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A CONTINUATION APPROACH TO SOME FORCED SUPERLINEAR STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS

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Dedicated, with admiration, to Jean Leray

1. Introduction

This paper is devoted to the existence and multiplicity of solutions of nonlinear ordinary differential equations of the form

$$(1) \qquad u''(t) + g(u(t)) = p(t, u(t), u'(t)), \qquad t \in [a, b],$$

satisfying boundary conditions of the Sturm-Liouville type at a and b, when $g : \mathbb{R} \to \mathbb{R}$ is continuous and *superlinear*, i.e.

$$(2) \qquad \frac{g(u)}{u} \to +\infty \qquad \text{as } |u| \to +\infty,$$

and $p : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies a linear growth condition in the last two arguments. Problems of this type have been considered since the late fifties, among others, by Ehrmann [7], Morris [14], Fučik-Lovicar [8], Struwe [16] using shooting arguments, and by Bahri-Berestycki [1], [2], Rabinowitz [15], Long [12], using critical point theory. More details and references can be found in [3], where this type of differential equation, with periodic boundary conditions, was treated using the celebrated Leray-Schauder continuation method [11]. It may look surprising that one had to wait for the nineties to see the method of Leray-Schauder applied to such problems. The reason can be found in the fact that the success of the Leray-Schauder method relies mostly upon obtaining a priori estimates for the possible solutions of a family of equations connecting (1) to a simpler problem for which the corresponding topological degree is not zero. But elementary considerations based upon the energy integral reveal that the equation

$$u''(t) + g(u(t)) = 0 \tag{3}$$

admits infinitely many solutions with arbitrary large amplitudes, satisfying classical boundary conditions like the Dirichlet, Neumann or periodic ones. And this lack of a priori bound also holds for the corresponding forced equation. Hence, the set of possible solutions of an associated family of equations like

$$u''(t) + g(u(t)) = \lambda p(t, u(t), u'(t)), \qquad \lambda \in [0,1], \tag{4}$$

satisfying the corresponding boundary conditions, is not necessarily a priori bounded.

A variant of the Leray-Schauder continuation theorem was introduced in [3] to overcome this difficulty in the case of periodic boundary conditions. Although this approach covers more general situations, we will just describe, in this introduction, the underlying ideas in the special case of equation (1) and for a bounded perturbation p. Let us introduce a continuous function $q : [a,b] \times \mathbb{R}^2 \times [0,1] \to \mathbb{R}$ such that

$$q(t,u,v,1) = p(t,u,v),$$

and such that the set of possible solutions of the periodic problem

$$\begin{aligned} u''(t) + g(u(t)) &= q(t, u(t), u'(t), \lambda), \\ u(a) - u(b) &= u'(a) - u'(b) = 0, \end{aligned} \tag{5}$$

with $\lambda = 0$, is a priori bounded and the corresponding topological degree of the set of solutions in the space $C^1_\sharp([a,b])$ of C^1 periodic functions on $[a,b]$ is different from zero. This is the case, in particular, if we choose

$$q(t,u,v,\lambda) = -(1-\lambda)\frac{v}{1+|v|} + \lambda p(t,u,v).$$

The Leray-Schauder theory implies then the existence of a continuum $\mathcal{C}$ of solutions (u,λ) of (5) which either connects $C^1_\sharp([a,b]) \times \{0\}$ to $C^1_\sharp([a,b]) \times \{1\}$ (in which case we obtain a periodic solution of (1)) or is unbounded in $C^1_\sharp([a,b]) \times [0,1]$. The second possibility is excluded by exhibiting a continuous functional $\varphi : C^1_\sharp([a,b]) \times [0,1] \to \mathbb{R}_+$ which is proper on the set $\Sigma \subset C^1_\sharp([a,b]) \times [0,1]$ of solutions (u,λ) of (5) and takes integer values on Σ when $\|u\|$ is sufficiently large. Namely, φ is chosen in such a way that it reduces, for $(u,\lambda) \in \Sigma$ with

$\|u\|$ sufficiently large, to the winding number around the origin of the curve $\{(u(t), u'(t)) : t \in [a, b]\}$. The reader can consult [3] for the rather lengthy technical details justifying the above assertions.

In contrast with the shooting or variational techniques mentioned above, the methodology of [3] can be and has been extended to the study of periodic solutions of functional differential equations of the form

$$u''(t) + g(u(t)) = p(t, u_t, u'_t),$$

and in particular to delay-differential equations (see [4]).

In the Hamiltonian case where p depends only upon t and u, Morris, Bahri-Berestycki, Rabinowitz and Long have succeeded in proving the existence of *infinitely many* periodic solutions of (1). One could think of adapting the above continuation technique to obtain such a result by considering instead of (5) the homotopy

$$\begin{gathered} u''(t) + g(u(t)) = \lambda p(t, u(t)), \qquad \lambda \in [0, 1], \\ u(a) - u(b) = u'(a) - u'(b) = 0, \end{gathered}$$

and showing the existence of distinct continua of solutions (u, λ) connecting infinitely many distinct periodic solutions of (3) to $C^1_\sharp([a, b]) \times \{1\}$. Unfortunately, the local index in $C^1_\sharp([a, b])$ of any nonconstant periodic solution of (3) is equal to zero (as shown in [5] in greater generality), and hence one is unable to prove that such continua merely start from $C^1_\sharp([a, b]) \times \{0\}$!

The situation is quite different in the case of Sturm-Liouville conditions as shown in [6] in the case of Dirichlet boundary conditions

$$u(a) = u(b) = 0.$$

The local index of the solutions of the unperturbed equation (3) satisfying those boundary conditions can be computed explicitly and an extension of the continuation theorem of [3] has been proposed, which allows one to prove the existence of infinitely many solutions with arbitrary large norms.

The aim of this paper is to extend this result to general Sturm-Liouville boundary conditions. The continuation theorem announced in [6] can still be applied and is developed in Section 2. Then the main task consists in computing the local index of the solutions of the unperturbed problem. This is done in Section 3 using some duality theorems of Krasnosel'skiĭ-Zabreĭko [10], and those results are of independent interest. In contrast to the shooting and variational techniques, the present approach can be easily extended to some boundary value problems for retarded second order equations in a way similar to that introduced in [4] for periodic boundary conditions.

We hope that this paper will help in convincing the readers that the Leray-Schauder continuation method, introduced sixty years ago, remains a powerful, efficient and versatile tool in proving the existence and multiplicity of solutions for nonlinear boundary value problems.

