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ON A POWER-TYPE COUPLED SYSTEM OF MONGE–AMPÈRE EQUATIONS

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Abstract. We study an elliptic system coupled by Monge–Ampère equations:

$$\begin{cases} \det D^2 u_1 = (-u_2)^\alpha & \text{in } \Omega, \\ \det D^2 u_2 = (-u_1)^\beta & \text{in } \Omega, \\ u_1 < 0,\ u_2 < 0 & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

here Ω is a smooth, bounded and strictly convex domain in $\mathbb{R}^N$, $N \geq 2$, $\alpha > 0$, $\beta > 0$. When Ω is the unit ball in $\mathbb{R}^N$, we use index theory of fixed points for completely continuous operators to get existence, uniqueness results and nonexistence of radial convex solutions under some corresponding assumptions on α, β. When $\alpha > 0$, $\beta > 0$ and $\alpha\beta = N^2$ we also study a corresponding eigenvalue problem in more general domains.

1. Introduction

Consider the following system coupled by Monge-Ampère equations:

$$\begin{cases} \det D^2 u_1 = (-u_2)^\alpha & \text{in } \Omega, \\ \det D^2 u_2 = (-u_1)^\beta & \text{in } \Omega, \\ u_1 < 0,\ u_2 < 0 & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

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Here Ω is a smooth, bounded and strictly convex domain in $\mathbb{R}^N$, $N \geq 2$, $\alpha > 0$, $\beta > 0$; $\det D^2 u$ stands for the determinant of Hessian matrix $(\frac{\partial^2 u}{\partial x_i \partial x_j})$ of u.

Monge–Ampère equations are fully nonlinear second order PDEs, and there are important applications in geometry and other scientific fields. Monge–Ampère equations have been studied in the past years [1], [6], [9], [12], [16]. However, to our best knowledge, only a few works have been devoted to coupled systems. We refer the reader to [10] where the author established a symmetry result for a system, which arises in studying the relationship between two noncompact convex surfaces in $\mathbb{R}^3$. It seems to be H. Wang [13], [14] who first considered systems for Monge–Ampère equations. He investigated the following system of equations:

$$\begin{cases} \det D^2 u_1 = f(-u_2) & \text{in } B, \\ \det D^2 u_2 = g(-u_1) & \text{in } B, \\ u_1 = u_2 = 0 & \text{on } \partial B. \end{cases} \tag{1.2}$$

Here and in the following $B := \{x \in \mathbb{R}^N : |x| < 1\}$. By reducing it to a system coupled by ODEs and using the fixed point index, the author obtained the following results:

Theorem 1.1 ([13, Theorem 1.1]). *Suppose $f, g \colon [0, \infty) \to [0, \infty)$ are continuous.*

(a) *If $f_0 = g_0 = 0$ and $f_\infty = g_\infty = \infty$, then* (1.2) *has at least one nontrivial radial convex solution.*
(b) *If $f_0 = g_0 = \infty$ and $f_\infty = g_\infty = 0$, then* (1.2) *has at least one nontrivial radial convex solution.*

The notations were

$$f_0 := \lim_{x \to 0^+} \frac{f(x)}{x^N}, \qquad f_\infty := \lim_{x \to \infty} \frac{f(x)}{x^N}.$$

The above theorem implies the solvability of (1.2) is related to the asymptotic behavior of f, g at zero and at infinity. Obviously, it asserts the existence of a radial convex solution for system (1.1) if $\Omega = B$ and one of the following cases holds:

(1) $\alpha > N$, $\beta > N$,
(2) $\alpha < N$, $\beta < N$.

What we are curious about is, for the sublinear-superlinear case, i.e. $\alpha < N$, $\beta > N$, does system (1.1) *admits a radial convex solution when $\Omega = B$?*

We obtain that:

Theorem 1.2. *Let $\Omega = B$, then* (1.1) *has a radial convex solution if $\alpha > 0$, $\beta > 0$ and $\alpha\beta \neq N^2$.*