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EQUATION WITH POSITIVE COEFFICIENT IN THE QUASILINEAR TERM AND VANISHING POTENTIAL

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Abstract. In this paper we study the existence of nontrivial classical solution for the quasilinear Schrödinger equation:

$$-\Delta u + V(x)u + \frac{\kappa}{2}\Delta(u^2)u = f(u),$$

in $\mathbb{R}^N$, where $N \geq 3$, f has subcritical growth and V is a nonnegative potential. For this purpose, we use variational methods combined with perturbation arguments, penalization technics of Del Pino and Felmer and Moser iteration. As a main novelty with respect to some previous results, in our work we are able to deal with the case $\kappa > 0$ and the potential can vanish at infinity.

1. Introduction

In this article, we consider the following quasilinear Schrödinger equations

$$-\Delta u + V(x)u + \frac{\kappa}{2}\Delta(u^2)u = f(u), \quad x \in \mathbb{R}^N \tag{1.1}$$

where $V \colon \mathbb{R}^N \to \mathbb{R}$ and $f \colon \mathbb{R} \to \mathbb{R}$ are continuous functions with V being a nonnegative function, f having a kind of subcritical growth at infinity and $\kappa > 0$ is a parameter.

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This equation arises in various branches of mathematical physics and has been the subject of extensive study in recent years. As it is well known, solutions of (1.1) are related to the existence of a standing wave solutions for quasilinear Schrödinger equation of the form:

$$i\partial_t z = -\Delta z + W(x)z - l(|z|^2)z + \frac{\kappa}{2}[\Delta\rho(|z|^2)]\rho'(|z|^2)z, \tag{1.2}$$

where $z\colon \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$, $W\colon \mathbb{R}^N \to \mathbb{R}$ is a given potential and l, ρ are real functions.

Quasilinear Schrödinger equations of the form (1.2) appear naturally in mathematical physics and have been derived as mathematical models of several physical phenomena corresponding to various types of the nonlinear term ρ. The case $\rho(s) = s$ was used for the superfluid film equation in plasma physics by Kurihara in [21]. In the case $\rho(s) = (1+s)^{1/2}$, considering solutions of the form $z(t,x) = e^{-i\xi t}u(x)$ where ξ is some real constant, equation (1.2) models the self-channeling of a highpower ultra short laser in matter, see [13], [16] and references in [18]. It is clear that $z(t,x)$ solves (1.2) if and only if $u(x)$ solves (1.1) with $V(x) = W(x) - \xi$ and $f(u) = l(u^2)u$.

Taking into account the values of κ, we find in the literature several papers devoted to the existence of solutions for equation (1.1) when the potential V vanishes at infinity.

The semilinear case corresponding to $\kappa = 0$, that is,

$$-\Delta u + V(x)u = f(u), \quad x\in\mathbb{R}^N, \tag{1.3}$$

has been studied extensively. See for example [3]–[7], [9]–[12], [14], [20] and the references therein. Among them, we recall the article due to Berestycki and Lions [12] that showed the existence of a positive solution in the case $V \equiv 0$, where the nonlinearity has a supercritical growth near the origin and subcritical growth at infinity. In [20] Ghimenti and Micheletti established existence of sign changing solutions. In [10] Benci, Grisanti and Micheletti established additional conditions on V which provide existence or non existence of the ground state solution. In the papers of Ambrosetti, Felli and Malchiodi [5], Ambrosetti and Wang [7], the nonlinearity $f(u)$ is replaced by a function $f(x,u)$ of the type $k(x)|u|^p$ where $k(x)\to 0$ as $|x|\to\infty$. In [3], Alves and Souto have introduced a new set of hypotheses on the potential V to show the existence of positive solution for (1.3) where f has a subcritical growth.

In the literature we also may cite the article due to Bastos, Miyagaki and Vieira [8] that has established the existence of positive solution for the following class of degenerate quasilinear elliptic problem

$$-\mathcal{L}u_{ap} + V(x)|x|^{-ap^*}|u|^{p-2}u = f(u), \quad \text{in } \mathbb{R}^N,$$