2. A continuation theorem

In this section, we shall introduce a generalization of the continuation theorem of [3]. Let X and Z be real Banach spaces, $L : X \supset D(L) \to Z$ a linear Fredholm mapping of index zero, $I = [0,1]$ and $N : X \times I \to Z$ an L-completely continuous operator (see [13] for the corresponding definitions). We consider the equation

$$Lu = N(u,\lambda), \qquad u \in D(L),\ \lambda \in I. \tag{6}$$

Let

$$\Sigma^* = \{(u,\lambda) \in D(L) \times I \,:\, Lu = N(u,\lambda)\}.$$

For any set $B \subset X \times I$ and any $\lambda \in I$, we denote by B_λ the section $\{u \in X : (u,\lambda) \in B\}$. Subsequently, we use the following conventions. For $\mathcal{O} \subset X \times I$, we denote by $\overline{\mathcal{O}}$ and $\partial\mathcal{O}$ its closure and boundary in $X \times I$, respectively. Similar notation is used for closure and boundary in X. If ω is an open subset of X (possibly not bounded) such that $S = \Sigma^*_\lambda \cap \overline{\omega}$ is compact and $S \subset \omega$ (i.e. there is no solution on $\partial\omega$), there exists $\mathcal{U}$ open bounded such that $S \subset \mathcal{U} \subset \overline{\mathcal{U}} \subset \omega$. For all such $\mathcal{U}$, the coincidence degree $D_L(L - N(\,\cdot\,,\lambda), \mathcal{U})$ (see [13] for its definition and notation) is the same, by excision property. We will denote it as

$$D_L(L - N(\,\cdot\,,\lambda), \omega).$$

Let $\mathcal{O} \subset X \times I$ be open in $X \times I$. Let us denote by Σ the (possibly empty) set of solutions (u,λ) of (6) which belong to $\overline{\mathcal{O}}$, i.e.

$$\Sigma = \{(u,\lambda) \in \overline{\mathcal{O}} \cap (D(L) \times I) \,:\, Lu = N(u,\lambda)\}.$$

We first suppose that

(i_1) *Σ_0 is bounded in X and $\Sigma_0 \subset \mathcal{O}_0$.*

We further assume that

(i_2) $D_L(L - N(\,\cdot\,,0), \mathcal{O}_0) \neq 0$, so that $\Sigma_0 \neq \emptyset$. Finally, we introduce a functional $\varphi : X \times I \to \mathbb{R}$ and suppose that

(i$_3$) *φ is continuous on $X \times I$ and proper on Σ.*

Consequently, the constants

$$\varphi_- = \min\{\varphi(u,0) : u \in \Sigma_0\}, \qquad \varphi_+ = \max\{\varphi(u,0) : u \in \Sigma_0\}$$

exist.

THEOREM 1. *Assume that conditions* (i$_1$), (i$_2$) *and* (i$_3$) *hold and that there exist constants c_-, c_+ with*

$$c_- < \varphi_- \leq \varphi_+ < c_+,$$

such that

$$\varphi(u,\lambda) \notin \{c_-, c_+\},$$

whenever $(u,\lambda) \in (D(L)\times]0,1[) \cap \mathcal{O} \cap \Sigma$, and

$$\varphi(u,\lambda) \notin [c_-, c_+],$$

whenever $(u,\lambda) \in (D(L)\times]0,1[) \cap \partial\mathcal{O} \cap \Sigma$. Then the equation

$$Lu = N(u,1) \tag{7}$$

has at least one solution in $D(L) \cap (\overline{\mathcal{O}})_1$.

PROOF. Assume that equation (7) has no solution. Then

$$\varphi(\mathcal{O} \cap \Sigma) \cap \{c_-, c_+\} = \emptyset, \tag{8}$$

and

$$\varphi(\partial\mathcal{O} \cap \Sigma) \cap [c_-, c_+] = \emptyset. \tag{9}$$

A version of the Leray-Schauder theorem (see e.g. [9]) asserts that there exists a continuum $\mathcal{C} \subset \Sigma$ with $\mathcal{C} \cap (\Sigma_0 \times \{0\}) \neq \emptyset$ such that either $\mathcal{C}$ is unbounded or $\mathcal{C}$ intersects $\partial\mathcal{O}$. By our assumption, $\varphi(\mathcal{C})$ is connected and intersects $[\varphi_-, \varphi_+]$ as $\mathcal{C} \cap (\Sigma_0 \times \{0\}) \neq \emptyset$. Suppose that $\mathcal{C}$ intersects $\partial\mathcal{O}$. Then the interval $\varphi(\mathcal{C})$ intersects at least one of the intervals $]-\infty, c_-[$ or $]c_+, +\infty[$. Hence $\{c_-, c_+\} \cap \varphi(\mathcal{C}) \neq \emptyset$ and if $(\widetilde{u}, \widetilde{\lambda}) \in \mathcal{C}$ is such that $\varphi(\widetilde{u}, \widetilde{\lambda}) = c \in \{c_-, c_+\}$, then $(\widetilde{u}, \widetilde{\lambda}) \in \overline{\mathcal{O}}$ but this is excluded by (8) and (9). Suppose now that $\mathcal{C}$ is unbounded, so that $\varphi(\mathcal{C})$ is unbounded too. Then, at least one of the unbounded intervals $]-\infty, \varphi_-], [\varphi_+, +\infty[$ is contained in $\varphi(\mathcal{C}) \subset \varphi(\Sigma)$, which implies that $\varphi(\Sigma) \cap \{c_-, c_+\} \neq \emptyset$, a contradiction. The proof is complete. $\square$

Remark. If $\Omega \subset X$ is open, bounded and such that

$$Lx \neq N(x, \lambda) \qquad \text{for all } (x, \lambda) \in \partial\Omega \times [0, 1[,$$

and if we take $\mathcal{O} = \Omega \times [0, 1]$ and

$$\varphi(x, \lambda) = \begin{cases} -\operatorname{dist}(x, \partial\Omega) & \text{if } x \in \Omega, \\ \operatorname{dist}(x, \partial\Omega) & \text{if } x \notin \Omega, \end{cases}$$

then, by assumption, $\Sigma_0 \subset \Omega$ is bounded, $-\operatorname{diam}\Omega \leq \varphi_+ < 0$, and we can take $c_+ = 0$, $c_- < -\operatorname{diam}\Omega$ in the above theorem to recover the classical Leray-Schauder continuation principle [11].

Let us now consider a consequence of Theorem 1 which is useful for the application we have in mind. Assume that $\varphi : X \times I \to \mathbb{R}_+$ is continuous and $(c_k)_{k \in \mathbb{N}}$ is an unbounded increasing sequence that satisfies the following conditions:

(i_4) *There exists $R > 0$ such that $\varphi(u, \lambda) \neq c_k$ for all $k \in \mathbb{N}$ and $(u, \lambda) \in \Sigma^*$ with $\|u\| \geq R$.*

(i_5) *$\varphi^{-1}([0, c_n[) \cap \Sigma^*$ is bounded for each $n \in \mathbb{N}$.*

Let k_0 be an integer such that

$$c_{k_0} > \sup\{\varphi(u, \lambda) : (u, \lambda) \in \Sigma^*, \|u\| \leq R\}.$$

Let, for $k \geq k_0$, $\mathcal{O}^k = \varphi^{-1}(]c_k, c_{k+1}[)$ and $\Sigma^k = \overline{\mathcal{O}^k} \cap \Sigma^*$. By ($i_5$), $(\Sigma^k)_0$ is bounded. But, by (i_4), $\varphi(x, \lambda) \neq c_k$ and $\varphi(x, \lambda) \neq c_{k+1}$ for all $(x, \lambda) \in \Sigma^k$, so $(\Sigma^k)_0 \subset (\mathcal{O}^k)_0$, and we have proved the condition (i_1).

