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ON CARISTI FIXED POINT THEOREM FOR SET-VALUED MAPPINGS

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Abstract. The aim of this paper is to discuss Penot's problem on a generalization of Caristi's fixed point theorem. We settle this problem in the negative and we present some new theorems on the existence of fixed points of set-valued mappings in ordered metric spaces and reflexive Banach spaces.

1. Introduction

The Caristi fixed point theorem is known as one of the most important results in metric fixed point theory [6]. It is not only a generalization of the Banach contraction principle [4] but it has also been proven to be equivalent to metric completeness [14, Theorem 6]. Moreover, it has been the subject of various generalizations and extensions (see e.g. [1], [5], [7] and the related references therein). For instance, in attempting to generalize Caristi's fixed point theorem, Kirk [12] raised the problem of whether a self-mapping T has a fixed point on a metric space (M, d) such that for all $x \in M$

$$\eta(d(x, Tx)) \leq \phi(x) - \phi(Tx),$$

where η is a function from $\mathbb{R}_+$, the set of all nonnegative reals, into $\mathbb{R}_+$, having appropriate properties. This problem has been settled in the negative by Khamsi in [12]. However, in order to generalize Caristi's fixed point theorem many works

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