by taking absolute values.

Relation (6.16) above was obtained after possibly modifying f through a homotopy h as indicated in Proposition 5.1. If $p \in X^G$ is a base point of f, Proposition 5.1 also ensures that h can be chosen so that p remains a base point of $h(t,\cdot)$, $0 \leq t \leq 1$. Thus, by homotopy invariance of the degree (see Remark 5.1), relation (6.16) valid with f replaced by $h(1,\cdot)$ remains valid with $f = h(0,\cdot)$. This proves (6.13).

Step 2: General case. Let $|G| = \wp^\alpha$. Since the case $\alpha = 1$ was solved in Step 1, suppose $\alpha \geq 2$. By induction, we assume that the result is true for all $\wp$-groups with order $\wp^\beta$, $1 \leq \beta \leq \alpha - 1$.

Since G is a $\wp$-group, the maximal proper subgroups of G have order $\wp^{\alpha-1}$ and are normal in G. Both properties are well known results from finite group theory. Let then $H < G$ be a maximal proper subgroup. From the hypothesis of induction, we have

$$|d|(f,\Omega,0) = \pm|d|(f^H,\Omega^H,0) \bmod \wp\mathbb{Z}. \tag{6.17}$$

As H is normal in G, the space X^H is G-invariant and the action of G in X^H factors through an action of G/H. Since $|H| = \wp^{\alpha-1}$ and $|G| = \wp^\alpha$, we have

$|G/H| = \wp$, i.e. $G/H \sim \mathbb{Z}_p$. By Theorem 3.3 and Remark 3.3, it follows that f^H is a G/H-regular covariant C^2 Fredholm mapping with index 0. Moreover, $f^H_{|\overline{\Omega}^H}$ is proper by properness of $f_{|\overline{\Omega}}$, and $0 \notin f^H(\partial\Omega^H)$. Also, $(X^H)^{G/H} = X^G$, whence $(\Omega^H)^{G/H} = \Omega^G$ and $(f^H)^{G/H} = f^G$. Thus, using Step 1 with $G/H \sim \mathbb{Z}_\wp$ replacing G and f^H replacing f, we find

$$|d|(f^H, \Omega^H, 0) = \pm|d|(f^G, \Omega^G, 0) \bmod \wp\mathbb{Z}, \tag{6.18}$$

so that (6.12) follows from (6.17) and (6.18).

If now $p \in X^G$ is a base point for f, then $p \in X^H$ for every subgroup $H \leq G$, and validity of (6.13) can be established from the case $G = \mathbb{Z}_\wp$ in Step 1 by a similar induction procedure. □

The generalization of Lemma 6.4 to the case when G is an arbitrary finite group and Ω need not be bounded is given in Theorem 6.1 below.

THEOREM 6.1. *Let the group G be finite and let $f : X \to X$ be a G-regular covariant C^2 Fredholm mapping with index 0. Let $\Omega \subset X$ be a G-invariant open subset of X such that $f_{|\overline{\Omega}}$ is proper and $0 \notin f(\partial\Omega)$. Finally, let $\mathcal{I}^G$ denote the ideal of $\mathbb{Z}$ generated by the integers $[G : G_x]$, $x \in X\backslash X^G$, where G_x is the isotropy subgroup of x. We have*

$$|d|(f, \Omega, 0) = \pm|d|(f^G, \Omega^G, 0) \bmod \sqrt{\mathcal{I}^G}. \tag{6.19}$$

In addition, if $p \in X^G$ is a base point of f (and hence f is automatically G-regular), we have

$$d_p(f, \Omega, 0) = d_p(f^G, \Omega^G, 0) \bmod \sqrt{\mathcal{I}^G}. \tag{6.20}$$

PROOF. Once again, the result is trivial if $X = X^G$. So, suppose $X \neq X^G$ and let Δ denote the g.c.d. of the integers $[G : G_x], x \in X\backslash X^G$. If $\Delta = 1$, then $\mathcal{I}^G = \sqrt{\mathcal{I}^G} = \mathbb{Z}$ and (6.19) and (6.20) hold trivially (but are useless). If $\Delta > 1$, let $\Delta = \wp_1^{\beta} \dots \wp_k^{\beta_k}$ be the decomposition of Δ into a product of distinct primes $\wp_1, \dots, \wp_k$. As $\wp_1^{\beta_1}, \dots, \wp_k^{\beta_k}$ all divide $|G|$, the decomposition of $|G|$ into a product of distinct primes has the form $|G| = \wp_1^{\alpha_1} \dots \wp_\ell^{\alpha_\ell}$ with $\ell \geq k$ and $\alpha_i \geq \beta_i$, $1 \leq i \leq k$.

First, we prove the validity of Theorem 6.1 under the additional assumption that Ω is *bounded*. For $1 \leq i \leq k$, let S_i denote a Sylow $\wp_i$-subgroup of G, i.e. a subgroup of G of order $\wp_i^{\alpha_i}$ (existence of S_i is another standard result from finite group theory). We claim that $X^{S_i} = X^G$. Indeed, if $X^G \subsetneq X^{S_i}$, pick

$x \in X^{S_i} \backslash X^G$. As $x \in X^{S_i}$, the isotropy subgroup G_x contains S_i, which implies that $\wp_i^{\alpha_i}$ divides $|G_x|$. But then, from $|G| = |G_x|[G : G_x]$, we see that $\wp_i$ does not divide $[G : G_x]$, hence does not divide Δ since $x \notin X^G$, a contradiction.