We now prove that φ is proper on Σ^k. Let K be a compact subset of $\mathbb{R}$. Then $\varphi^{-1}(K) \cap \Sigma^k$ is closed and included in Σ^k which is compact, so it is also compact.

Let us assume that

(i_6) $D_L(L - N(\cdot, 0), \mathcal{O}^k) \neq 0$.

Thus, all conditions of Theorem 1 with $\Sigma = \Sigma^k$, $\mathcal{O} = \mathcal{O}^k$ are satisfied and equation (7) will have at least one solution $u \in D(L) \cap (\overline{\mathcal{O}^k})_1$. We have therefore the following result:

Corollary 1. *Assume that conditions* (i_4) *and* (i_5) *hold and that* (i_6) *is satisfied for each integer $k > k_0$. Then, for each of those integers, equation* (7) *has at least one solution u_k such that $\varphi(u_k, 1) \in \,]c_k, c_{k+1}[$. Moreover,* $\lim_{j \to \infty} \|u_j\| = +\infty$.

Proof. Only the last part of the Corollary has still to be proved. If this conclusion is not true, we can find a bounded subsequence (u_{k_j}) of solutions of (7) with $\varphi(u_{k_j}) \in]c_{k_j}, c_{k_j+1}[$. So $\varphi(u_{k_j}) \to \infty$ as $j \to \infty$. Thus we get a contradiction, as the sequence (u_{k_j}) is precompact. $\square$

3. Computation of the degree for Sturm-Liouville problems

To apply this abstract theory, we have to be able to compute the degree for some "simple" problems.

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, odd and such that $f(x)x > 0$ for all $x \in \mathbb{R}_0 :=]0, +\infty)$. We consider the Sturm-Liouville boundary value problem for the scalar autonomous equation

$$x''(t) + f(x(t)) = 0, \tag{10}$$

$$ax(0) + bx'(0) = cx(1) + dx'(1) = 0, \tag{11}$$

with $a^2 + b^2 > 0$ and $c^2 + d^2 > 0$.

Writing (10) as an equivalent system in the phaseplane $(x, y) = (x, x')$, we see that all solutions of (10) conserve the energy

$$H(x, y) = F(x) + \frac{y^2}{2},$$

where $F(x) = \int_0^x f(s)\,ds$. For $\alpha > 0$ we define F^α as the set of points with energy smaller than $\alpha^2/2$, i.e.

$$F^\alpha = \{(x, y) \in \mathbb{R}^2 \, : \, H(x, y) < \alpha^2/2\}.$$

We denote by $C(\alpha)$ the abscissa of the intersection in the right half plane of the x-axis and the energy level $\alpha^2/2$. This means that $F(C(\alpha)) = \alpha^2/2$ and $C(\alpha) > 0$. The intersection of the y-axis with this energy level is just the pair of points $(0, \alpha)$ and $(0, -\alpha)$ (see Figure 1).

Because of the conditions on a, b, c and d, $ax + by = 0$ and $cx + dy = 0$ are the equations of straight lines, which we call respectively D and A.

We denote respectively by $A(\alpha)$ and $D(\alpha)$ the abscissas of intersection in the upper half plane of A and D with the level line Γ_α of energy $\alpha^2/2$. This means that

$$2b^2 F(D(\alpha)) + a^2 D(\alpha)^2 = b^2\alpha^2$$

and $-(a/b)D(\alpha) \geq 0$ ($D(\alpha) = 0$ if $b = 0$). In the case where $a = 0$, there are two solutions, and we choose arbitrarily $D(\alpha) = -C(\alpha)$. And similarly,

$$2d^2 F(A(\alpha)) + c^2 A(\alpha)^2 = d^2\alpha^2$$

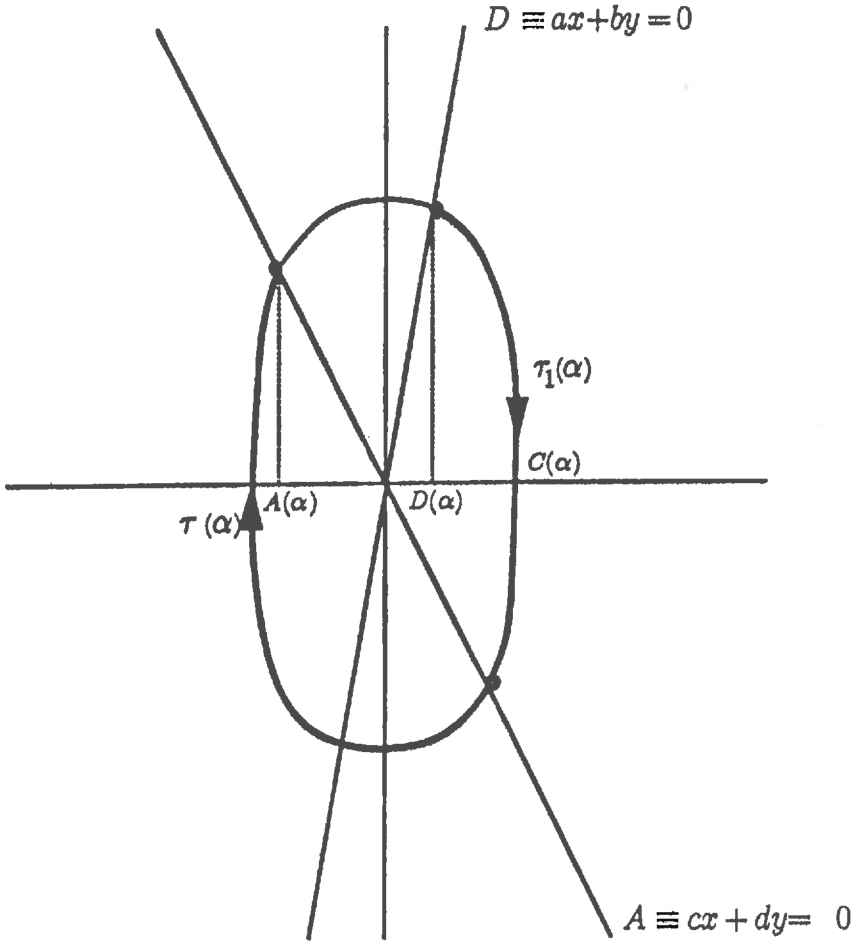

FIGURE 1. Picture of $C(\alpha)$, $A(\alpha)$ and $D(\alpha)$

and $-(c/d)A(\alpha) \geq 0$ ($A(\alpha) = 0$ if $d = 0$). In the case where $c = 0$, we take $A(\alpha) = C(\alpha)$.

