Topological **M**ethods in **N**onlinear **A**nalysis
Volume 60, No. 2, 2022, 491–516
DOI: 10.12775/TMNA.2022.005

THE BORSUK–ULAM PROPERTY FOR HOMOTOPY CLASSES OF MAPS FROM THE TORUS TO THE KLEIN BOTTLE PART 2

Daciberg Lima Gonçalves — John Guaschi
Vinicius Casteluber Laass

Abstract. Let M be a topological space that admits a free involution τ, and let N be a topological space. A homotopy class $\beta \in [M, N]$ is said to have *the Borsuk–Ulam property with respect to* τ if for every representative map $f\colon M \to N$ of β, there exists a point $x \in M$ such that $f(\tau(x)) = f(x)$. In this paper, we determine the homotopy class of maps from the 2-torus $\mathbb{T}^2$ to the Klein bottle $\mathbb{K}^2$ that possess the Borsuk–Ulam property with respect to any free involution of $\mathbb{T}^2$ for which the orbit space is $\mathbb{K}^2$. Our results are given in terms of a certain family of homomorphisms involving the fundamental groups of $\mathbb{T}^2$ and $\mathbb{K}^2$. This completes the analysis of the Borsuk–Ulam problem for the case $M = \mathbb{T}^2$ and $N = \mathbb{K}^2$, and for any free involution τ of $\mathbb{T}^2$.

1. Introduction

The classical Borsuk–Ulam theorem states that for all $n \in \mathbb{N}$ and any continuous map $f\colon \mathbb{S}^n \to \mathbb{R}^n$, there exists a point $x \in \mathbb{S}^n$ such that $f(-x) = f(x)$

2020 *Mathematics Subject Classification.* Primary: 55M20, 57M07; Secondary: 20F36.
Key words and phrases. Borsuk–Ulam theorem; homotopy class; braid groups; surfaces.
This paper was completed during the Postdoctoral Internship of the third author at IME-USP from March 2020 to August 2021. He was supported by Capes/INCTMat project no. 8887.136371/2017-00-465591/2014-0.
The first author is partially supported by the Projeto Temático FAPESP, *Topologia Algébrica, Geométrica e Diferencial*, grant no. 2016/24707-4.