By Proposition 5.1 and homotopy invariance of the absolute degree, we may assume that f has a base point $p \in X^G$ without affecting $|d|(f, \Omega, 0)$ or $|d|(f^G, \Omega^G, 0)$. As $X^{S_i} = X^G$ implies $\Omega^{S_i} = \Omega^G$ and $f^{S_i} = f^G$, Lemma 6.4 with S_i replacing G yields (note that $\sqrt{\mathcal{I}^{S_i}} = \wp_i\mathbb{Z}$ since $X^{S_i} = X^G \neq X$)

$$d_p(f, \Omega, 0) = d_p(f^G, \Omega^G, 0) \bmod \wp_i\mathbb{Z}, \qquad 1 \leq i \leq k, \tag{6.21}$$

and hence

$$d_p(f, \Omega, 0) = d_p(f^G, \Omega^G, 0) \bmod \wp_1 \dots \wp_k\mathbb{Z}. \tag{6.22}$$

Thus, (6.19) follows from $\wp_1 \dots \wp_k\mathbb{Z} = \sqrt{\mathcal{I}^G}$. If f has a base point $p \in X^G$ in the first place, then Proposition 5.1 need not be used and (6.21) and (6.22) hold without modifying f. This proves (6.20).

Finally, suppose that Ω is unbounded. As $f_{|\overline{\Omega}}$ is proper and $0 \notin f(\partial\Omega)$, the set $f^{-1}(0) \cap \Omega$ is compact. By Lemma 5.1 with $Z = X$, there is a G-invariant open neighborhood N of 0 in X such that $N \subset B(0,1)$. Changing N into λN, $\lambda > 0$ large enough, we may assume $f^{-1}(0) \cap \Omega \subset N \cap \Omega$.

Let $p \in X^G$ (resp. $q \in X$) be a base point of f^G (resp. f). By additivity of the degree with respect to the domain, $d_q(f, \Omega, 0) = d_q(f, N \cap \Omega, 0)$ and $d_p(f^G, \Omega^G, 0) = d_p(f^G, (N \cap \Omega)^G, 0)$, whence $|d|(f, \Omega, 0) = |d|(f, N \cap \Omega, 0)$ and $|d|(f^G, \Omega^G, 0) = |d|(f^G, (N \cap \Omega)^G, 0)$. In fact, these relations remain valid when either f or f^G has no base point, for they reduce to the trivial $0 = 0$ in this case. That (6.19) holds thus follows from the first part of the proof with Ω replaced by the bounded $N \cap \Omega$. Likewise, (6.20) is immediate from the validity of the same relation with Ω replaced by $N \cap \Omega$ and additivity of d_p with respect to the domain. $\square$

In a number of cases, $\mathcal{I}^G$ can be substituted for $\sqrt{\mathcal{I}^G}$ in Theorem 6.1; a few of them are given in

Theorem 6.2. *Retain the hypotheses of Theorem 6.1 and assume further that one of the following conditions holds:*

(i) 0 *is a regular value of* f.
(ii) *The restriction of the representation R to each Sylow subgroup of G is semi-free (in particular, trivial).*
(iii) *The order of G is not divisible by $\wp^3$ for any prime number $\wp \geq 2$.*

Then Theorem 6.1 *holds with* $\mathcal{I}^G$ *replaced by* $\sqrt{\mathcal{I}^G}$.

PROOF. As in the proof of Theorem 6.1, the problem reduces to the case when Ω is bounded and G is a $\wp$-group, i.e. it suffices to show that Lemma 6.4 holds with $\mathcal{I}^G$ replacing $\sqrt{\mathcal{I}^G}$.

In both cases (i) and (ii) of the theorem, the induction procedure of Lemma 6.4 may be bypassed: after possibly modifying f through a homotopy according to Proposition 5.1, we find $d_p(f,\Omega,0) = d_p(f,\omega,0)+d_p(f,\Omega\backslash\overline{\omega},0)$, where $p \in X^G$ is a base point and ω is the G-invariant open neighborhood of Ω_G given by Theorem 5.1. Next, $d_p(f,\Omega\backslash\overline{\omega},0) = 0 \operatorname{mod} \mathcal{I}^G$ directly from Lemma 6.1 in case (i), or after using Lemma 6.3 when R is semi-free in case (ii).

Thus, to complete the proof when (i) or (ii) holds, it suffices to show that

$$d_p(f,\omega,0) = d_p(f,\Omega^G,0) \operatorname{mod} \mathcal{I}^G.$$

If R is semi-loose, this follows at once from Theorem 5.1(i). Otherwise, Theorem 5.1(ii) just yields $d_p(f,\omega,0) = d_p(f^G,\Omega^G,0) \operatorname{mod} 2\mathbb{Z}$. However, we claim that $\mathcal{I}^G = 2\mathbb{Z}$ in this case. To see this, note first that every $\wp$-group is (obviously) 2-nilpotent. Thus, as R is not semi-loose, it follows from Theorem 4.3 that there is a subgroup $H < G$ with $X^G \subsetneq X^H$ and $[G:H] = 2$. The latter relation implies at once that $\wp = 2$. On the other hand, H being a maximal proper subgroup of G, every point $x \in X^H\backslash X^G \neq \emptyset$ has isotropy subgroup $G_x = H$. It follows that $[G:H] = 2 \in \mathcal{I}^G$, whence $2\mathbb{Z} \subset \mathcal{I}^G \subset \sqrt{\mathcal{I}^G} \subset 2\mathbb{Z}$ (recall that $\sqrt{\mathcal{I}^G} = \wp\mathbb{Z}$ or $\{0\}$ when G is a $\wp$-group), so that $\mathcal{I}^G = 2\mathbb{Z}$, as claimed.