The time needed by one solution of energy $\alpha^2/2$ to rotate from the point of abscissa x in the upper half plane to the point of abscissa $C(\alpha)$ is denoted by $\tau(x, \alpha)$. This time can be explicitly computed by the integral

$$\tau(x, \alpha) = \int_x^{C(\alpha)} \frac{1}{\sqrt{\alpha^2 - 2F(s)}}\, ds.$$

We define, for each energy level Γ_α, two times which will be useful in the computation of the degree.

The first one, $\tau(\alpha) = \tau(-C(\alpha), \alpha)$ is the time needed to make one half turn along the orbit Γ_α in the phaseplane. Since f is odd, this is the time of a half turn starting at any point of this energy level, in particular, $\tau(\alpha)$ is the time to move from any of the two points of intersection of Γ_α with the straight line A to the other point of intersection.

The second time $\tau_1(\alpha)$ is the one needed to go from the line D to the line A along Γ_α. Since the lines can take different positions with respect to each other, we have to consider several cases to define it. If $D(\alpha) \leq A(\alpha)$, then $\tau_1(\alpha) = \tau(D(\alpha), \alpha) - \tau(A(\alpha), \alpha)$, and if $D(\alpha) > A(\alpha)$, then $\tau_1(\alpha) = \tau(\alpha) - \tau(A(\alpha), \alpha) + \tau(D(\alpha), \alpha)$.

Let $u(\,\cdot\,; x, y)$ be the solution of the Cauchy problem

$$u''(t) + f(u(t)) = 0, \tag{12}$$

$$u(0) = x, \quad u'(0) = y. \tag{13}$$

The solution $u(\,\cdot\,; D(\alpha), -(a/b)D(\alpha))$ (or $u(\,\cdot\,; 0, \alpha)$ if $b = 0$) will be a solution of (10)–(11) if and only if

$$\tau_1(\alpha) + m\tau(\alpha) = 1$$

for some $m \in \mathbb{N}$. Let

$$\mathcal{S} = \{(x, y) \in \mathbb{R}^2 : x > y \geq 0 \text{ and } y + mx = 1 \text{ for some } m \in \mathbb{N}\}.$$

So, such a solution satisfies the boundary conditions (11) if and only if $(\tau(\alpha), \tau_1(\alpha)) \in \mathcal{S}$.

Set $C^k := C^k(I, \mathbb{R})$. The operator $\mathcal{L} : C^1 \supset C^2 \to C(I, \mathbb{R}) \times \mathbb{R}^2$, $u \mapsto (u'', 0, 0)$, is a linear Fredholm operator of index zero. The norm in C^1 is $\|x\|_1 = \sup_{t \in I} |(x(t), x'(t))|$, where $|P|$ denotes the euclidean norm of a point $P = (x, y) \in \mathbb{R}^2$. Set $e_1(u) = au(0) + bu'(0)$ and $e_2(u) = cu(1) + du'(1)$. The non-linear operator $\mathcal{N} : C^1 \to C(I, \mathbb{R}) \times \mathbb{R}^2$, $u \mapsto (Nu, e_1(u), e_2(u))$, where N is the Nemytskiĭ operator associated with $-f$, is $\mathcal{L}$-completely continuous, as is easily checked (see [13]).

The set

$$\Omega^\alpha = \{u \in C^1 : (\forall t \in \mathbb{R})\, (u(t), u'(t)) \in F^\alpha\}$$

is an open bounded subset of C^1.

So if $\alpha > 0$ is such that $(\tau(\alpha), \tau_1(\alpha)) \notin \mathcal{S}$, then the equation (10)–(11) has no solution on $\partial\Omega^\alpha$ and the degree (see [13])

$$D_{\mathcal{L}}(\mathcal{L} - \mathcal{N}, \Omega^\alpha)$$

is well defined. We notice that to obtain this conclusion, we have also used the fact that f is odd, so that u and $-u$ are both solutions of (10)–(11).

We now want to compute this last degree. We will use a duality theorem due to Krasnosel'skiĭ and Zabreĭko [10] to prove that this degree is the same as that of a one-dimensional map. And for such a map, the degree will be easy to compute.

THEOREM 2. *If $\alpha > 0$ is such that $(\tau(\alpha), \tau_1(\alpha)) \notin \mathcal{S}$ then*

$$D_{\mathcal{L}}(\mathcal{L} - \mathcal{N}, \Omega^\alpha) = \deg_B(\phi,]-l, l[, 0),$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is the function

$$\zeta \mapsto cu\left(1; \frac{1}{a^2+b^2}(-b\zeta, a\zeta)\right) + du'\left(1; \frac{1}{a^2+b^2}(-b\zeta, a\zeta)\right)$$

and $l > 0$ is such that

$$2F\frac{bl}{a^2+b^2} + \frac{a^2 l^2}{(a^2+b^2)^2} = \alpha^2.$$

PROOF. For the proof we will proceed by steps. We will first, by definition of the degree of coincidence, write this degree as the Leray-Schauder degree of some compact perturbation of the identity.

More generally, we can study the problem

$$u^{[m]}(t) + a_1(t)u^{[m-1]}(t) + \cdots + a_m(t)u(t) = -f(t, u(t), \ldots, u^{[m-1]}(t)),$$
$$l_1(u) = 0, \ldots, l_{mn}(u) = 0,$$

where $l_i : C^{m-1}(I, \mathbb{R}^n) \to \mathbb{R}$ are continuous linear functionals and $f : I \times \mathbb{R}^{mn} \to \mathbb{R}^n$ is continuous. We define $Lu(t) = u^{[m]}(t) + a_1(t)u^{[m-1]}(t) + \cdots + a_m(t)u(t)$.

Let $X = C^{m-1}(I, \mathbb{R}^n)$, $D(\mathcal{L}) = C^m(I, \mathbb{R}^n)$ and $\mathcal{Z} = C(I, \mathbb{R}^n) \times \mathbb{R}^{mn}$. So $\mathcal{L} : X \supset D(\mathcal{L}) \to \mathcal{Z}$, $u \mapsto (Lu, 0)$, is a linear Fredholm operator of index zero, and $\mathcal{N} : X \to \mathcal{Z}$, $u \mapsto (Nu, \Lambda u)$, where N is the Nemytskiĭ operator associated with $-f$ and $\Lambda : C^{m-1}(I, \mathbb{R}^n) \to \mathbb{R}^{mn}$, $u \mapsto (l_1(u), \ldots, l_{mn}(u))$, is an $\mathcal{L}$-completely continuous operator.

