Topological **M**ethods in **N**onlinear **A**nalysis
Volume 60, No. 1, 2022, 185–201
DOI: 10.12775/TMNA.2022.004

EMBEDDABILITY OF JOINS AND PRODUCTS OF POLYHEDRA

Sergey A. Melikhov

Abstract. We present a short proof of S. Parsa's theorem that there exists a compact n-polyhedron P, $n \geq 2$, non-embeddable in $\mathbb{R}^{2n}$, such that $P * P$ embeds in $\mathbb{R}^{4n+2}$. This proof can serve as a showcase for the use of geometric cohomology. We also show that a compact n-polyhedron X embeds in $\mathbb{R}^m$, $m \geq 3(n+1)/2$, if either

- $X * K$ embeds in $\mathbb{R}^{m+2k}$, where K is the $(k-1)$-skeleton of the $2k$-simplex; or
- $X * L$ embeds in $\mathbb{R}^{m+2k}$, where L is the join of k copies of the 3-point set; or
- X is acyclic and $X \times (\text{triod})^k$ embeds in $\mathbb{R}^{m+2k}$.

1. Introduction

It was shown by Flores, van Kampen and Grünbaum [9] that every n-dimensional join of k_i-skeleta of $(2k_i+2)$-simplexes does not embed into $\mathbb{R}^{2n}$ (see also [11, Examples 3.3, 3.5], [12], [20]). Some other k_i-polyhedra with this property are constructed in [12].

As noted by S. Parsa [15], it is implicit in a paper by Bestvina, Kapovich and Kleiner [5] that if compact polyhedra P^n and Q^m both have non-zero mod 2 van Kampen obstruction, then $P * Q$ does not embed in $\mathbb{R}^{2(n+m+1)}$. An n-dimensional polyhedron, non-embeddable in $\mathbb{R}^{2n}$ but with vanishing mod 2 van Kampen obstruction was constructed by the author for each $n \geq 2$ [11], settling

2020 *Mathematics Subject Classification.* Primary: 57Q35; Secondary: 57N35.
Key words and phrases. Polyhedron; embedding; join; the van Kampen obstruction.