If now assumption (iii) of the theorem holds (and G is a $\wp$-group), then $|G| = \wp$ or $|G| = \wp^2$. If $|G| = \wp$, then $\mathcal{I}^G = \wp\mathbb{Z}$ or $\{0\}$, whence $\mathcal{I}^G = \sqrt{\mathcal{I}^G}$ and the conclusion follows from Theorem 6.1 itself. Next, if $|G| = \wp^2$ and R is not semi-free, there is $x \in X\backslash X^G$ and there is $g \in G$ such that $R_g x = x$. But then the isotropy subgroup G_x is a nontrivial proper subgroup of G, so that $|G_x| = \wp$ and also $[G:G_x] = \wp$ since $|G| = \wp^2$. Thus, $\wp\mathbb{Z} \subset \mathcal{I}^G \subset \sqrt{\mathcal{I}^G} \subset \wp\mathbb{Z}$, i.e. $\mathcal{I}^G = \wp\mathbb{Z} = \sqrt{\mathcal{I}^G}$, and Theorem 6.1 yields the desired conclusion. On the other hand, if R is semi-free, the result follows from part (ii) of the theorem. $\square$

It is easy to exhibit other cases when Theorem 6.1 holds with $\mathcal{I}^G$ replacing $\sqrt{\mathcal{I}^G}$. For instance, since the degree of this paper is equivalent to Leray-Schauder's when f is a compact perturbation of the identity, Theorem 1.1 gives another example. But we do not know whether this improvement is always true. Theorems 1.1 and 6.2 show that if counterexamples exist, they must be fairly complicated. The results of Borisovich et al. [2] for the degree of Elworthy and

Tromba when $G = \mathbb{Z}_\wp$ are covered by Theorem 6.2(ii) and (iii), and also by Theorem 6.1 since $\sqrt{\mathcal{I}^G} = \mathcal{I}^G$ when $G = \mathbb{Z}_\wp$, and hence do not help clarify this issue.

For the record, we explicitly mention the simplest particular case of Theorem 6.1 when $G = \mathbb{Z}_2$ is represented by $\{I, -I\}$. If so, $\Phi_{0G}(X) = \Phi_{0G}^{\mathrm{reg}}(X)$ (see Section 3).

COROLLARY 6.1 (Borsuk's theorem). *Let $f : X \to X$ be an odd C^2 Fredholm mapping with index 0 and let $\Omega \subset X$ be an open subset symmetric with respect to the origin and such that $f_{|\overline{\Omega}}$ is proper, $0 \notin f(\partial\Omega)$. If $0 \in \Omega$ (resp. $0 \notin \Omega$), then $|d|(f, \Omega, 0)$ is odd (resp. even) and hence $d_q(f, \Omega, 0)$ is odd (resp. even) for every base point $q \in X$ of f.*

NOTE. If $0 \in \Omega$, existence of base points is ensured by oddness of $|d|(f, \Omega, 0)$ since $|d|(f, \Omega, 0) = 0$ if no base point exists (see [10]).

REMARK 6.1. Because the group structure and the action are so simple in Borsuk's theorem, its validity does not require any restriction on the domain of definition of f except simple connectivity already needed in the noncovariant theory (see [10] and Remark 1.1). This variant must be proved directly by the method of this paper, but considerable simplifications can be incorporated. □

7. The case of compact Lie groups with arbitrary dimension

In this section, G denotes a compact Lie group with identity component G^0. We prove a generalization of Theorem 6.1 based upon Theorem 6.1 itself and the following lemma.

LEMMA 7.1. *Suppose that G is a torus. Let $f : X \to X$ be a G-regular covariant C^2 Fredholm mapping with index 0, and let $\Omega \subset X$ be a G-invariant open subset such that $f_{|\overline{\Omega}}$ is proper and $0 \notin f(\partial\Omega)$. Then*

$$|d|(f, \Omega, 0) = |d|(f^G, \Omega^G, 0). \tag{7.1}$$

In addition, if $p \in X^G$ is a base point of f, we have

$$d_p(f, \Omega, 0) = d_p(f^G, \Omega^G, 0). \tag{7.2}$$

PROOF. Since a torus is a m.i.i.s. of itself, the representation R of G is semi-loose (see Remark 4.1(ii)). First, we consider the case when Ω is *bounded*. By Proposition 5.1, we may assume that f has a base point $p \in X^G$ without changing $|d|(f, \Omega, 0)$ or $|d|(f^G, \Omega^G, 0)$, that 0 is a regular value of $f^G_{|\Omega^G}$ and that

$Df(x) \in GL_G(X)$ for all $x \in (f^G)^{-1}(0) \cap \Omega^G$. If so, Theorem 5.1 ensures that there is a G-invariant open neighborhood ω of Ω^G such that $0 \notin f(\partial\omega)$ and

$$f^{-1}(0) \cap \omega = (f^G)^{-1}(0) \cap \Omega^G, \tag{7.3}$$

$$d_p(f, \omega, 0) = d_p(f^G, \Omega^G, 0). \tag{7.4}$$

Let g_* be a generator of the torus G, and let $g_n \in G$ be a sequence tending to g_* such that the subgroup H_n generated by g_n has prime order $\wp_n$ with $\lim_{n\to\infty} \wp_n = \infty$. Existence of such a sequence g_n is easily shown (see e.g. [18] where a similar argument is used). For n large enough, we claim that

$$f^{-1}(0) \cap \omega = (f^{H_n})^{-1}(0) \cap \Omega^{H_n}. \tag{7.5}$$

Indeed, "$\subset$" is trivial from (7.3) and from $X^G \subset X^{H_n}$ (whence $\Omega^G \subset \Omega^{H_n}$). Conversely, suppose by contradiction that for a subsequence n_k there is $x_k \in (f^{H_{n_k}})^{-1}(0) \cap \Omega^{H_{n_k}}$ such that $x_k \notin f^{-1}(0) \cap \omega$ (i.e. $x_k \notin \omega$). Extracting another subsequence, we may assume $\lim_{k\to\infty} x_k = x \in (f^G)^{-1}(0) \cap \Omega$ (use properness of $f_{|\overline{\Omega}}$ and $0 \notin f(\partial\Omega)$). Also, $x \in X^G$. Indeed, from $R_{g_{n_k}} x_k = x_k$ for all $k \in \mathbb{N}$, we find