Then $\operatorname{Im} \mathcal{L} = C(I, \mathbb{R}^n) \times \{0\}$. If we denote by $V(t)$ the fundamental matrix of $Lu = 0$, then $\operatorname{Ker} \mathcal{L} = \{u \in C^{m-1} : (\exists z \in \mathbb{R}^{mn}),\ u(t) = V(t)z\}$. With these sets, we associate the projection $Q : \mathcal{Z} \to \mathcal{Z}$, $(u, r) \mapsto (0, r)$, $J : \operatorname{Ker} \mathcal{L} \to \operatorname{Im} Q$, $u \mapsto (0, u(0))$, and $\mathcal{P} : X \to X$, $u \mapsto V(t)u(0)$.

Thus, for any open set Ω on which the degree is defined,

$$D_{\mathcal{L}}(\mathcal{L} - \mathcal{N}, \Omega) = \deg_{LS}(I - \mathcal{M}, \Omega),$$

where $\mathcal{M} = \mathcal{P} + J^{-1}Q\mathcal{N} + K_{\mathcal{P}Q}\mathcal{N}$ (see [13]).

But

$$\mathcal{M}u(t) = (\mathcal{P}u + J^{-1}(0, \Lambda u) + K_{\mathcal{P}}(Nu, 0))(t) \tag{14}$$

$$= V(t)u(0) + V(t)\Lambda u - \int_0^t h_L(t,s) f(s, u(s), \dots, u^{[m-1]}(s))\, ds \tag{15}$$

where h_L is the Cauchy kernel associated with L.

This last operator is defined in [10, p. 170].

Coming back to problem (10)–(11), we now assume that for each initial condition $(u(0), u'(0)) = z$, there exists a unique solution $u(t,z)$ of the Cauchy problem (12)–(13) which is defined on I (this follows from the hypotheses we have put on f).

We consider the operator $\mathcal{U} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathcal{U}(z) = (z_1 + e_1(u(\cdot, z)), z_2 + e_2(u(\cdot, z))),$$

whose fixed points coincide with the initial values of the solutions of (10)–(11).

DEFINITION. We say that the bounded open sets $\Omega \subset C^1$ and $G \subset \mathbb{R}^2$ *have a common core* with respect to (10)–(11), if there are no fixed points of $\mathcal{M}$ on $\partial\Omega$ and no one of $\mathcal{U}$ on ∂G, and each solution u of (10)–(11) is in Ω if and only if $(u(0), u'(0)) \in G$.

Then we have, according to [10, Theorem 29.4]:

LEMMA 1. *Let $\Omega \subset C^1$ and $G \subset \mathbb{R}^2$ be open bounded sets having a common core with respect to* (10)–(11). *Then*

$$\deg_B(I - \mathcal{U}, G, 0) = \deg_{LS}(I - \mathcal{M}, \Omega).$$

We now show that Ω^α and F^α have a common core with respect to (10)–(11) when $(\tau(\alpha), \tau_1(\alpha)) \notin S$.

Note that $u \in \partial\Omega^\alpha$ if and only if $(u(t), u'(t)) \in \overline{F^\alpha}$ for all $t \in I$, and $(u(t_0), u'(t_0)) \in \partial F^\alpha$ for some $t_0 \in I$. But $H(u(t), u'(t))$ is constant on I, hence $(u(0), u'(0)) \in \partial F^\alpha$ too. But, by the choice of α, there is no solution with energy level $\alpha^2/2$. So we have no fixed point of $\mathcal{M}$ on $\partial\Omega^\alpha$, and no one of $\mathcal{U}$ on ∂F^α. Similarly, $u \in \Omega^\alpha$ if and only if $(u(0), u'(0)) \in F^\alpha$. Therefore, $D_{\mathcal{L}}(\mathcal{L} - \mathcal{N}, \Omega^\alpha) = \deg_B(I - \mathcal{U}, F^\alpha, 0)$.

We now have to compute

$$\deg_B(I - \mathcal{U}, F^\alpha, 0) = (-1)^2 \deg_B(\mathcal{U} - I, F^\alpha, 0) = \deg_B(\mathcal{U} - I, F^\alpha, 0).$$

Let
$$T = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}^{-1} = \frac{1}{a^2+b^2}\begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$
$B = T^{-1}(F^\alpha)$ and $\widetilde{U} = (\mathcal{U} - I) \circ T$. By the product formula
$$\deg_B(\widetilde{U}, B, 0) = \deg_B(\mathcal{U} - I, F^\alpha, 0),$$
and by the choice of T,
$$\widetilde{U}(z) = (z_1, cu(1, Tz) + du'(1, Tz)).$$
Consider the straight line $D:\ ax + by = 0$, which can be now written, after the change of variables, as $D = \{(0, z_2) : z_2 \in \mathbb{R}\}$, and set $B_D = \{z_2 \in \mathbb{R} : (0, z_2) \in B\}$ and $\mathcal{R} = \{(z_1, z_2) \in \mathbb{R}^2 : |z_1| < R \text{ and } z_2 \in B_D\}$ for some $R > 0$. We show that
$$\deg_B(\widetilde{U}, B, 0) = \deg_B(\widetilde{U}, \mathcal{R}, 0).$$
Indeed, if $\widetilde{U}z = 0$, then $z \in D$. But, by construction, $B \cap D = \mathcal{R} \cap D$; so by excision, the degree on these two sets must be the same (see Figure 2).

Then we use invariance by homotopy to compute this last degree. Let $h(z, \lambda) = (z_1, \widetilde{U}_2(\lambda z_1, z_2))$, so that $h(\cdot, 1) = \widetilde{U}$ and $h(\cdot, 0) = I_{\mathbb{R}} \times \phi$. Moreover, $h(z, \lambda) = 0$ if and only if $h(z, 1) = 0$. By the choice of $\mathcal{R}$, $h(z, \lambda) \neq 0$ for all $\lambda \in I$ and $z \in \partial\mathcal{R}$. Hence
$$\begin{aligned}\deg_B(\widetilde{U}, \mathcal{R}, 0) &= \deg_B(I_{\mathbb{R}} \times \phi,]-R, R[\times B_D, 0) \\ &= \deg_B(I_{\mathbb{R}},]-R, R[, 0) \deg_B(\phi, B_D, 0) \\ &= \deg_B(\phi, B_D, 0).\end{aligned}$$

The last thing to prove is that $B_D =]-l, l[$. Accordingly, observe that $\zeta \in \partial B_D$ if $(0, \zeta) \in \partial B$. By construction of B, this is equivalent to $T(0, \zeta) \in \partial F^\alpha$. It means that $(a^2 + b^2)^{-1}(-b\zeta, a\zeta)$ must satisfy the relation $2F(x) + y^2 = \alpha^2$. And this is exactly the definition of l. Therefore, Theorem 2 is proved. □

LEMMA 2. *If $\alpha > 0$ is such that $(\tau(\alpha), \tau_1(\alpha)) \notin S$ then there exists $m \in \mathbb{N}$ such that $\tau_1(\alpha) + (m-1)\tau(\alpha) < 1$ and $\tau_1(\alpha) + m\tau(\alpha) > 1$ and*
$$\deg_B(\phi,]-l, l[, 0) = (-1)^m \sigma$$
where $\sigma = \operatorname{sgn}(cz_1 + dz_2)$ with $z = T(0, l)$.

