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FUNCTIONS AND VECTOR FIELDS ON $C(\mathbb{C}P^n)$-SINGULAR MANIFOLDS

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Abstract. In this paper we study functions and vector fields with isolated singularities on a $C(\mathbb{C}P^n)$-singular manifold. In general, a $C(\mathbb{C}P^n)$-singular manifold is obtained from a smooth $(2n+1)$-manifold with boundary which is a disjoint union of complex projective spaces $\mathbb{C}P^n \cup \ldots \cup \mathbb{C}P^n$ and subsequent capture of the cone over each component $\mathbb{C}P^n$ of the boundary. We calculate the Euler characteristic of a compact $C(\mathbb{C}P^n)$-singular manifold M^{2n+1} with finite isolated singular points. We also prove a version of the Poincaré–Hopf Index Theorem for an almost smooth vector field with finite number of zeros on a $C(\mathbb{C}P^n)$-singular manifold.

1. Introduction

A manifold with isolated singularities is a topological space M which has the structure of a smooth (C^∞) manifold in $M \setminus S$, where S is the discrete set of singular points of M. A diffeomorphism between two such manifolds M and N is a homeomorphism from M into N such that sends the set of singular points of M onto the set of singular points of N and is a diffeomorphism outside of them. We say that M has a cone-like singularity at a (singular) point $P \in S$ if there exists a neighbourhood of the point P diffeomorphic to a cone over

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a smooth manifold W_P (W_P is called the link at the point P). In what follows we assume all manifolds have only isolated cone-like singularities, more precisely $C(\mathbb{C}P^n)$-singular manifolds.

In this work it is considered almost Morse functions on $C(\mathbb{C}P^n)$-singular manifolds and it is given an answer for a particular case of the following unsolved problem: for any $C(\mathbb{C}P^n)$-singular manifold M^{2n+1} with singular points $m_1, \ldots, m_k$ and any collection of almost Morse functions $St = \pi_*(f_1), \ldots, \pi_*(f_k)$ in the neighbourhoods $U(m_1), \ldots, U(m_k)$ *find exact values* of Morse number $\mathcal{M}_\lambda(M^{2n+1}, St)$ of index λ. We point out that the notion of almost Morse function is close related to the notion of a stratified Morse function. (See the classical book of Goresky–MacPherson [7].)

In this setting, one has the following result:

THEOREM 3.10. *Let M^{2n+1}, $(2n \geq 5)$, be a compact simply connected $C(\mathbb{C}P^n)$-singular manifold with singular points $m_1, \ldots, m_k$. Let σ be a permutation of $(1, \ldots, k)$ and let A (with s points) and B (with $k-s$ points) be the split of the singular points $m_1, \ldots, m_k$ into two disjoint sets*:

$$A = m_{\sigma(1)}, \ldots, m_{\sigma(s)}, \qquad B = m_{\sigma(s+1)}, \ldots, m_{\sigma(k)}.$$

We fix a collection of almost Morse functions

$$St = \underbrace{\pi_*(f_1), \ldots, \pi_*(f_1)}_{s}, \underbrace{\pi_*(f_2), \ldots, \pi_*(f_2)}_{k-s}$$

in the neighbourhoods $U(m_{\sigma(1)}), \ldots, U(m_{\sigma(s)}), U(m_{\sigma(s+1)}), \ldots, U(m_{\sigma(k)})$ respectively, where

$$f_1 = \sum_{i=1}^{2n} |z_i|^2, \qquad f_2 = 1 - \sum_{i=1}^{2n} |z_i|^2.$$

Then

$$\mathcal{M}_\lambda(M^{2n+1}, St) = \mu(H_\lambda(M^{2n+1} \setminus B, A, \mathbb{Z})) + \mu(\operatorname{Tors} H_{\lambda-1}(M^{2n+1} \setminus B, A, \mathbb{Z})),$$

where $\mu(H)$ is the minimal number of generators of the group H.

A (smooth) vector field on a manifold M with isolated singularities is a (smooth) vector field on the set of regular points of M. The set of singular points S_X of a vector field X on a (singular) manifold M is the union of the set of usual singular points of X on M (i.e. points at which X tends to zero) and of the set S of singular points of M itself. For an isolated usual singular point P of a vector field X there is defined its index $\mathrm{ind}_P X$.

Inspired on the book "Vector Fields on Singular Varieties" [5] we study vector fields on manifolds with isolated cone-like singularities and present a proof of a version of the Poincaré–Hopf Theorem. We recall that M.-H. Schwartz was the first to consider the index of vector fields on singular varieties. For her purposes,