Topological **M**ethods in **N**onlinear **A**nalysis
Volume 46, No. 2, 2015, 731–754
DOI: 10.12775/TMNA.2015.065

POSITIVE SOLUTIONS TO p-LAPLACE REACTION-DIFFUSION SYSTEMS WITH NONPOSITIVE RIGHT-HAND SIDE

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Abstract. The aim of the paper is to show the existence of positive solutions to the elliptic system of partial differential equations involving the p-Laplace operator

$$\begin{cases} -\Delta_p u_i(x) = f_i(u_1(x), u_2(x), \ldots, u_m(x)), & x \in \Omega,\ 1 \le i \le m, \\ u_i(x) \ge 0, & x \in \Omega,\ 1 \le i \le m, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

We consider the case of nonpositive right-hand side f_i, $i = 1, \ldots, m$. The sufficient conditions entails spectral bounds of the matrices associated with $f = (f_1, \ldots, f_m)$. We employ the degree theory from [5] for tangent perturbations of maximal monotone operators in Banach spaces.

1. Introduction

In the recent paper [5] the following nonlinear boundary value problem was discussed:

$$\begin{cases} -\Delta_p u(x) = f(x, u(x)), & x \in \Omega, \\ u(x) \ge 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

2010 *Mathematics Subject Classification.* Primary: 35J57; Secondary: 35B09, 35D30, 35J92, 47H10.

Key words and phrases. Degree theory, tangency condition, cone, quasilinear elliptic system, positive weak solution.

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$, for $p \geq 2$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is called the p-Laplace operator or p-Laplacian and $f\colon \Omega \times [0,+\infty) \to \mathbb{R}$ is a Carathéodory function, which is *not* necessarily positive. The authors, exploiting the idea originated in [4], introduced the topological degree Deg_M for mappings that are tangent to the set of constraints M and they applied it to obtain sufficient conditions under which the problem (1.1) possesses at least one weak solution:

Theorem 1.1. *Suppose that the following conditions are satisfied*:

(a) *there is* $C > 0$ *such that* $|f(x,s)| \leq C(1+s^{p-1})$ *for all* $s \geq 0$ *and alomost all* $x \in \Omega$,

(b) $\displaystyle\lim_{s\to 0+} \frac{f(x,s)}{s^{p-1}} = \rho_0(x)$ *and* $\displaystyle\lim_{s\to\infty} \frac{f(x,s)}{s^{p-1}} = \rho_\infty(x)$ *uniformly with respect to* $x \in \Omega$,

where $\rho_0, \rho_\infty \in L^\infty(\Omega)$. *If the principal eigenvalue* $\lambda_{1,p}$ *of* p*-Laplacian lies between* ρ_0 *and* ρ_∞, *i.e. either* $\rho_0 < \lambda_{1,p} < \rho_\infty$ *or* $\rho_\infty < \lambda_{1,p} < \rho_0$ *almost everywhere, then the problem* (1.1) *admits at least one nontrivial weak solution* $u \in W_0^{1,p}(\Omega)$.

The question we are concerned with is whether or not the results obtained in [5] can be generalised to the case of the system of equations of the type (1.1). In this paper we are focused on an autonomous case. Let us consider the system

$$\begin{cases} -\Delta_p u(x) = f(u(x)), & x \in \Omega, \\ u_i(x) \geq 0, & x \in \Omega,\ 1 \leq i \leq m, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $u = (u_1,\ldots,u_m)$, $\Delta_p u = (\Delta_p u_1,\ldots,\Delta_p u_m)$ and $f\colon [0,+\infty)^m \to \mathbb{R}^m$ is a continuous function. The problem was investigated in [14] for $p = 2$ and a multivalued right-hand side. The existence of a positive solutions to systems involving p-Laplacian has been investigated by means of a topological approach, for example [2], [12], [13], [15], [19], as well as of a variational approach, for example [1], [18]. In all these papers, the right-hand side of the quasilinear elliptic system is assumed to be nonnegative. It seems this assumption is present in most of the articles related to the subject. One of the goals of our paper is to drop this assumption in favour of a weaker one.

Let us denote by θ the real function $\theta\colon \mathbb{R} \ni s \mapsto |s|^{p-2}s$ and let $\Theta = \theta\times\ldots\times\theta$ be the Cartesian product of m copies of θ.

The following assumption reflects the assumption (b) of Theorem 1.1 from the one-dimensional case:

$$f(u) = D_0\Theta(u) + o_0(u), \quad f(u) = D_\infty\Theta(u) + o_\infty(u), \quad u \in \mathbb{R}^m_+ \tag{1.3}$$