PROOF. Since f is odd, ϕ is also odd. So
$$\deg_B(\phi,]-l, l[, 0) = \operatorname{sgn}(\phi(l)).$$
The conditions on $\tau(\alpha)$ and $\tau_1(\alpha)$ imply that the solution makes at least $m-1$ half turns after its first contact with A and at most m. If m is even, then

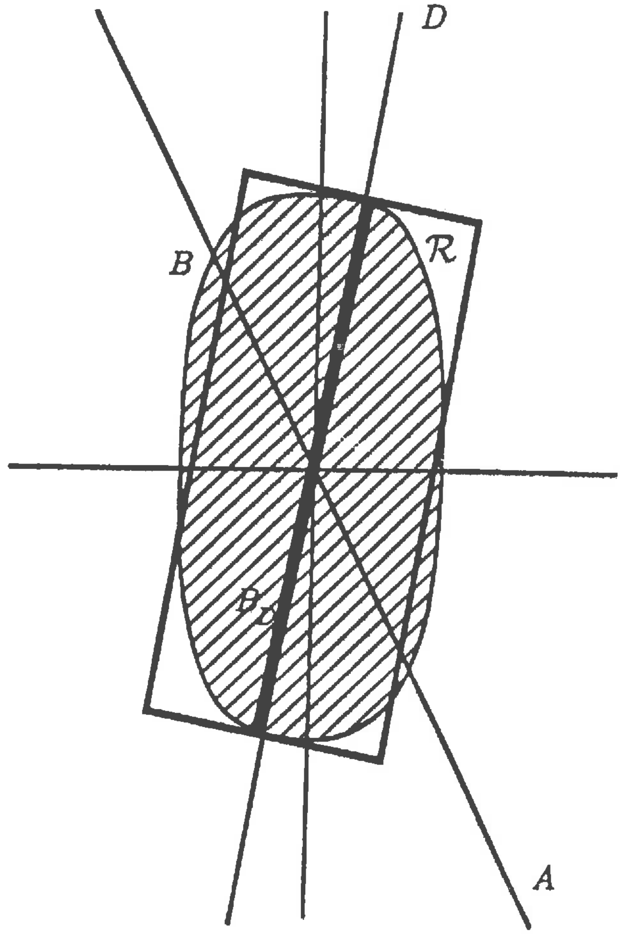

FIGURE 2. The sets B, B_D and $\mathcal{R}$

$(u(1,z), u'(1,z))$ will be on the same side of A as z. So $\mathrm{sgn}\phi(l)$ will be σ. And if m is odd, $\mathrm{sgn}\phi(l) = -\sigma$. □

Figure 3 gives a more visual statement of this lemma.

So we have proved

THEOREM 3. *If $\alpha > 0$ is such that $(\tau(\alpha), \tau_1(\alpha)) \notin S$, then there exists $m \in \mathbb{N}$ such that $\tau_1(\alpha) + (m-1)\tau(\alpha) < 1$ and $\tau_1(\alpha) + m\tau(\alpha) > 1$, and*

$$D_{\mathcal{L}}(\mathcal{L} - \mathcal{N}, \Omega^{\alpha}) = \sigma(-1)^m,$$

where $\sigma \in \{-1, 1\}$ is defined in Lemma 2.

REMARK. The computation of the degree for the same problem when f is not odd has also been made. But it is rather long, because instead of involving

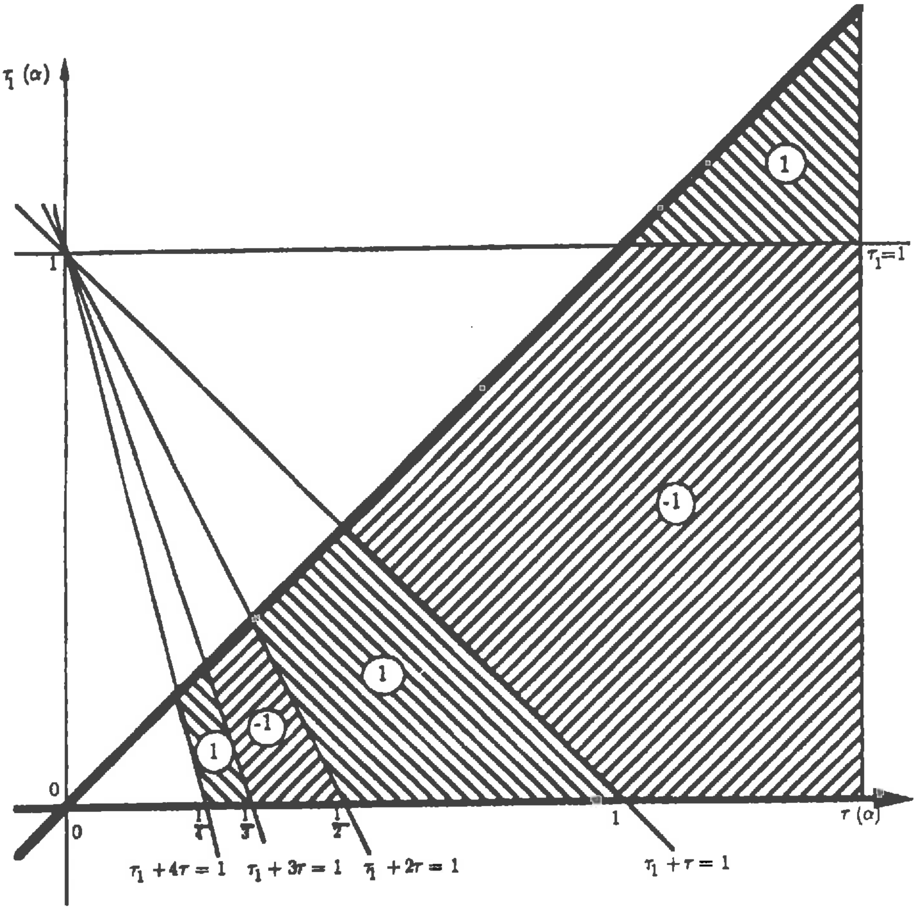

FIGURE 3. Visual statement of Lemma 2

two times (τ and τ_1), it involves four, two for each side of the line A. So we prefer to give here this simplest version and make a homotopy to a symmetric equation (see equation (19)). But in the case where non-symmetric equations are studied, like in "jumping non-linearities", the degree for non-symmetric functions is needed.