$$\begin{aligned} \|R_{g_*}x - x\| &\le \|R_{g_*}x - R_{g_{n_k}}x\| + \|R_{g_{n_k}}(x - x_k)\| + \|x_k - x\| \\ &\le \|R_{g_*}x - R_{g_{n_k}}x\| + (1 + \|R_{g_{n_k}}\|)\|x_k - x\|, \end{aligned}$$

and the right-hand side tends to 0 since $\|R_{g_{n_k}}\|$ is bounded (Proposition 2.1(i)) and $G \ni g \mapsto R_g x \in X$ is continuous. This implies $R_{g_*}x = x$, whence $x \in X^G$ since g_* is a generator of G. From the above, $x \in (f^G)^{-1}(0) \cap \Omega^G = f^{-1}(0) \cap \omega$ (see (7.3)). As ω is open and $x \in \omega$, we must have $x_k \in \omega$ for k large enough, a contradiction.

Thus, as claimed, (7.5) holds for n large enough, and of course it is not restrictive to assume $\wp_n \ge 3$. As $H_n \sim \mathbb{Z}_{\wp_n}$, H_n acts freely, hence loosely, in $\widetilde{X}^H$ (Remark 4.1(ii)). In addition, from (7.3) and (7.5) and since $Df(x) \in GL_G(X)$ for every $x \in (f^G)^{-1}(0) \cap \Omega^G$, we have $Df(x) \in GL_{H_n}(X)$ for every $x \in (f^{H_n})^{-1}(0) \cap \Omega^{H_n}$. In particular, 0 is a regular value of $f^{H_n}_{|\Omega^{H_n}}$. Finally, $p \in X^G \subset X^{H_n}$. It follows from all this that we may apply Theorem 5.1 once again, but now with H_n replacing G. This yields existence of an H_n-invariant open neighborhood ω_n of Ω^{H_n} in Ω such that $0 \notin f(\partial\omega_n)$ and

$$f^{-1}(0) \cap \omega_n = (f_{|H_n})^{-1}(0) \cap \Omega^{H_n}, \tag{7.6}$$

$$d_p(f, \omega_n, 0) = d_p(f^{H_n}, \Omega^{H_n}, 0). \tag{7.7}$$

Comparing (7.5) and (7.6), we see that $f^{-1}(0)\cap\omega = f^{-1}(0)\cap\omega_n = f^{-1}(0)\cap(\omega\cap\omega_n)$. As $0\notin f(\partial(\omega\cap\omega_n))\subset f(\partial\omega)\cup f(\partial\omega_n)$, additivity of the degree with respect to the domain yields $d_p(f,\omega\cap\omega_n,0) = d_p(f,\omega,0) = d_p(f,\omega_n,0)$, whence $|d|(f,\omega,0) = |d|(f,\omega_n,0)$. Together with (7.4) and (7.7), this implies

$$|d|(f^{H_n},\Omega^{H_n},0) = |d|(f^G,\Omega^G,0). \tag{7.8}$$

Now, Theorem 6.1 with $H_n\sim\mathbb{Z}_{\wp_n}$ replacing G provides $|d|(f,\Omega,0) = \pm|d|(f^{H_n},\Omega^{H_n},0)\bmod\wp_n\mathbb{Z}$, that is, with (7.8), $|d|(f,\Omega,0) = \pm|d|(f^G,\Omega^G,0)\bmod\wp_n\mathbb{Z}$. Choosing n such that $\wp_n>\max\{|d|(f,\Omega,0),|d|(f^G,\Omega^G,0)\}$ we infer that $|d|(f,\Omega,0) = \pm|d|(f^G,\Omega^G,0)$, whence $|d|(f,\Omega,0) = |d|(f^G,\Omega^G,0)$, since the absolute degree is nonnegative. This proves (7.1), and (7.2) follows in a similar way, using the "furthermore..." part of Proposition 5.1(iii) and Remark 5.1.

When Ω is unbounded, the problem can easily be reduced to the bounded case via Lemma 5.1 and additivity of the base point degree with respect to the domain, as in the proof of Theorem 6.1. □

From now on, $\mathcal{I}^G$ denotes the ideal of $\mathbb{Z}$ generated by the integers $\chi(G/G_x)$ (Euler-Poincaré characteristic) for $x\in X\backslash X^G$ and $\operatorname{rank}G_x = \operatorname{rank}G$ (recall that the rank of a compact Lie group G is the dimension of any maximal torus of the identity component G^0). If $T\le G^0$ is a maximal torus and $N(T)$ its normalizer in G, the fixed point space X^T is $N(T)$-invariant and the action of $N(T)$ in X^T factors through an action of the factor group $N(T)/T$ (the Weyl group of G when G is connected).

LEMMA 7.2.

