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THE R_∞ PROPERTY FOR ABELIAN GROUPS

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Abstract. It is well known there is no finitely generated abelian group which has the R_∞ property. We will show that also many non-finitely generated abelian groups do not have the R_∞ property, but this does not hold for all of them! In fact we construct an uncountable number of infinite countable abelian groups which do have the R_∞ property. We also construct an abelian group such that the cardinality of the Reidemeister classes is uncountable for any automorphism of that group.

1. Introduction

Let G be a group and φ be an endomorphism of G. Then two elements x, y of G are said to be Reidemeister equivalent (with respect to φ), if there exists an element $z \in G$ such that $y = zx\varphi(z)^{-1}$. The equivalence classes are called the Reidemeister classes or twisted conjugacy classes.

Definition 1.1. The Reidemeister number of a homomorphism φ, denoted by $R(\varphi)$, is the cardinality of the Reidemeister classes of φ.

We remark here that most authors define the Reidemeister number as either a positive integer or ∞. This latter definition of course coincides with ours in the finite case, but does not allow to make a distinction between the various infinite cases.

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The Reidemeister number is a relevant ingredient in connection with many parts of mathematics. See for example [4] and references therein. This is for instance also the case in the study of the fixed point properties of the homotopy class of a self map on a topological space. In this situation, the group G will be the fundamental group $\pi_1(X)$ of the space and the homomorphism $\varphi = f_\sharp$ is the one which is induced by the map f on the fundamental group G. Under certain hypothesis the Reidemeister number $R(\varphi)$ is then exactly the number of essential fixed point classes of f if $R(\varphi)$ is finite and the number of essential fixed point classes of f is zero if $R(\varphi)$ is infinite. See [7] and [12] and the references therein for more information.

A group G has the R_∞ property if for every automorphism φ of G the Reidemeister number is not finite. In recent years many works have studied the question of which groups G have the R_∞ property. We refer to [4] for an overview of the results which have been obtained in this direction. Some more recent results, e.g. dealing with (extensions of) linear groups and lattices in semisimple Lie groups, can be found in [3], [9], [10], [11]. In [1], we treated the case of free nilpotent and free solvable groups. It turns out that in this sitution, infinitely generated groups behave quite differently than finitely generated groups.

The present work will also give a contribution for this problem, where we will consider infinite abelian groups. If an abelian group A is finitely generated then it is well known that A does not have the R_∞ property, since it is easy to see that the automorphism $\varphi\colon A \to A\colon a \mapsto -a$ has a finite Reidemeister number in this case ([2]). So, in this paper, we will focus on abelian groups which are not finitely generated. For information about infinite abelian groups in general we refer to [5], [6] and [8].

To the best of our knowledge, up till now, there is no example in literature of an abelian group having the R_∞ property. In this paper we do construct an uncountable number of countable abelian groups which do have the R_∞ property.

Before we announce the main results of this paper, let us fix some notation:

• Let p be a prime, then with $\mathbb{Z}_p$, we will denote the additive group of p-adic integers.

• For any positive integer $n \geq 2$, $\mathbb{Z}/n\mathbb{Z}$ will denote the additive group of integers modulo n.

• Let $\mathcal{P}$ be any set of primes, then $\mathbb{Z}_{\mathcal{P}}$ denotes the additive group of rational numbers which can be written as a fraction whose denominator is relative prime with all primes in $\mathcal{P}$. When p is a prime, then $\widehat{p}$ is the set of all primes which are different from p and hence $\mathbb{Z}_{\widehat{p}}$ is the group of all rational numbers whose denominator is a power of p.

• Finally, when p is a prime $\mathbb{Z}(p^\infty)$ is the Prüfer group $\mathbb{Z}_{\widehat{p}}/\mathbb{Z}$.