4. Application to superlinear problems

Now, we will use the continuation theorem and the computation of the degree to prove the existence of solutions for the problem

$$u''(t) + g(u(t)) = p(t, u(t), u'(t)), \tag{16}$$

$$ax(0) + bx'(0) = 0, \qquad cx(1) + dx'(1) = 0, \tag{17}$$

when g is continuous and satisfies (2) and $p : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies a linear growth condition in the last two arguments, i.e.

$$|p(t,x,y)| \le \eta|x| + \beta|y| + \gamma \qquad \text{for all } (t,x,y) \in I \times \mathbb{R}^2 \tag{18}$$

and for some constants η, β and γ. To avoid some technical problems, we suppose that $|g(x)| \ge |x|$. Thanks to the superlinearity of g, this condition is satisfied for $|x|$ sufficiently large. If this is not the case for all x, take $E = \operatorname{conv}\{x \in \mathbb{R} : |g(x)| < |x|\}$. Then the function

$$\widetilde{g}(x) = \begin{cases} x & \text{if } x \in E, \\ g(x) & \text{otherwise,} \end{cases}$$

has this property and the growth condition (18) is still valid for the function $\widetilde{p}(t,x,y) = p(t,x,y) + \widetilde{g}(x) - g(x)$. Let $f(x) = x + x^3$. This (odd) function satisfies the conditions imposed in Section 3 for the computation of the degree.

We consider the homotopy

$$u''(t) + f(u(t)) = \lambda q(t, u(t), u'(t)), \tag{19}$$

where $q(t,x,y) = p(t,x,y) - g(x) + f(x)$. For $\lambda = 1$, this is the equation (16).

Suppose for the moment that $ad - bc \neq 0$; this means that the two lines D and A are distinct. Let

$$r_1(u) = \frac{au + bu'}{|ad - bc|^{1/2}}, \qquad r_2(u) = \frac{cu + du'}{|ad - bc|^{1/2}}$$

and $\delta : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\delta(x,y) = \min\left\{1, \frac{1}{x^2 + y^2}\right\}.$$

We now define the continuous functional φ on $C^1(I, \mathbb{R}) \times I$ by

$$\varphi(u,\lambda) = \frac{2}{\pi}\left|\int_0^1 [u'(t)^2 + u(t)(f(u(t)) - \lambda q(t,u(t),u'(t)))]\delta(r_1(u)(t), r_2(u)(t))\,dt\right|.$$

If (u, λ) is a solution of (19) such that $r_1(u)(t)^2 + r_2(u)(t)^2 \geq 1$ for all $t \in [0, 1]$, we get

$$\begin{aligned}\varphi(u, \lambda) &= \frac{2}{\pi}\left|\int_0^1 \frac{u'(t)^2 - u(t)u''(t)}{r_1(u)(t)^2 + r_2(u)(t)^2}\, dt\right| \\ &= \frac{2}{\pi}\left|\int_0^1 \frac{d}{dt} \arctan\left(\frac{r_2(u)(t)}{r_1(u)(t)}\right) dt\right|.\end{aligned}$$

Thus, for a solution of (19) satisfying the boundary condition (17), it turns out that $\varphi(u, \lambda)$ counts the number of quarters of lap if this is understood as the passage from one of the lines to the other.

When we are in the "pathological" case where $ad - bc = 0$, we define φ by

$$\varphi(u, \lambda) = \frac{2}{\pi}\left|\int_0^1 [u'(t)^2 + u(t)(f(u(t)) - \lambda q(t, u(t), u'(t)))]\delta(u(t), u'(t))\, dt\right| + 1.$$

And if (u, λ) is a solution of (16)–(17) with $u(t)^2 + u'(t)^2 \geq 1$ for all $t \in I$, then

$$\varphi(u, \lambda) = \frac{2}{\pi}\left|\int_0^1 \frac{d}{dt} \arctan\left(\frac{u'(t)}{u(t)}\right) dt\right| + 1$$

is just the number of quarters of lap plus one.

Consistently with the notations of Section 2, we set

$$\Sigma^* = \{(u, \lambda) \in C^1([0, 1]) \times I : (u, \lambda) \text{ is a solution of (16)–(17)}\}.$$

We now prove some properties of the solutions and of φ. Those properties are similar to the ones of [3].

LEMMA 3. *Let $V_\lambda \in C^1(\mathbb{R}^2, \mathbb{R})$, with $(\lambda, z) \mapsto V_\lambda(z)$ continuous, be such that $\lim_{z\to\infty} |V_\lambda(z)| = +\infty$ uniformly in $\lambda \in I$. If there exist constants $K \geq 0$ and $d > 0$ such that for all $t \in I$, $\lambda \in I$ and $(x, y) \in \mathbb{R}^2$ with $|(x, y)| > K$,*

$$|V'_{\lambda x}(x, y)y + V'_{\lambda y}(x, y)(-f(x) + \lambda q(t, x, y))| \leq d|V_\lambda(x, y)|,$$

then for all $R_1 \geq 0$, there exists $R_2 \geq R_1$ such that for all $(x, \lambda) \in \Sigma^*$ with

$$\min_{t\in I} |(x(t), x'(t))| \leq R_1,$$

one has $\|x\|_1 \leq R_2$.

PROOF. We show this for $R_1 > K$. Put

$$W_\lambda(x, y) = \ln |V_\lambda(x, y)|.$$

Let $(x, \lambda) \in \Sigma^*$ be such that $\min_{t\in I} |(x(t), x'(t))| \leq R_1$. If $\|x\|_1 \leq R_1$, just take $R_2 = R_1$. Otherwise, take t_1 such that $|(x(t_1), x'(t_1))| = R_1$ and t_0 such that $|(x(t_0), x'(t_0))| = \|x\|_1$ with $|(x(t), x'(t))| > R_1$ between t_0 and t_1. Let $w(t) = W_\lambda(x(t), x'(t))$. Then

$$\begin{aligned} w(t_0) &= w(t_1) + \int_{t_1}^{t_0} w'(s)\, ds \\ &\leq w(t_1) + \left| \int_{t_1}^{t_0} |w'(s)|\, ds \right| \\ &\leq w(t_1) + \left| \int_{t_1}^{t_0} \left| \frac{\langle \nabla V_\lambda(x(s), x'(s)) | (x'(s), x''(s)) \rangle}{V_\lambda(x(s), x'(s))} \right| ds \right| \\ &\leq w(t_1) + |t_1 - t_0| d \\ &\leq c_2 + d \end{aligned}$$

with $c_2 = \sup\{W_\lambda(z) : |z| = R_1, \lambda \in I\}$. As $\lim_{z\to\infty} W_\lambda(z) = +\infty$, uniformly in λ, there exists K_2 such that $W_\lambda(z) > c_2 + d$ for all $\lambda \in I$, if $|z| \geq K_2$. So, $\|x\|_1 \leq K_2$. □

We apply this result to the equation (19).