(i) *The group $N(T)/T$ is finite and independent of T (i.e. two different choices of T yield isomorphic factor groups).*

(ii) *The condition $X^{N(T)} = X^G$ is independent of T.*

For $x\in X^T$, let Γ_x be the (finite from part (i) above) isotropy subgroup of x relative to the action of $N(T)/T$ in X^T. Denote by $\mathcal{I}^{N(T)/T}$ the ideal generated by the integers $[N(T)/T:\Gamma_x]$, $x\in X\backslash X^T$. Then:

(iii) *$\mathcal{I}^{N(T)/T}$ is independent of T.*

(iv) *If $X^{N(T)} = X^G$ (a condition independent of T from part (ii) above) we have $\mathcal{I}^{N(T)/T} = \mathcal{I}^G$.*

(v) *If $X^{N(T)}\ne X^G$ (a condition independent of T from part (ii) above) we have $\mathcal{I}^G = \mathbb{Z}$.*

PROOF. Part (i) is well-known (see e.g. Bredon [3]). Parts (ii) through (v) are proved in [20, Theorem 7.1(i), (ii) and Theorem 8.1(ii), (iii)]. In fact, (ii) and (iii) follow easily from conjugacy of maximal tori while (iv) and (v) are due to the (known) formula $\chi(A) = \chi(A^T)$ for every compact G-manifold A. $\square$

THEOREM 7.1. *Let G be a compact Lie group and let $f : X \to X$ be a G-regular covariant C^2 Fredholm mapping with index 0. Let $\Omega \subset X$ be a G-invariant open subset of X such that $f_{|\overline{\Omega}}$ is proper and $0 \notin f(\partial\Omega)$. Finally, let $\mathcal{I}^G$ denote the ideal of $\mathbb{Z}$ generated by the integers $\chi(G/G_x)$, $x \in X \backslash X^G$ with $\operatorname{rank} G_x = \operatorname{rank} G$. We have*

$$|d|(f, \Omega, 0) = \pm |d|(f^G, \Omega^G, 0) \bmod \sqrt{\mathcal{I}^G}. \tag{7.9}$$

In addition, if $p \in X^G$ is a base point of f (and hence f is automatically G-regular) we have

$$d_p(f, \Omega, 0) = d_p(f^G, \Omega^G, 0) \bmod \sqrt{\mathcal{I}^G}. \tag{7.10}$$

PROOF. Let $T \leq G^0$ be a maximal torus. From Lemma 7.2(v), $\mathcal{I}^G = \mathbb{Z}$ if $X^{N(T)} \neq X^G$, hence $\sqrt{\mathcal{I}^G} = \mathbb{Z}$ and both (7.9) and (7.10) are trivial (but useless). Accordingly, we shall henceforth assume that

$$X^{N(T)} = X^G. \tag{7.11}$$

From Lemma 7.1 with T replacing G (that f is T-regular follows from Theorem 3.2), we find

$$|d|(f, \Omega, 0) = |d|(f^T, \Omega^T, 0). \tag{7.12}$$

On the other hand, replacing X by X^T and f by f^T, it is trivial to check that the hypotheses of Theorem 6.1 are satisfied with G replaced by $N(T)/T$. It is equally trivial that $X^{N(T)} = (X^T)^{N(T)/T}$, $\Omega^{N(T)} = (\Omega^T)^{N(T)/T}$ and $f^{N(T)} = (f^T)^{N(T)/T}$. Thus,

$$|d|(f^T, \Omega^T, 0) = \pm |d|(f^{N(T)}, \Omega^{N(T)}, 0) \bmod \sqrt{\mathcal{I}^{N(T)/T}}. \tag{7.13}$$

Now, in view of (7.11), we have $\Omega^{N(T)} = \Omega^G$ and $f^{N(T)} = f^G$, whereas $\mathcal{I}^{N(T)/T} = \mathcal{I}^G$ by Lemma 7.2(iv). Thus, (7.13) reads

$$|d|(f^T, \Omega^T, 0) = \pm |d|(f^G, \Omega^G, 0) \bmod \sqrt{\mathcal{I}^G},$$

which together with (7.12) yields (7.9). The proof of (7.10) is similar, using $p \in X^G \subset X^{N(T)} \subset X^T$. $\square$

REMARK 7.1. From the above proof, Theorem 7.1 is useful only when $X^{N(T)} = X^G$ for some (or equivalently every, by Lemma 7.2(ii)) maximal torus T of G^0. Furthermore, in this case, the ideal $\mathcal{I}^G = \mathcal{I}^{N(T)/T}$ can be calculated directly from the action of the finite group $N(T)/T$ in X^T. In other words, precise information about the geometry of the orbits G/G_x is not required. □

REMARK 7.2. It can be shown that the ideal $\mathcal{I}^G$ is generated by *all* the integers $\chi(G/G_x)$, $x \in X \backslash X^G$ (because $\chi(G/G_x) = 0$ if $\operatorname{rank} G_x < \operatorname{rank} G$; see [20, Lemma 8.2(ii)], or [12] for the case when G is connected). □

REMARK 7.3. When G is finite, Theorem 7.1 gives again Theorem 6.1 and Theorem 7.1 is the same as Lemma 7.1 when G is a torus since $\mathcal{I}^G = \{0\}$ (the ideal of $\mathbb{Z}$ generated by the empty set of generators) in this case. □

REMARK 7.4. In all the cases when $\mathcal{I}^{N(T)/T}$ can be substituted for $\sqrt{\mathcal{I}^{N(T)/T}}$ in Theorem 6.1 (with X replaced by X^T and f by f^T), e.g. when Theorem 6.2 can be used instead of Theorem 6.1, Theorem 7.1 holds with $\mathcal{I}^G$ replacing $\sqrt{\mathcal{I}^G}$.□

COROLLARY 7.1. *If $X^G = \{0\}$ in Theorem 7.1 and $0 \in \Omega$ (resp. $0 \notin \Omega$), then $|d|(f, \Omega, 0) = \pm 1$ (resp. 0) $\operatorname{mod} \sqrt{\mathcal{I}^G}$. In addition, if $Df(0) \in GL(X)$, then 0 is a base point of f and $d_0(f, \Omega, 0) = 1$ (resp. 0) $\operatorname{mod} \sqrt{\mathcal{I}^G}$.*

PROOF. Trivial. □

Appendix: Proofs of Theorems 2.2, 2.3, 3.1 and Corollary 3.3

PROOF OF THEOREM 2.2. Let $P = \int_G R_g \, dg$ (i.e. $Px = \int_G R_g x \, dg$ for all $x \in X$; see (2.1)). By Proposition 2.1(ii) we have $P \in \mathcal{L}(X)$, and it follows at once from the properties of invariant integration that P is a G-covariant projection onto X^G. Hence, $Z = \ker P$ is a closed G-invariant subspace, and $X = \ker(I - P) \oplus \ker P = X^G \oplus Z$.

