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MULTIPLICITY OF SOLUTIONS OF ASYMPTOTICALLY LINEAR DIRICHLET PROBLEMS ASSOCIATED TO SECOND ORDER EQUATIONS IN $\mathbb{R}^{2n+1}$

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Abstract. We present a result about multiplicity of solutions of asymptotically linear Dirichlet problems associated to second order equations in $\mathbb{R}^{2n+1}$, $n \geq 1$. Under an additional technical condition, the number of solutions obtained is given by the gap between the Morse indexes of the linearizations at zero and infinity. The additional condition is stable with respect to small perturbations of the vector field. We show with a simple example that in some cases the size of the perturbation can be explicitly estimated.

1. Introduction

In this paper we are interested on the existence of multiple solutions to the problem

$$\begin{aligned} x'' + A(t,x)x &= 0, \\ x(0) = x(\pi) &= 0, \end{aligned} \tag{1.1}$$

$x \in \mathbb{R}^{2n+1}$, $n \geq 1$, $t \in [0,\pi]$.

We will assume that $A\colon [0,\pi] \times \mathbb{R}^{2n+1} \to \mathrm{GL}_s(\mathbb{R}^{2n+1})$ is a continuous function with values in the set of the real symmetric matrices of order $2n+1$, denoted

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by $\mathrm{GL}_s(\mathbb{R}^{2n+1})$, and that there exist $A_i(\,\cdot\,)\colon [0,\pi] \to \mathrm{GL}_s(\mathbb{R}^{2n+1})$, $i = 0, \infty$, also continuous such that

$$\begin{aligned} \lim_{|x|\to 0} A(t,x) &= A_0(t) \quad \text{uniformly in } t \in [0,\pi], \\ \lim_{|x|\to \infty} A(t,x) &= A_\infty(t) \quad \text{uniformly in } t \in [0,\pi], \end{aligned}$$

that is, we assume asymptotically linear conditions at the origin and at infinity. If the indexes of $A_0(\,\cdot\,)$, $i(A_0)$, and of $A_\infty(\,\cdot\,)$, $i(A_\infty)$, are different we prove, under an extra technical assumption, the existence of multiple solutions to the boundary value problem.

The existence of solutions of boundary value problems associated to asymptotically linear problems is a subject which has been studied by many authors for more than 30 years. We mention the seminal papers by Amann and Zehnder [1], [2] as maybe the first in which existence of solutions to asymptotically linear boundary value problems was studied. In those papers the authors introduce an index depending on the linearizations at zero and infinity and prove, when that index is positive, the existence of one or, in nondegenerate cases, two solutions. After those works many authors studied this problem both for Hamiltonian systems and for second order equations, assuming periodic or two-point boundary conditions (see [5] for details on the bibliography). We note that, for scalar second order equations, the multiplicity of solutions for the Dirichlet problem in terms of the gap between the Morse indexes of the linearizations can be easily proved using the link between Morse index and rotation number (see [9], where this approach was used for the Neumann and the periodic BVPs). For higher dimensions and when no symmetry or convexity conditions are assumed usually the existence of at most two solutions is guaranteed. One of the exceptions is [10] where multiple periodic solutions to a planar Hamiltonian system were obtained under no additional conditions by using the Poincaré–Birkhoff theorem. Also, in what concerns second order equations recently, in [5], (1.1) was considered for $x \in \mathbb{R}^2$ and multiple solutions were obtained for the Dirichlet problem assuming sign conditions on the entries of $A(t,x)$. In both of these papers the number of solutions increases when the gap of the indexes of the linearizations at zero and at infinite (Maslov index in the first paper, Morse index in the second) increases.

As far as we know in the case of second order equations in dimension larger than two, multiple solutions were only obtained in [4]. However the number of solutions obtained depends on the number of elements of a set which one has to check to be nonempty.

In this paper we consider the problem in $\mathbb{R}^{2n+1}$, $n \geq 1$ and obtain a result similar to that in [5] by imposing a technical assumption concerning the space of solutions of some linear problems associated to $x''+A(t,x)x = 0$, see Theorem 2.3