LEMMA 4. *For each $R_1 > 0$ there is $R_2 > R_1$ such that for any $(x, \lambda) \in \Sigma^*$ verifying $\min_{t\in I} |(x(t), x'(t))| \leq R_1$ we have $\|x\|_1 \leq R_2$.*

PROOF. Let $G(x) = \int_0^x g(s)\, ds$ and

$$V_\lambda(x, y) = (1 - \lambda)F(x) + \lambda G(x) + \frac{y^2}{2} \geq \min\{F(x), G(x)\} + \frac{y^2}{2}.$$

Then $\lim_{z\to\infty} V_\lambda(z) = +\infty$ uniformly in $\lambda \in I$. We show that there exist constants d and K such that for $|(x, y)| > K$, we have $|yp(t, x, y)| \leq dV_\lambda(x, y)$. Indeed, for $|(x, y)|$ large enough,

$$|yp(t, x, y)| \leq |y|(\eta|x| + \beta|y| + \gamma) \leq M(x^2 + y^2) \leq 2MV_\lambda(x, y)$$

because $F(x)$ and $G(x)$ are greater than $x^2/2$. We can then apply the preceding lemma. □

For $ad - bc \neq 0$, the function $((au + bv)^2 + (cu + dv)^2)/|ad - bc|$ is a quadratic form. Let us denote by μ_- and μ_+ the (positive) eigenvalues associated with this form. So if $u^2 + v^2 > 1/\mu_-$, the quadratic form is greater than 1. We take $R = R_2(1/\mu_-)$ (for the particular case where $ad - bc = 0$, $R = R_2(1)$). So we have proved the following lemma.

LEMMA 5. *There is a $R > 0$ such that for any $(x, \lambda) \in \Sigma^*$ verifying $\|x\|_1 > R$ there exist a $k \in \mathbb{N}$ such that we have $\varphi(u, \lambda) = 2k + 1$.*

LEMMA 6. *For each $N > 0$ there is $R_1(N) > 0$ such that for all $(u, \lambda) \in \Sigma^*$ verifying $\min_{t\in I} |(u(t), u'(t))| \geq R_1(N)$ we have $\varphi(u, \lambda) \geq N$.*

PROOF. By superlinearity of f and g and linear growth of p, for all $K_0 > 1 + \eta + \beta^2/2$, there exists $c_{K_0} > 0$ such that

$$\begin{aligned} x(f(x) - \lambda q(t, x, y)) + y^2 &\geq K_0 x^2 - c_{K_0} - \eta x^2 - \beta|x||y| - \gamma|x| + y^2 \\ &\geq K_0 x^2 - c_{K_0} - \eta x^2 - \frac{\beta}{2}\left(\beta x^2 + \frac{y^2}{\beta}\right) - \gamma|x| + y^2 \\ &\geq \frac{y^2}{2} + \left(K_0 - \eta - \frac{\beta^2}{2}\right)x^2 - \gamma|(x, y)| - c_{K_0} \\ &\geq \frac{K}{2}x^2 + \frac{y^2}{2} - \gamma|(x, y)| - c_{K_0}, \end{aligned}$$

where we put $K = 2(K_0 - \eta - \beta^2/2)$.

If $(u, \lambda) \in \Sigma^*$ and $\min_{t\in I} |(u(t), u'(t))| \geq \max\{\sqrt{c_{K_0}}K^{1/4}, \gamma K^{1/2}\}$, passing to the polar coordinates, we have

$$\begin{aligned} -\theta'(t) &= \frac{u(t)(f(u(t)) - \lambda q(t, u(t), u'(t))) + u'(t)^2}{|(u(t), u'(t))|^2} \\ &\geq \frac{K}{2}\left(\frac{u(t)^2 + u'(t)^2/K}{|(u(t), u'(t))|^2}\right) - 2K^{-1/2}. \end{aligned}$$

If we let

$$\Theta(x, y) = \frac{x^2 + y^2/K}{|(x, y)|^2}$$

and

$$\sigma = \min_{(x,y)\in\mathbb{R}^2} \Theta(x, y) = 1/K,$$

then

$$\begin{aligned} -\int_{\theta(0)}^{\theta(1)} \frac{1}{\Theta(\cos\theta, \sin\theta)}\, d\theta &= \int_0^1 \frac{-\theta'(t)}{\Theta(\cos\theta(t), \sin\theta(t))}\, dt \\ &\geq \frac{K}{2} - 2K^{-1/2}\frac{1}{\sigma} \\ &\geq \frac{K}{2} - 2\sqrt{K}. \end{aligned}$$

So if $\|u\|_1 > R$ with R given by Lemma 5, then we have $(\theta(0) - \theta(1) + 2\pi)\sqrt{K} \geq K/2 - 2\sqrt{K}$, and as $(\varphi(u, \lambda) + 1)\pi/2 \geq \theta(0) - \theta(1)$,

$$\varphi(u, \lambda) \geq \frac{\sqrt{K}}{\pi} - \frac{4}{\pi} - 5 = \alpha(K).$$

Then, for N fixed, we take K_0 such that $\alpha(K) \geq N$. So if $\min_{t\in I} |(u(t), u'(t))| \geq \max\{\sqrt{c_{K_0}}K^{1/4}, \gamma K^{1/2}\}$, then $\varphi(u, \lambda) \geq N$. □

We can now prove the main theorem of this section.

THEOREM 4. *Assume that* $g : \mathbb{R} \to \mathbb{R}$ *is continuous and superlinear* (*see* (2)), *and that* $p : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ *is continuous and has at most linear growth in the last two variables* (*see* (18)). *Then there exists* $k_0 \in \mathbb{N}$ *such that, for each* $j > k_0$, *the problem* (16)–(17) *has at least one solution* u_j *such that* $\varphi(u_j, 1) \in]2j, 2(j+1)[$. *Moreover,* $\|u_j\| \to \infty$ *as* $j \to \infty$.

PROOF. We apply Corollary 1 with the abstract functional setting needed to write problem (16)–(17) in the form $Lx = N(x, \lambda)$, according to [13] and Section 3.

Take $c_k = 2k$. Hence condition (i_4) is satisfied with R given in Lemma 5. For condition (i_5), suppose that $(x, \lambda) \in \Sigma^*$ and $\varphi(x, \lambda) < c_n$. Then we have $\min |(u(t), u'(t))| < R_1(c_n)$ and thus $\|x\|_1 < R_2(R_1(c_n))$ and $\|x\|_1$ is bounded.

Moreover, $D_L(L - N(\cdot, 0), \mathcal{O}^k) = D_L(L - N(\cdot, 0), \Omega^\alpha \setminus \overline{\Omega^{\alpha'}})$ where α and α' are such that $(k+1)\tau(\alpha) = 1$ and $k\tau(\alpha') = 1$ ($(k+1/2)\tau(\alpha) = 1$ and $(k-1/2)\tau(\alpha') = 1$ in the "pathological" case). Then $D_L(L - N(\cdot, 0), \mathcal{O}^k) = -2\sigma(-1)^k$. Therefore, all the conditions of Corollary 1 are satisfied. □

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