Next, we show that if Z is any G-invariant closed complement of X^G and $A \in \mathcal{L}_G(X)$, then $A(Z) \subset Z$. Indeed, let $U \subset Z$ be any irreducible G-module of type V. Necessarily, $V \nsim V_0$ where V_0 denotes the trivial irreducible G-module, for otherwise $U \subset X^G$, a contradiction. Let $P : X \to X^G$ denote the projection onto X^G relative to the splitting $X = X^G \oplus Z$. As $P \in \mathcal{L}_G(X)$ (Theorem 2.1) we have $PA \in \mathcal{L}_G(X)$. Hence, by irreducibility of U, we have either $\ker PA_{|U} = \{0\}$ or $\ker PA_{|U} = U$. But in the first case, PA is a G-covariant isomorphism of U onto $PA(U)$ (recall $\dim U < \infty$). Hence $PA(U)$ is an irreducible G-module of type V, and $V \sim V_0$ since $PA(U) \subset X^G$, a contradiction. Thus, $\ker PA_{|U} = U$, i.e. $PA(U) = \{0\}$ or, equivalently, $A(U) \subset Z$. Now, Z is a Banach space and hence equals the closure of the algebraic sum S of all the irreducible G-modules contained in Z (see [4]). As $A(S) \subset Z$ from the above, we find $A(Z) \subset Z$ by continuity of A, as claimed.

This property implies uniqueness of the G-invariant closed complement Z of X^G, for if Z_1 and Z_2 are two possible choices, the projection Q_1 onto Z_1 associated with $X = X^G \oplus Z_1$ is G-covariant (Theorem 2.1). Choosing $A = Q_1$ and $Z = Z_2$ above, we find $Q_1(Z_2) \subset Z_2$. But $Q_1(Z_2) = Q_1(X)$ since $Q_1(X^G) = \{0\}$, and obviously $Q_1(X) = Z_1$. As a result, $Z_1 \subset Z_2$, i.e. $Z_1 = Z_2$ by exchanging the roles of Z_1 and Z_2.

Finally, if $A \in \mathcal{L}_G(X)$, and $\widetilde{X}^G$ denotes the unique G-invariant complement of X^G, then $A(X^G) \subset X^G$ is obvious, and $A(\widetilde{X}^G) \subset \widetilde{X}^G$ is just the previous relation $A(Z) \subset Z$ with $Z = \widetilde{X}^G$. This is equivalent to the desired block-diagonal decomposition of A. □

PROOF OF THEOREM 2.3. (i) Let $U \subset \overline{X}_V$ be an irreducible G-module of type W, and let $x \in U \backslash \{0\}$ be chosen once and for all ($U \neq \{0\}$ since $W \neq \{0\}$). As $x \in \overline{X}_V$, there is a sequence $x_n \in X_V$ tending to x. For each n, let $S_n \subset X_V$ be a (finite-dimensional) V-isotypical subspace of X containing x_n. Because $\dim U < \infty$ and U is G-invariant, there is a G-invariant closed complement Z of U in X, and the projection P onto U associated with $X = U \oplus Z$ is G-covariant (Theorem 2.1). From G-invariance of U_n, it follows that $P(S_n)$ is a G-invariant subspace of U, whence $P(S_n) = \{0\}$ or $P(S_n) = U$ since U is irreducible. As x_n tends to x, Px_n tends to $Px = x \neq 0$, which shows that $P(S_n) \neq \{0\}$ for n large enough. Thus, $P(S_n) = U$ and since S_n is V-isotypical, all its G-invariant subspaces are V-isotypical too. In particular, this is true of any G-invariant complement Z_n of $\ker P_{|S_n}$ (recall $\dim S_n < \infty$). But $P_{|S_n}$ is a G-covariant linear isomorphism of Z_n onto U, so that U is V-isotypical. As U is irreducible of type W, this implies $V \sim W$.

(ii) Obviously, X_V is G-invariant and hence $\overline{X}_V$ is G-invariant too. To prove $A(\overline{X}_V) \subset \overline{X}_V$, it suffices to show that $A(X_V) \subset X_V$, for then $A(\overline{X}_V) \subset \overline{A(X_V)} \subset \overline{X}_V$. If $U \subset X$ is an irreducible G-module of type V, then $A(U)$ is G-invariant and $\ker A_{|U} = U$ or $\{0\}$ by irreducibility of U. In the first case, $A(U) = \{0\} \subset X_V$. In the second case, $A_{|U}$ is a G-covariant linear isomorphism of U onto $A(U)$, whence $A(U)$ has the same type V as U and, once again, $A(U) \subset X_V$. This clearly implies $A(X_V) \subset X_V$.

Lastly, if $\overline{X}_V$ is split and Z is a closed G-invariant complement of $\overline{X}_V$, the relation $A(Z) \subset Z$ follows from a straightforward modification of the proof given in Theorem 2.2 for the special case $V = V_0$ (the trivial irreducible G-module).□

PROOF OF THEOREM 3.1. That (i) $\Rightarrow$ (ii) is trivial, and (ii) $\Rightarrow$ (iii) follows by choosing a G-covariant linear isomorphism T in the first part of the proof of Proposition 3.1 (as assumption (ii) permits) and letting $B = A$ in that proof. (iii) $\Rightarrow$ (iv) was shown in Proposition 3.2.

