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MULTIPLE SOLUTIONS TO THE BAHRI–CORON PROBLEM IN A BOUNDED DOMAIN WITHOUT A THIN NEIGHBORHOOD OF A MANIFOLD

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Abstract. We show that the critical problem

$$-\Delta u = |u|^{4/(N-2)}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

has at least

$$\max\{\operatorname{cat}(\Theta, \Theta \setminus B_r M), \operatorname{cupl}(\Theta, \Theta \setminus B_r M) + 1\} \geq 2$$

pairs of nontrivial solutions in every domain Ω obtained by deleting from a given bounded smooth domain $\Theta \subset \mathbb{R}^N$ a thin enough tubular neighborhood $B_r M$ of a closed smooth submanifold M of Θ of dimension $\leq N-2$, where "cat" is the Lusternik–Schnirelmann category and "cupl" is the cup-length of the pair.

1. Introduction

Let Θ be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, and let M be a compact smooth submanifold of $\mathbb{R}^N$, without boundary, contained in Θ. Consider the problem

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \Theta_r, \\ u = 0 & \text{on } \partial\Theta_r, \end{cases} \tag{1.1}$$

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where $2^* := 2N/(N-2)$ is the critical Sobolev exponent and

$$\Theta_r := \{x \in \Theta : \operatorname{dist}(x, M) > r\}, \quad r > 0.$$

Our aim is to establish multiplicity of solutions for r small.

If M is a point and r is small enough, Coron showed in [9] that this problem has at least one positive solution. The existence of at least two solutions was established by Clapp and Weth in [8]. More recently, Ge, Musso and Pistoia [14] proved that the number of sign changing solutions becomes arbitrarily large as r goes to zero. Their solutions are bubble-towers, i.e. they look like superpositions of standard bubbles with alternating signs concentrating at the point M. Under additional assumptions, positive and sign changing solutions which look like a sum of standard bubbles one of which concentrates at the point M and the others at some points in $\Theta \setminus M$ were constructed in [6]. There are also various results on the existence and shape of solutions to this problem when M is a finite set of points and r is small enough, see e.g. [16], [17], [18], [19].

In contrast to this, if M has positive dimension only few results are known. Hirano and Shioji established the existence of two solutions in an annular domain with a thin straight tunnel in [15]. Some multiplicity results were recently obtained by Clapp, Grossi and Pistoia in [5] when both Θ and M are invariant under the action of some group of symmetries. They also showed that, without any symmetry assumption, this problem has at least $\operatorname{cat}(\Theta, \Theta_r)$ positive solutions for small enough r, where $\operatorname{cat}(\Theta, \Theta_r)$ is the Lusternik-Schnirelmann category of the pair (Θ, Θ_r).

Here we show that for some domains there is an additional solution. We write $\operatorname{cupl}(\Theta, \Theta_r)$ for the cup-length of the pair (Θ, Θ_r). The definitions of category and cup-length are given in appendix A. We prove the following result.

Theorem 1.1. *Assume that* $\dim M \leq N-2$. *Then there exists* $r_0 > 0$ *such that, if* Ω *is a bounded smooth domain in* $\mathbb{R}^N$ *which satisfies*

$$M \cap \overline{\Omega} = \emptyset \quad \text{and} \quad \Theta_r \subset \Omega \subset \Theta,$$

for some $r \in (0, r_0)$, *then problem*

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

has at least $\max\{\operatorname{cat}(\Theta, \Theta_r), \operatorname{cupl}(\Theta, \Theta_r) + 1\} \geq 2$ *pairs of nontrivial solutions.*

It is well known that $\operatorname{cat}(\Theta, \Theta_r) \geq \operatorname{cupl}(\Theta, \Theta_r)$ (see Lemma A.3). So Theorem 1.1 improves Corollary 1.2 in [5] when $\operatorname{cat}(\Theta, \Theta_r) = \operatorname{cupl}(\Theta, \Theta_r)$. There are some interesting situations in which this occurs. For example, the following ones.