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SOLUTIONS TO A NONLINEAR SCHRÖDINGER EQUATION WITH PERIODIC POTENTIAL AND ZERO ON THE BOUNDARY OF THE SPECTRUM

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Abstract. We study the following nonlinear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = g(x,u) & \text{for } x \in \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases}$$

where $V\colon \mathbb{R}^N \to \mathbb{R}$ and $g\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are periodic in x. We assume that 0 is a right boundary point of the essential spectrum of $-\Delta + V$. The superlinear and subcritical term g satisfies a Nehari type monotonicity condition. We employ a Nehari manifold type technique in a strongly indefitnite setting and obtain the existence of a ground state solution. Moreover, we get infinitely many geometrically distinct solutions provided that g is odd.

1. Introduction

We are concerned with the following nonlinear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = g(x,u) & \text{for } x \in \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

where $V\colon \mathbb{R}^N \to \mathbb{R}$ is a periodic potential and $g\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ has superlinear growth. This equation appears in mathematical physics, e.g. when one studies

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standing waves $\Phi(x,t)=u(x)e^{-iEt/\hbar}$ of the time-dependent Schrödinger equation of the form

$$i\hbar\frac{\partial\Phi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\Phi+W(x)\Phi-f(x,|\Phi|)\Phi.$$

If the potential V is periodic, then (1.1) is of particular interest since it has a wide range of physical applications, e.g. in photonic crystals, where one considers periodic optical nanostructures (see [18] and references therein). It is well-known that the spectrum $\sigma(-\Delta+V)$ of $-\Delta+V$ is purely continuous and may contain gaps, i.e. open intervals free of spectrum (see [21]). When $\inf\sigma(-\Delta+V)>0$ or 0 lies in a gap of the spectrum $\sigma(-\Delta+V)$ then nonlinear Schrödinger equations have been widely investigated by many authors (see [8], [20], [1], [7], [26], [13], [9] and references therein) and nontrivial solutions to (1.1) have been obtained. Ground state solutions, i.e. nontrivial solutions with the least possible energy, play an important role in physics and their existence has been studied e.g. in [14], [18], [24], [15]. If $V=0$ then $\sigma(-\Delta+V)=[0,+\infty)$ and the problem has been investigated in a classical work [6] or in a recent one [2] (see also references therein). If V is constant and negative then 0 is an interior point of $\sigma(-\Delta+V)$ and solutions to (1.1) have been found in [10].

In the present work, we focus on the situation when 0 lies in the spectrum of $-\Delta+V$ and is the left endpoint of a spectral gap. As far as we know there are only three papers dealing with this case. In [4] Bartsch and Ding obtained a nontrivial solution to (1.1) assuming, among others, the following Ambrosetti–Rabinowitz condition:

$$g(x,u)u\ge\gamma G(x,u)>0\quad\text{for some } \gamma>2 \text{ and all } u\in\mathbb{R}\setminus\{0\},\ x\in\mathbb{R}^N, \tag{1.2}$$

and a lower bound estimate:

$$G(x,u)\ge b|u|^\mu\quad\text{for some } b>0,\ \mu>2 \text{ and all } u\in\mathbb{R},\ x\in\mathbb{R}^N, \tag{1.3}$$

where G is the primitive of g with respect to u. Applying a generalized linking theorem due to Kryszewski and Szulkin [13], they proved that there is a solution in $H^2_{\mathrm{loc}}(\mathbb{R}^N)\cap L^t(\mathbb{R}^N)$ for $\mu\le t\le 2^*$, where $2^*=2N/(N-2)$ if $N\ge 3$, and $2^*=\infty$ if $N=1,2$. If g is odd then the existence of infinitely many geometrically distinct solutions was obtained as well by means of an abstract critical point theory involving the $(\mathrm{PS})_I$-attractor concept (see Section 4 in [4] for details). In [28] Willem and Zou relaxed condition (1.2) and they dealt with the lack of boundedness of Palais–Smale sequences. The authors developed the so-called monotonicity trick for strongly indefinite problems and established weak linking results. Recently Yang, Chen and Ding in [29] considered a Nehari-type monotone condition (see (G5) below) instead of (1.2) and obtained a solution to (1.1) using a variant of weak linking due to Schechter and Zou [23]. The lower bound estimate (1.3) has been assumed so far.