(iv) $\Rightarrow$ (v): Let $A_n \in GL_G(X)$ be a sequence tending to A. From Theorem 2.3(ii), we have $A_{|\overline{X}_V}, A_{n_{|\overline{X}_V}} \in \mathcal{L}_G(\overline{X}_V)$, and $A_{n_{|\overline{X}_V}}$ tends to $A_{|\overline{X}_V}$ in $\mathcal{L}_G(\overline{X}_V)$. Clearly, $A_{n_{|\overline{X}_V}}$ is one-to-one, and since $A_n^{-1} \in GL_G(X)$, Theorem 2.3(ii) implies that for $y \in \overline{X}_V$ we have $A_n^{-1}y \in \overline{X}_V$, i.e. $A_{n_{|\overline{X}_V}}$ is onto $\overline{X}_V$. Thus, $A_{n_{|\overline{X}_V}} \in GL_G(\overline{X}_V) \subset \Phi_{0G}(\overline{X}_V)$. Local constancy of the index will imply $A_{|\overline{X}_V} \in \Phi_{0G}(\overline{X}_V)$ (the desired result) provided that we show that $A_{|\overline{X}_V}$ is Fredholm of any index.

First, $\ker A_{|\overline{X}_V} \subset \ker A$, and hence $\dim\ker A_{|\overline{X}_V} < \infty$. To show that $\operatorname{codim}\operatorname{rge} A_{|\overline{X}_V} < \infty$, let us first observe that the method of proof of part (ii) of Theorem 2.3 shows that, more generally, if X and Y are Banach spaces equipped with an action of G, and if $A \in \mathcal{L}_G(X,Y)$, then $A(\overline{X}_V) \subset \overline{Y}_V$ for every irreducible G-module V. In addition, if $A \in GL_G(X,Y)$ then $A_{|\overline{X}_V} \in GL_G(\overline{X}_V, \overline{Y}_V)$ by the same arguments as above when $X = Y$.

Using this remark with X and Y replaced by X_1 and Y_1, respectively, where $Y_1 = \operatorname{rge} A$ and X_1 is any G-invariant closed complement of $\ker A$ (recall $\dim\ker A < \infty$ and use Theorem 2,1), we infer that $A_{|\overline{X}_{1V}} = \overline{Y}_{1V}$ and hence that for every $y \in \overline{Y}_{1V}$, there is $x \in \overline{X}_{1V} \subset \overline{X}_V$ such that $Ax = y$. In other words, $\overline{Y}_{1V} \subset \operatorname{rge} A_{|\overline{X}_V}$. As a result, to prove that $\operatorname{rge} A_{|\overline{X}_V}$ has finite codimension, it suffices to show that $\overline{Y}_{1V} \subset \overline{X}_V$ has finite codimension in $\overline{X}_V$.

To do this, we show $\overline{X}_V = Y_{0V} \oplus \overline{Y}_{1V}$ where Y_0 is any (finite-dimensional) G-invariant complement of $Y_1 = \operatorname{rge} A$ (Theorem 2.1). Let Q_0 and Q_1 denote the projections onto Y_0 and Y_1 associated with the splitting $X = Y_0 \oplus Y_1$. Both Q_0 and Q_1 are G-covariant. Let $U \subset X$ be an irreducible G-module of type V. Then $Q_0(U)$ is G-invariant, and $\ker Q_{0_{|U}} = \{0\}$ or U. Equivalently, $Q_0(U) = \{0\}$ or is an irreducible G-module of type V. In both cases, $Q_0(U) \subset Y_{0V}$ and hence $Q_0 X_V \subset Y_{0V}$. Likewise, $Q_1(X_V) \subset Y_{1V}$. Let now $x \in \overline{X}_V$ be fixed, and consider a sequence $x_n \in X_V$ tending to x. From the above, $x_n = Q_0 x_n + Q_1 x_n$ with $Q_\alpha x_n \in Y_{\alpha V}$, $\alpha = 0, 1$. In the limit as n tends to ∞, we find $x = y_0 + y_1$ where $y_\alpha = \lim_{n\to\infty} Q_\alpha x_n \in \overline{Y}_{\alpha V}$, $\alpha = 0, 1$. This shows that $\overline{X}_V = \overline{Y}_{0V} + \overline{Y}_{1V} = Y_{0V} + \overline{Y}_{1V}$ (recall $\dim Y_0 < \infty$), and the sum is direct since $Y_{0V} \subset Y_0$, $\overline{Y}_{1V} \subset Y_1$ and $Y_0 \cap Y_1 = \{0\}$. It follows that $\operatorname{codim} \overline{Y}_{1V} = \dim \overline{Y}_{0V} < \infty$.

(v) $\Rightarrow$ (i): Since $A \in \Phi_{0G}(X)$ by hypothesis, both $\ker A$ and $\operatorname{rge} A$ are G-invariant subspaces of X, and $\operatorname{rge} A = Y_1$ has a G-invariant complement Y_0

(Theorem 2.1) with $\dim \ker A = \dim Y_0 < \infty$. With no loss of generality, we assume $\ker A \neq \{0\}$ since the problem is trivial otherwise.

Since $X_0 = \ker A$ is G-invariant and finite-dimensional, it is the direct sum $X_0 = X_{01} \oplus X_{02} \oplus \ldots \oplus X_{0k}$ of finitely many G-invariant nontrivial spaces X_{0i}, each of which is V_i-isotypical for some family $V_1, \ldots, V_k$ of nonisomorphic irreducible G-modules. By definition of X_{V_i}, we have $X_{0i} \subset X_{V_i}$, and $X_{0i} \subset \ker A_{|X_{V_i}} \subset \ker A_{|\overline{X}_{V_i}}$. In fact, $X_{0i} = \ker A_{|\overline{X}_{V_i}}$: first, $A_{|\overline{X}_{V_i}}$ is Fredholm with index 0 by hypothesis, and hence $\ker A_{|\overline{X}_{V_i}}$ is also an algebraic sum of isotypical components. Next, by Theorem 2.3(i), $\overline{X}_{V_i}$ contains no nonzero irreducible G-module nonisomorphic to V_i, so that $\ker A_{|\overline{X}_{V_i}}$ is either $\{0\}$ or a V_i-isotypical subspace of X contained in $\ker A = X_0$. But obviously, X_{0i} is the largest V_i-isotypical subspace of X contained in $\ker A = X_0$, so that $A_{|\overline{X}_{V_i}} \subset X_{0i}$, i.e. $\ker A_{|\overline{X}_{V_i}} = X_{0i}$, as claimed.

Observe also that if $V \neq \{0\}$ is any irreducible G-module nonisomorphic to V_i, $1 \leq i \leq k$, then $\ker A_{|\overline{X}_V} = \{0\}$, for otherwise $\ker A_{|\overline{X}_V} \subset \ker A$ implies that $\ker A = X_0$ contains a V-isotypical component (again by Theorem 2.3(i)), which is not the case.

In a similar way, we may write $Y_0 = Y_{01} \oplus Y_{02} \oplus \ldots \oplus Y_{0\ell}$ where Y_{0i} is a nontrivial W_i-isotypical subspace of X and $W_1, \ldots, W_\ell$ is some family of non-isomorphic irreducible G-modules. Since $Y_{0i} \subset Y_0$, we have $Y_{0i} \cap \operatorname{rge} A_{|\overline{X}_{W_i}} \subset Y_0 \cap \operatorname{rge} A = \{0\}$, and $A_{|\overline{X}_{W_i}}$ being Fredholm with index 0 by hypothesis, we infer that $\ker A_{|\overline{X}_{W_i}} \neq \{0\}$. As noted above, this is possible only if $W_i \sim V_j$ for some index $1 \leq j \leq k$. Furthermore, as the W_i's are nonisomorphic, different indices j correspond to different indices i, whence $\ell \leq k$, and after reordering $W_1, \ldots, W_\ell$, we may assume $W_i \sim V_i$, $1 \leq i \leq \ell$, i.e. $\overline{X}_{W_i} = \overline{X}_{V_i}$, $1 \leq i \leq \ell$.

The relation $Y_{0i} \cap \operatorname{rge} A_{|\overline{X}_{V_i}} = \{0\}$ obtained earlier implies $\operatorname{codim} A_{|\overline{X}_{V_i}} \geq \dim Y_{0i}$, that is, $\dim Y_{0i} \leq \dim X_{0i}$, $1 \leq i \leq \ell$, since $\ker A_{|\overline{X}_{V_i}} = X_{0i}$ (as shown above) and $A_{|\overline{X}_{V_i}}$ has index 0 by hypothesis. If either $\ell < k$ or the above inequality is strict for at least one index $1 \leq i \leq \ell$, then $\sum_{i=1}^{\ell} \dim Y_{0i} < \sum_{i=1}^{k} \dim X_{0i}$. But this contradicts the relation $\sum_{i=1}^{\ell} \dim Y_{0i} = \dim Y_0 = \dim X_0 = \sum_{i=1}^{k} \dim X_{0i}$. Therefore, $k = \ell$ and $\dim Y_{0i} = \dim X_{0i}$, $1 \leq i \leq k$.

In summary, for $1 \leq i \leq k$, both X_{0i} and Y_{0i} are V_i-isotypical subspaces of X with equal dimension, and hence they can be written as the direct sum of the same number of irreducible G-modules of type V_i. This makes it clear that X_{0i} and Y_{0i} are isomorphic through a G-covariant isomorphism. The corresponding k isomorphisms induce a G-covariant isomorphism between $\ker A = X_{0i} \oplus \ldots \oplus X_{0k}$ and $Y_0 = Y_{01} \oplus \cdots \oplus Y_{0k}$ in the obvious way. This completes the proof. □

PROOF OF COROLLARY 3.3. Let $A \in \Phi_{0G}(X)$. Since $\dim W < \infty$ for every irreducible G-module $W \not\sim V$, we have $A_{|X_W} \in \Phi_0(X_W)$. Thus, by Theorem 3.1 (v) $\Rightarrow$ (iii), it suffices to show that $A_{|\overline{X}_V} \in \Phi_0(\overline{X}_V)$ to prove that $A \in \Phi_{0G}^{\mathrm{reg}}(X)$.

As $\overline{X}_V$ is split, it has a G-invariant closed complement Z (Theorem 2.2), and both $A_{|\overline{X}_V}$ and $A_{|Z}$ are Fredholm with

$$0 = \operatorname{ind} A = \operatorname{ind} A_{|\overline{X}_V} + \operatorname{ind} A_{|Z}.$$

Thus, $\operatorname{ind} A_{|\overline{X}_V} = 0$ if and only if $\operatorname{ind} A_{|Z} = 0$. To prove the latter relation, note that if $W \subset Z$ is an irreducible G-module, then $W \not\sim V$ and hence $\dim W < \infty$. On the other hand, $\dim Z_W \leq \dim X_W < \infty$, whence $\overline{Z}_W = Z_W$ and $A_{|Z}$ is an isomorphism of Z_W for every $W \subset Z$ that does not appear in $\ker A_{|Z}$ or $\operatorname{coker} A_{|Z}$ (for convenience, we denote by $\operatorname{coker} A_{|Z}$ any chosen G-invariant closed complement of $\operatorname{rge} A_{|Z}$ in Z). Thus, the index of $A_{|Z}$ equals the finite sum of the indices of $A_{|Z_W}$ with $W \subset \ker A_{|Z}$ or $W \subset \operatorname{coker} A_{|Z}$, whence $\operatorname{ind} A_{|Z} = 0$ using once again $\dim Z_W < \infty$. □

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